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On diameters of algebraic integers

by

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1. Introduction. Let α be an algebraic integer of degree $n \geqslant 2$ with conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. The *diameter* of α, denoted by $\operatorname{diam}(\alpha)$, is defined by

$$\operatorname{diam}(\alpha) = \max_{i,j} |\alpha_i - \alpha_j|.$$

Lower bounds for $\operatorname{diam}(\alpha)$ were first found by Favard [3], [4], [5], and more recently by several other authors. For the recent history of this and related problems, see [1], [6]. Among other results, it was shown by Lloyd-Smith [7], using a very simple argument, that

$$\operatorname{diam}(\alpha) > 3/2.$$

Using a more complicated argument, McAuley showed in his thesis [9] that

$$\operatorname{diam}(\alpha) > 1.659.$$

On the basis of partial results and numerical results, it is conjectured that

$$\operatorname{diam}(\alpha) \geqslant \sqrt{3}.$$

In this paper we show this to be true under certain restrictions.

THEOREM 1. *If α is an algebraic integer of the form $\alpha = \beta + k$ where β is reciprocal and k an integer, then*

$$\operatorname{diam}(\alpha) \geqslant \sqrt{3}. \tag{1}$$

Moreover, when α is reciprocal, we have equality in (1) *if and only if $\alpha = \pm\zeta$ where ζ is a primitive cube root of unity.*

Remarks. (a) An algebraic number β is said to be *reciprocal* if β^{-1} is a conjugate of β.

(b) This result is due to Lloyd-Smith and may be found in his thesis [6]. We reproduce his proof in this paper.

We are also able to give the following partial result.

THEOREM 2. *There exists* $\varepsilon > 0$ *and a positive integer* n_0 *(depending on* ε*) such that*

$$\operatorname{diam}(\alpha) > \sqrt{3}+\varepsilon \tag{2}$$

whenever the degree of α *is at least* n_0. *In particular, we have*

$$\operatorname{diam}(\alpha) > 1.7321 \tag{3}$$

whenever the degree of α *exceeds* 200000.

Remarks. (a) The constants in the statement of Theorem 2 are of no particular significance. By further refinements of our arguments, we could improve the specific values of ε and n_0 that are given.

(b) It will be clear from the proof of Theorem 2 that our result could be sharpened if good estimates were available for the discriminant of a finite set of complex numbers lying in a region whose boundary is a suitably modified version of a Reuleaux triangle. It is hoped to pursue these ideas elsewhere.

Before proceeding to the proofs of these results it is convenient to define the *circumdiameter* of α to be the diameter of the smallest closed disc containing the conjugates of α. It will be denoted by $D(\alpha)$. We have the following consequence of a classical inequality due to Jung (see [10], pp. 17, 18):

$$\operatorname{diam}(\alpha) \leqslant D(\alpha) \leqslant \frac{2}{\sqrt{3}}\operatorname{diam}(\alpha). \tag{4}$$

In general this inequality is best possible, but it can be strengthened under certain restrictions, and this is implicit in our proof of Theorem 2.

2. Preliminaries. We prove several lemmas which are necessary for the proofs of Theorems 1 and 2.

LEMMA 1. *If* $r, \varrho > 0$, *then*

$$|re^{i\theta}-\varrho e^{i\varphi}| \geqslant 2r^* |\sin\tfrac{1}{2}(\theta-\varphi)|,$$

where $r^* = \min\{r, \varrho\}$.

Proof. The easy proof is omitted.

LEMMA 2. *Let* z *be a complex number satisfying the conditions*

$$|\operatorname{Im} z| \leqslant \sqrt{3}/2, \quad |z| \geqslant 1, \quad |z-z^{-1}| \leqslant \sqrt{3}, \quad \operatorname{Re} z > 0.$$

Then

$$|(z-1)(z^{-1}-1)| \leqslant 1, \tag{5}$$

with equality if and only if $z = (1\pm\sqrt{-3})/2$.

Proof. Firstly, observe that

$$|z|-|z|^{-1} \leqslant |z-z^{-1}| \leqslant \sqrt{3},$$

implying that

$$|z| \leqslant (\sqrt{3}+\sqrt{7})/2 < 2.2. \tag{6}$$

It readily follows from the inequalities $|\operatorname{Im} z| \leqslant \sqrt{3}/2$ and $1 \leqslant |z| < 2.2$ that

$$|z-2| \leqslant \sqrt{3}. \tag{7}$$

However, (7) is equivalent to

$$\sqrt{3}\,|z-1| \leqslant |z+1|.$$

Thus if z satisfies the hypotheses of Lemma 2, it follows that

$$|(z-1)(z^{-1}-1)| = |z|^{-1}|z-1|^2 = |z|^{-1}|z^2-1|\left|\frac{z-1}{z+1}\right|$$

$$\leqslant \frac{1}{\sqrt{3}}|z|^{-1}|z^2-1| = \frac{1}{\sqrt{3}}|z-z^{-1}| \leqslant 1.$$

If we had $|(z-1)(z^{-1}-1)| = 1$, then it would follow that $|z-2| = \sqrt{3}$, and the conditions $|\operatorname{Im} z| \leqslant \sqrt{3}/2$ and $|z| \geqslant 1$ would imply that $z = (1\pm\sqrt{-3})/2$ or $\operatorname{Re} z \geqslant 7/2$. The latter case is excluded by (6). It is easily checked that $z = (1\pm\sqrt{-3})/2$ satisfies the hypotheses of the lemma and also satisfies $|(z-1)(z^{-1}-1)| = 1$. This completes the proof of Lemma 2.

LEMMA 3. *Let* d_n *be the least absolute value for discriminants of algebraic number fields of degree* n *over the rationals. If* α *is an algebraic integer of degree* n, *then*

$$D(\alpha) \geqslant 2\left(\frac{|d_n|}{n^n}\right)^{1/n(n-1)}. \tag{8}$$

Proof. This inequality is implicit in the work of McAuley [9]. If the circumcircle of the set of conjugates of α has centre μ and radius ϱ, then μ is real and $\varrho = \frac{1}{2}D(\alpha)$. Putting $z_j = \varrho^{-1}(\alpha_j-\mu)$ for $1 \leqslant j \leqslant n$, we get

$$|d_n| \leqslant \prod_{i\neq j}|\alpha_i-\alpha_j| = \prod_{i\neq j}\varrho\,|z_i-z_j|.$$

The result follows after using Hadamard's inequality to estimate the Vandermonde determinant, since $|z_j| \leq 1$ $(1 \leq j \leq n)$.

LEMMA 4. *Suppose $z_1, z_2, \ldots, z_n$ are complex numbers satisfying $|z_j| \leq 1$ $(1 \leq j \leq n)$, and suppose that there is an integer s satisfying $1 \leq s \leq n$ such that $|z_j| \leq r$ for $1 \leq j \leq s$ where r is a given real number in the range $0 < r < 1$. Then*

$$\prod_{i \neq j} |z_i - z_j| \leq n^n r^{(n-1)(2s-n)}.$$

Proof. Using Hadamard's inequality on the corresponding Vandermonde determinant we get

$$\prod_{i \neq j} |z_i - z_j| = r^{n(n-1)} \prod_{i \neq j} \left|\frac{z_i}{r} - \frac{z_j}{r}\right| \leq r^{n(n-1)} \prod_{j=1}^{n} \sum_{k=0}^{n-1} \left|\frac{z_j}{r}\right|^{2k}$$
$$\leq r^{n(n-1)} n^s \{nr^{-2(n-1)}\}^{n-s} \leq n^n r^{(n-1)(2s-n)}.$$

Remark. This lemma is sufficient for the purpose of proving Theorem 2, but a stronger result would enable a sharpening of the theorem.

3. Proof of Theorem 1. To prove Theorem 1 we may clearly suppose that α is reciprocal and that

$$|\mathrm{Im}\,\alpha_j| \leq \sqrt{3}/2 \quad (1 \leq j \leq n).$$

As α is reciprocal and since we may assume $\mathrm{diam}(\alpha) \leq \sqrt{3}$, we have

$$|\alpha_j - \alpha_j^{-1}| \leq \sqrt{3} \quad (1 \leq j \leq n).$$

Also, by replacing α with $-\alpha$ if necessary, we may suppose that there exists an integer m $(1 \leq m \leq n)$ such that

$$|\alpha_m| \geq 1, \quad \mathrm{Re}\,\alpha_m > 0.$$

We now conclude that all conjugates of α have positive real part, for otherwise we may suppose that for some r $(1 \leq r \leq n)$

$$|\alpha_r| \geq 1 \quad \text{and} \quad \mathrm{Re}\,\alpha_r < 0.$$

We would then have

$$\mathrm{diam}(\alpha) \geq \max\{|\alpha_m - \alpha_r|, |\alpha_m - \bar{\alpha}_r|\} \geq \sqrt{3} \quad \text{by Lemma 1}.$$

This is a contradiction, since equality would correspond to the condition

$$\alpha_m = (1 \pm \sqrt{-3})/2, \quad \alpha_r = -1 \quad \text{(or vice versa)},$$

which is impossible.

We now invoke Lemma 2 to see that $|(\alpha_m - 1)(\alpha_m^{-1} - 1)| \leq 1$ for all j $(1 \leq j \leq n)$. It follows that the norm of $\alpha - 1$ satisfies

$$|N(\alpha - 1)| \leq 1.$$

Since $\alpha - 1 \neq 0$, we deduce from Lemma 2 that $\alpha = (1 \pm \sqrt{-3})/2$. This completes the proof of Theorem 1.

4. Proof of Theorem 2. Using the notations introduced in the proof of Lemma 3, we write

$$\tau = \max_{i,j} |z_i - z_j| = \varrho^{-1}\,\mathrm{diam}(\alpha).$$

Since $|d_n| > 1$, the result of Lemma 3 implies that, in order to prove the theorem, it will suffice to prove the existence of an $\varepsilon > 0$ such that

$$\tau \geq \sqrt{3} + \varepsilon$$

for sufficiently large n.

In particular, to prove (3), it suffices to show that

$$\tau \geq 1.7322. \tag{9}$$

This follows since for $n \geq 200\,000$ we have

$$\varrho \geq \left(\frac{22}{n}\right)^{1/(n-1)} \geq 0.999954 \tag{10}$$

(using bounds for d_n derived from the tables of Diaz y Diaz [2], formula (1), p. 2).

By Jung's inequality (4), we see that (3) also follows from the inequality

$$\varrho \geq 1.0000285. \tag{11}$$

We first establish that there are z_j on certain arcs of the unit circle. Let S be the subset of $\{z_1, z_2, \ldots, z_n\}$ containing those z_j with unit modulus. Both sets are closed under complex conjugation. Without loss of generality, we may suppose that $\pm i$ are not in S, otherwise $\tau = 2$ and (9) is satisfied. We show that the real parts of members of S cannot all be of the same sign.

Indeed, suppose that $\mathrm{Re}\,z > 0$ for all $z \in S$. Now those z_j not in S lie strictly inside the unit circle. It easily follows that there exists a circle centred on some $\delta > 0$ of radius less than 1 which contains $\{z_1, z_2, \ldots, z_n\}$. But then the unit circle would not be the circumcircle of $\{z_1, z_2, \ldots, z_n\}$. This is a contradiction. The case where $\mathrm{Re}\,z < 0$ for all $z \in S$ is handled similarly. It follows that there exist z_i and z_j belonging to S such that $\mathrm{Re}\,z_i > 0$ and $\mathrm{Re}\,z_j < 0$. Further we can assume that $\max_{z \in S} |\mathrm{Im}\,z|$ occurs for $z = z_j$ where j is such that $\mathrm{Re}\,z_j < 0$.

If x is a small positive number, $|z_j| = 1$ and

$$\pi/3 + x \leqslant |\arg z_j| \leqslant 2\pi/3 - x$$

then it follows from Lemma 1 that

$$|z_j - \bar{z}_j| \geqslant 2\left|\sin\left(\frac{\pi}{3} + x\right)\right|.$$

Thus if we take $x = 1/1500$, we deduce

$$\tau \geqslant |z_j - \bar{z}_j| \geqslant 1.7327,$$

and the result follows since (9) is satisfied.

However, suppose the subscript j is such that $|z_j| = 1$,

$$\operatorname{Im} z_j = \max_{z \in S} |\operatorname{Im} z|, \qquad \operatorname{Re} z_j < 0$$

and for this value of x,

$$2\pi/3 + 2x \leqslant \arg z_j \leqslant 4\pi/3 - 2x.$$

From previous arguments, there exists z_i with $|z_i| = 1, \operatorname{Re} z_i > 0$ and $|\operatorname{Im} z_i| \leqslant |\operatorname{Im} z_j|$. Without loss of generality, we may assume that $\operatorname{Im} z_j > 0 > \operatorname{Im} z_i$ and a simple argument shows that

$$\tau \geqslant |z_j - z_i| \geqslant |z_j - 1| \geqslant 2\cos(\pi/3 + x) \geqslant 1.7322$$

and the result again follows.

We have thus shown that if $\operatorname{diam}(\alpha) \leqslant 1.7321$ we have some z_j on each of the arcs of the unit circle

$$A_1 = \{z\colon\ |z| = 1, 2\pi/3 - x < \arg z < 2\pi/3 + 2x\}$$

and

$$A_2 = \{z\colon\ |z| = 1, 4\pi/3 - 2x < \arg z < 4\pi/3 + x\}.$$

Using an analogous argument and the assumption that $\operatorname{diam}(\alpha) \leqslant 1.7321$, we deduce from Lemma 1 that we also have some z_j on the arc

$$A_3 = \{z\colon\ |z| = 1, -3x < \arg z < 3x\}.$$

Thus we may assume that there are z_j on each of these three specified arcs of the unit circle. Note that each arc contains a cube root of unity; this is not surprising since Jung's inequality is precise for vertices of an equilateral triangle.

We can further reduce the problem as follows. Set

(12) $$r = 0.999702$$

and consider the three regions

$$B_1 = \{z\colon\ r \leqslant |z| \leqslant 1, 3x \leqslant \arg z \leqslant 2\pi/3 - x\},$$

$$B_2 = \{z\colon\ r \leqslant |z| \leqslant 1, 2\pi/3 + 2x \leqslant \arg z \leqslant 4\pi/3 - 2x\},$$

$$B_3 = \{z\colon\ r \leqslant |z| \leqslant 1, 4\pi/3 + x \leqslant \arg z \leqslant 2\pi - 3x\}.$$

Using Lemma 1, the existence of z_j in each of the arcs A_1, A_2 and A_3, and the fact that the set $\{z_1, z_2, \ldots, z_n\}$ is closed under complex conjugation, we deduce that whenever there is a z_j within one of the regions B_1, B_2 or B_3, then

$$\tau \geqslant 2r\sin(\pi/3 + x) > 1.7322.$$

Thus we conclude that if $\operatorname{diam}(\alpha) \leqslant 1.7321$, then

$$z_j \in R = M_1 \cup M_2 \cup M_3 \cup S \qquad (1 \leqslant j \leqslant n),$$

where

$$M_1 = \{z\colon\ r < |z| \leqslant 1, 2\pi/3 - x \leqslant \arg z \leqslant 2\pi/3 + 2x\},$$

$$M_3 = \{z\colon\ r < |z| \leqslant 1, -3x \leqslant \arg z \leqslant 3x\},$$

$$M_2 = \bar{M}_1, \qquad S = \{z\colon\ |z| \leqslant r\}.$$

The remainder of the proof rests on an estimation of $\Delta = \prod |z_i - z_j|$ for $z_j \in R$ $(1 \leqslant j \leqslant n)$. As remarked earlier other shaped regions would be more appropriate, but the estimation of Δ for such regions is more difficult.

Let m_k be the number of the z_j in M_k $(1 \leqslant k \leqslant 3)$ and let s be the number of the z_j in S. Let $m = m_1 + m_2 + m_3$ and note that $m + s = n$. The remainder of the proof is divided into two cases depending upon the proportion of z_j in S.

Suppose first that

$$s \geqslant \tfrac{5}{8}n.$$

As in the proof of Lemma 3,

$$|d_n| \leqslant \varrho^{m(n-1)}\Delta$$

and so by Lemma 4,

$$|d_n| \leqslant \varrho^{m(n-1)} n^n r^{1/4n(n-1)}.$$

Thus

$$\varrho \geqslant r^{-1/4}\left(\frac{|d_n|}{n^n}\right)^{1/m(n-1)} > 1.0000285$$

by (12) and the estimates involved in establishing (10). This settles the first case, by (11).

Now consider the complementary case where

$$m > \tfrac{3}{8}n. \tag{13}$$

This is handled by a somewhat more tedious estimation of Δ. It can be checked by a straightforward computation that M_1 and M_2 can each be covered by a disc of radius

$$\delta = 1/980 \tag{14}$$

and further, M_3 can be covered by a disc of radius 2δ. To estimate Δ we give upper bounds for the discriminants of the z_j in each of the four subregions M_1, M_2, M_3 and S, and a trivial upper bound for the cross-products. Writing $y = 1.7322$, we get

$$\Delta \leqslant (m_1^{m_1}\delta^{m_1(m_1-1)})(m_2^{m_2}\delta^{m_2(m_2-1)})(m_3^{m_3}(2\delta)^{m_3(m_3-1)}) \times (s^s r^{s(s-1)})\, y^{2ms}\, y^{2(m_1m_2+m_1m_3+m_2m_3)}, \tag{15}$$

as we may clearly assume that any $|z_i - z_j| \leqslant y$ else (9) is satisfied. Now

$$\delta^{m_1^2+m_2^2+m_3^2}\, 2^{m_3^2} = \delta^{m_1^2+m_2^2+(1-\beta)m_3^2}$$

where $\beta = \log 2/\log\delta^{-1}$. For β satisfying $0 \leqslant \beta \leqslant 1$, a straightforward application of Lagrange multipliers shows that, given the constraint $m_1+m_2+m_3 = m$, we have

$$m_1^2+m_2^2+(1-\beta)m_3^2 \geqslant \frac{1-\beta}{3-2\beta}m^2.$$

When β is specified as above, we have

$$\frac{1-\beta}{3-2\beta} \geqslant \frac{m^2}{3.112}.$$

Taking $\beta = 0$, we readily obtain

$$m_1m_2+m_1m_3+m_2m_3 \leqslant \tfrac{1}{3}m^2.$$

Also we have the trivial bound

$$s^s\prod_{j=1}^{3} m_j^{m_j} \leqslant n^{s+m_1+m_2+m_3} = n^n.$$

Thus, by (13), (14) and (15)

$$\Delta \leqslant n^n \delta^{m^2/3.112}\delta^{-m} y^{2m(n-m)} y^{2m^2/3}$$
$$= n^n\left\{\left(\frac{\delta^{1/3.112}}{y^{4/3}}\right)^{3/8}\delta^{-1/n}y^2\right\}^{mn}$$
$$\leqslant n^n\{\delta^{3/25}\delta^{-1/n}y^{3/2}\}^{3n^2/8} < n^n(0.9992)^{n^2}.$$

As before

$$|d_n| \leqslant \varrho^{n(n-1)}\Delta$$

and using (10) we deduce

$$\varrho > 1.0007.$$

Hence the result follows in this case too, by (11).

The proof of Theorem 2 is now complete.

Note added in proof by the editor. The Favard problem has been recently solved by M. Langevin, E. Reyssat, G. Rhin in the paper *Diamètres transfinis et problème de Favard*, Ann. Int. Fourier 38, fasc. 1, 1–16; see also M. Langevin, *Solution des problèmes de Favard*, ibid. fasc. 2, 1–10.

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