**6. Remarks.** Let us note that the factor $1+O(1/\log x)$ in Theorems 1 and 2 can be improved, namely it can be replaced by

$$1+b_1(1/\log x)+\ldots+b_m(1/\log^m x)+O(1/\log^{m+1} x)$$

It would be interesting to prove Theorem 1 by the elementary methods from [2]. However, it seems that the elementary approach cannot be used for Theorems 2 and 3, because for problems of distribution of values of arithmetical functions in sectorial regions the elementary techniques do not give satisfactory accuracy.

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# Sub-bases of pleasant $h$-bases

by

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Given an integral basis

$$A_k=\{a_1, a_2, \ldots, a_k\},\quad 1=a_1<a_2<\ldots<a_k$$

for a positive integer $h$, we form all the combinations

$$\sum_{i=1}^{k} x_i a_i,\quad x_i\geqslant 0,\quad \sum_{i=1}^{k} x_i\leqslant h,$$

and ask for the smallest integer $N_h(A_k)$ which is not represented by such a combination. The number $n_h(A_k)=N_h(A_k)-1$ is called the *h-range* of $A_k$. In this connection, $A_k$ is often denoted as *h-basis*.

A popular interpretation arises if we consider the integers $a_i$ as *stamp* denominations, and $h$ as the "size of the envelope". More information on the postage stamp problem can be found for instance in [4]. A comprehensive treatment of this problem is contained in the author's research monograph [5] (freely available on request). We only give here some more definitions which will be needed below.

A representation $n=\sum_{i=1}^{k} x_i a_i$ is called *regular* if we first use $a_k$ as often as possible, then $a_{k-1}$ as often as possible, etc. This means to impose the additional condition

$$\sum_{i=1}^{j} x_i a_i<a_{j+1},\quad j=1, 2, \ldots, k-1.$$

If only such representations are allowed, still restricted to at most $h$ addends, we speak of the *regular h-range* $g_h(A_k)$. Clearly $n_h(A_k)\geqslant g_h(A_k)$ for all $A_k$ and $h$. In contrast to $n_h(A_k)$, the general determination of $g_h(A_k)$ is fairly simple, see for instance [3].

A given integer may have several representations by a basis $A_k$. A *minimal* representation (not necessarily unique) is one with the smallest number of addends from the basis. Djawadi [1] called a basis *pleasant* (German:

"angenehm") if one minimal representation of $n$ always coincides with the (unique) regular representation, for all natural numbers $n$. In such cases, we clearly have $n_h(A_k) = g_h(A_k)$ for all $h$. This equality also holds for certain non-pleasant bases, which are then denoted as *weakly pleasant*.

Let $A_i = \{1, a_2, \ldots, a_i\}$, $2 \leqslant i \leqslant k$, be a "partial basis" of $A_k$. Then $A_2$ is always pleasant, and Djawadi [1] gave the following criterion for pleasantness in general: Let $\langle x \rangle$ denote the smallest integer $\geqslant x$, and put

$$(1) \qquad a_i = \gamma_i a_{i-1} - \underbrace{\sum_{j=1}^{i-2} \beta_j^{(i)} a_j}_{\text{regular by } A_{i-2}}, \quad \gamma_i = \left\langle \frac{a_i}{a_{i-1}} \right\rangle.$$

Let further $A_{i-1}$ be pleasant. Then $A_i$ is pleasant if and only if

$$(2) \qquad \gamma_i > \sum_{j=1}^{i-2} \beta_j^{(i)}.$$

Djawadi's proof has been simplified by the author ([5], Ch. X).

If the condition (2) is satisfied for all $i = 3, 4, \ldots, k$, then all partial bases $A_i$ are pleasant, and we call $A_k$ *completely pleasant*.

Zöllner [6] showed that

$$(3) \qquad k \geqslant 4, \quad A_k \text{ pleasant} \Rightarrow \{1, a_2, a_i\} \text{ pleasant}, \quad 3 \leqslant i \leqslant k.$$

The condition was weakened to "$A_k$ *weakly pleasant*" by Kirfel [3].

In particular, a pleasant $A_k$ always has a pleasant partial basis $A_3$, and a pleasant $A_4$ is thus completely pleasant. For $k \geqslant 5$, there are pleasant $A_k$ which are not completely pleasant. For $k = 5$, all such bases were determined by Djawadi [2]:

$$(4) \qquad A_5 = \{1, 2, b, b+1, 2b\}, \quad b \geqslant 4$$

(where $A_4$ is non-pleasant for $b \geqslant 4$).

For $k = 6$, the similar bases were characterized by Zöllner [6]. On the average, probably "most" pleasant bases are completely pleasant.

Even if the complete set of conditions (2), for $i = 4, 5, \ldots, k$, is not always necessary for pleasantness of $A_k$, there are some cases of necessity. Djawadi writes (1) as

$$(5) \qquad a_i + \sum_{j=1}^{i-2} \beta_j^{(i)} a_j = \gamma_i a_{i-1},$$

where the left-hand side is a regular representation by $A_i$. If then (2) fails, this representation has a larger coefficient sum than the non-regular representation $\gamma_i a_{i-1}$, and $A_i$ is then not pleasant by definition. In particular, the condition (2) for $i = k$ is thus always necessary for pleasantness of $A_k$ (whether $A_{k-1}$ is pleasant or not).



We have observed the following trivial but perhaps useful generalization: If $i < k$, and $\gamma_i a_{i-1} < a_{i+1}$, the left-hand side of (5) is also a regular representation by the full basis $A_k$. Hence, *if*

$$(6) \qquad \left\langle \frac{a_i}{a_{i-1}} \right\rangle a_{i-1} < a_{i+1} \quad (i < k),$$

*the condition* (2) *is necessary for pleasantness of* $A_k$.

If $k > 3$, and we remove the basis elements $a_3, a_4, \ldots, a_{k-1}$, it follows from (3) with $i = k$ that the "sub-basis" $\{1, a_2, a_k\}$ is pleasant if $A_k$ is pleasant (or only weakly pleasant by [3]). We can prove the following generalization:

THEOREM. *If* $k \geqslant 5$, $3 \leqslant \varkappa \leqslant k-2$, *and the partial bases* $A_i$, $i = \varkappa, \varkappa+1, \ldots, k$, *are all pleasant, then*

$$A_k^{(\varkappa)} = \{1, a_2, \ldots, a_\varkappa, a_k\}$$

*is also pleasant. If in particular* $A_k$ *is completely pleasant, so is* $A_k^{(\varkappa)}$ *for all* $\varkappa$.

Before proving this, we make some comments:

(i) We must remove a "block" $a_{\varkappa+1}, \ldots, a_{k-1}$ of elements in $A_k$ up to $a_{k-1}$. The simplest counterexample is given by the completely pleasant basis $A_5 = \{1, 2, 3, 5, 7\}$. Removing $a_3$, we get the non-pleasant basis $\{1, 2, 5, 7\}$.

(ii) The condition $A_i$ pleasant for all $i = \varkappa, \varkappa+1, \ldots, k$ is not always necessary. For instance, the Djawadi basis (4) leads to $A_5^{(3)} = \{1, 2, b, 2b\}$, which is pleasant by (2).

(iii) As an example where the Theorem fails when $A_i$ is not pleasant for all $i = \varkappa, \varkappa+1, \ldots, k$, consider the following extension of (4):

$$A_6 = \{1, 2, b, b+1, 2b, a_6\}, \quad b \geqslant 4,$$

which is pleasant if $a_6 > 2b$ is chosen such that (2) holds for $i = 6$. However, $A_6^{(4)} = \{1, 2, b, b+1, a_6\}$ is not of the form (4), and is consequently not pleasant since the partial basis $A_4$ is not.

To prove the Theorem, it will clearly suffice to use repeated removal of the next largest element, hence to show that

$$(7) \qquad A_k^{(k-2)} = \{1, a_2, \ldots, a_{k-2}, a_k\}$$

is pleasant. For this purpose, we substitute $a_{k-1}$ from (1) with $i = k-1$ into (1) with $i = k$, and get $a_k$ expressed by $A_{k-2}$ as

$$(8) \qquad a_k = (\gamma_k \gamma_{k-1} - \beta_{k-2}^{(k)}) a_{k-2} - \sum_{j=1}^{k-3} (\gamma_k \beta_j^{(k-1)} + \beta_j^{(k)}) a_j$$

$$= \tilde{\gamma} a_{k-2} - \sum_{j=1}^{k-3} \tilde{\beta}_j a_j \quad \text{(say)}.$$

Using (2) for $i = k-1$ and $i = k$, this gives

$$\tilde{\gamma} - \sum_{j=1}^{k-3} \tilde{\beta}_j = \gamma_k\Big(\gamma_{k-1} - \sum_{j=1}^{k-3} \beta_j^{(k-1)}\Big) - \sum_{j=1}^{k-2} \beta_j^{(k)} \geqslant \gamma_k - \sum_{j=1}^{k-2} \beta_j^{(k)} > 0,$$

in analogy with (2). However, we do not know if (8) corresponds to the form (1) for the basis (7), where we now need

$$a_k = \gamma a_{k-2} - \underbrace{\sum_{j=1}^{k-3} \beta_j a_j}_{\text{regular by } A_{k-3}}, \qquad \gamma = \left\langle \frac{a_k}{a_{k-2}} \right\rangle. \tag{9}$$

Equating the two expressions for $a_k$, we get

$$\tilde{\gamma} a_{k-2} + \sum_{j=1}^{k-3} \beta_j a_j = \gamma a_{k-2} + \sum_{j=1}^{k-3} \tilde{\beta}_j a_j.$$

The left-hand side is a regular representation by the pleasant basis $A_{k-2}$, and thus has a minimal coefficient sum:

$$\tilde{\gamma} + \sum_{j=1}^{K-3} \beta_j \leqslant \gamma + \sum_{j=1}^{k-3} \tilde{\beta}_j,$$

$$\gamma - \sum_{j=1}^{k-3} \beta_j \geqslant \tilde{\gamma} - \sum_{j=1}^{k-3} \tilde{\beta}_j > 0.$$

This shows that (2) is satisfied for the form (9). Since $A_{k-2}$ is pleasant, so is also the basis (7), and the Theorem is proved.

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# A simple construction of minimal asymptotic bases

by

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**1. Introduction.** Let $N$ be the set of all nonnegative integers. A subset $A$ of $N$ is called an *asymptotic basis of order h* if every sufficiently large integer can be represented as a sum of $h$ not necessarily distinct elements in $A$. An asymptotic basis $A$ of order $h$ is called *minimal* if no proper subset of $A$ is an asymptotic basis of order $h$. Stöhr [4] introduced this concept of minimality. Härtter [1] showed by a nonconstructive argument that there exist minimal asymptotic bases. Nathanson [2] constructed the first nontrivial examples of minimal asymptotic bases of order $h \geqslant 2$. In this paper we give a simple and explicit construction of minimal asymptotic bases of order $h$ for every $h \geqslant 2$. In particular, it is proved that if $h \geqslant 2$ and $1/h \leqslant \alpha < 1$, then there exists a minimal asymptotic basis of order $h$ whose counting function has order of magnitude $x^{\alpha}$.

**2. Results.** Let $W$ be a subset of $N$. Denote by $\mathscr{F}^*(W)$ the set of all finite, nonempty subsets of $W$. Let $A(W)$ be the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathscr{F}^*(W)$. Note that $\emptyset \notin \mathscr{F}^*(W)$, hence $0 \notin A(W)$. For any real number $x$, let $[x]$ denote the greatest integer $n$ such that $n \leqslant x$, and $\langle x \rangle$ the least integer $n$ such that $n \geqslant x$. If $A$ is a subset of $N$, let $hA$ denote the set of all sums of $h$ elements of $A$. Let $A(x)$ denote the counting function of $A$.

THEOREM 1. *Let $h \geqslant 2$, and let $t = \langle \log(h+1)/\log 2 \rangle$. Partition $N$ into $h$ pairwise disjoint subsets $W_0, \ldots, W_{h-1}$ such that each set $W_r$ contains infinitely many intervals of $t$ consecutive integers. Then*

$$A = A(W_0) \cup \ldots \cup A(W_{h-1})$$

*is a minimal asymptotic basis of order h.*

The proof uses the following two lemmas of Nathanson [3].

LEMMA 1. (a) *If $W_1$ and $W_2$ are disjoint subsets of $N$, then* $A(W_1) \cap A(W_2) = \emptyset$.