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On metrical theory of diophantine approximation over imaginary quadratic field

by

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For a complex number z, we consider a diophantine inequality

$$(1)\qquad \left|z-\frac{p}{q}\right| < \frac{k}{|q|^2}, \quad k > 0,\ p,\ q \in o_d \text{ and } (p, q) = 1,$$

where d is a square free negative integer, o_d denotes the ring of integers of $Q(\sqrt{d})$ and $(p, q) = 1$ means ideals generated by p and by q are relatively prime.

In the case of $d = -1$, LeVeque [8] proved Khintchine type metrical theorem by using complex continued fractions. Moreover, Sullivan [11] established Khintchine theorem for any square free negative integer d by using the notion of disjoint spheres on the three-dimensional hyperbolic space $H^3 = \{(x_1, x_2, y)\colon x_1, x_2 \in R,\ y > 0\}$. His method shows that the asymptotic property of solutions of (1) is related to an excursion of the geodesic flow on $H^3/\mathrm{PSL}(2, o_d)$ to the point of infinity.

In this paper, we estimate the asymptotic number of solutions of (1) by using (generalized) Ford balls. Such a family of balls was first considered by Ford [4] and [5] to determine an approximating constant of (1) in the case of $d = -1$.

MAIN THEOREM. *For almost all $z \in C$, we have*

$$\lim_{N\to\infty} \frac{\#\left\{\frac{p}{q} \in Q(\sqrt{d})\colon \frac{p}{q} \text{ satisfies (1) and } |q| \leqslant N\right\}}{\log N} = C_d\cdot k^2$$

for any $k > 0$, where C_d is a constant depending on d.

Similar theorem also holds for some discrete subgroups, with a cusp at infinity, of $\mathrm{PSL}(2, C)$ acting on H^3. (Such a class of subgroups corresponds to the class of zonal subgroups of $\mathrm{PSL}(2, R)$ acting on the upper half plane H^2.) We prove the assertion in the general case in Section 3 and next apply it to the case of $\mathrm{PSL}(2, o_d)$ and its subgroups in Section 4. We also determine the constant C_d. For preparation, we introduce the notion of Ford balls in Section 2.

Our argument is also available for an approximation of real numbers by cusps of a zonal Fuchsian group Γ. To do this, we use Ford disks and the geodesic flow on H^2/Γ. Then we see that the asymptotic numbers of solutions are in proportion to k (instead of k^2), see [10].

1. Preliminaries. Let H^3 be the upper half-space. We write $w \in H^3$ by

$$w = x_1 + x_2 i + yj \quad \text{with } x_1, x_2 \in R \text{ and } y > 0,$$

where $1, i, j, k$ are the basis of the quaternion fields. We consider

$$P^1(C) = C \cup \{\infty\} \quad (= \partial H^3).$$

The hyperbolic metric s on H^3 and its associated measure μ are defined by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2} \quad \text{and} \quad d\mu = \frac{dx_1 dx_2 dy}{y^3},$$

respectively. Furthermore, the induced measure $\bar{\mu}$ on the unit tangent space $\mathrm{UT}(H^3) = H^3 \times S^2$ is given by

$$d\bar{\mu} = d\mu \otimes (\cos\varphi\, d\theta\, d\varphi),$$

where θ and φ denote longitude and latitude, respectively. It follows that the geodesics are half-circles and half-lines perpendicular to C.

For any matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, C)$, we define the Möbius transformation on H^3 by

$$g(w) = (aw+b)(cw+d)^{-1}$$

for $w \in H^3$. This preserves the hyperbolic metric and measure. It is also possible to define the transformation on $\mathrm{UT}(H^3)$ by g. Since the transformation g is conformal, it follows that g also preserves the measure $\bar{\mu}$ on $\mathrm{UT}(H^3)$. An element g of $\mathrm{SL}(2, C)$ is said to be a *parabolic element* if (trace of $g)^2 = 4$.

For any $(w, \theta, \varphi) \in \mathrm{UT}(H^3)$ with $w = x_1 + x_2 i + yj$, there is a unique geodesic curve passing through w with the tangent vector (θ, φ). We denote such a geodesic by $\overrightarrow{(z_1, z_2)}$, where z_1 and z_2 denote the initial and the terminal points in $P^1(C)$, respectively. We also denote by u the "signed" hyperbolic distance from the top of the geodesic $\overrightarrow{(z_1, z_2)}$ to w, if $\overrightarrow{(z_1, z_2)}$ is a half-circle, from $x_1 + x_2 i + j$ to w, if $\overrightarrow{(z_1, z_2)}$ is a half-line. Then we may regard

$$\mathrm{UT}(H^3) = \{(z_1, z_2, u) : (z_1, z_2) \in P^1(C) \times P^1(C) \backslash \{\text{diagonal}\}, u \in R\}.$$

In this representation, it follows by a simple calculation of the change of variables that

$$(2) \qquad d\bar{\mu} = \frac{4\, dx_1 dx_2 dx_1' dx_2' du}{|z_1 - z_2|^4}$$

where $z_1 = x_1 + x_2 i$ and $z_2 = x_1' + x_2' i$.

For any $w = (z_1, z_2, u) \in \mathrm{UT}(H^3)$, we consider the horosphere tangent to $P^1(C)$ at z_2 and perpendicular to $\overrightarrow{(z_1, z_2)}$ at the base point of (z_1, z_2, u). Then for any z_1' $(\neq z_2) \in P^1(C)$, there is a unique point, say $\hat{w}' = (z_1', z_2, \tilde{u})$, in $\overrightarrow{(z_1', z_2)}$, whose base point is on the above horosphere.

Lemma 1.1. *The hyperbolic distance between the base points of* $(z_1, z_2, u+t)$ *and* $(z_1', z_2, \tilde{u}+t)$ *converges to zero exponentially as* t *tends to* ∞.

We consider a discrete subgroup Γ of the group

$$\mathrm{PSL}(2, C) = \mathrm{SL}(2, C) / \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

which is the group of all orientation preserving isometries for the hyperbolic metric. Γ acts discontinuously on H^3 as Möbius transformations. A parabolic fixed point $z \in P^1(C)$ of Γ is said to be a *cusp of* Γ if

$$\left\{ w : \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \in g\Gamma g^{-1} \right\}$$

is a lattice in C, where g is an element of $\mathrm{PSL}(2, C)$ such that $z = g^{-1}(\infty)$.

Suppose that ∞ is a cusp of Γ. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, C)$ with $c \neq 0$, we denote by K_g the isometric hemi-sphere of g, that is,

$$K_g = \{ w \in H^3 : u_1^2 + u_2^2 + v^2 = 1 \text{ with } cw + d = u_1 + u_2 i + vj \}.$$

We put

$$I(K_g) = \{ w \in H^3 : u_1^2 + u_2^2 + v^2 < 1 \text{ with } cw + d = u_1 + u_2 i + vj \},$$

$$E(K_g) = \{ w \in H^3 : u_1^2 + u_2^2 + v^2 > 1 \text{ with } cw + d = u_1 + u_2 i + vj \}$$

and r_g be the (Euclidean) radius of K_g. We also define $D_* \subset C$ by a fundamental region of the lattice induced from $\Gamma_\infty = \{ g \in \Gamma : g(\infty) = \infty \}$ and put

$$D^* = \{ w = x_1 + x_2 i + yj : y > 0,\ x_1 + x_2 i \in D_* \}.$$

Lemma 1.2. *If we put*

$$D = \Big(\bigcap_g E(K_g) \Big) \cap D^*,$$

then D *is a fundamental region of* Γ, *where* g *runs over all elements in* $\Gamma \backslash \Gamma_\infty$.

Proof. This follows from the facts: (i) $g(I(K_g)) = E(K_{g^{-1}})$ and (ii) $y < v$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $c \neq 0$, $w = x_1 + x_2 i + yj \in H^3$ and $cw+d = u_1 + u_2 i + vj$.

2. Ford balls. Let Γ be a finitely generated discrete subgroup of $\mathrm{PSL}(2, C)$ with $\mu(H^3/\Gamma) < +\infty$ and suppose that ∞ is a cusp of Γ. We put

$$F_k(\infty) = \{w \in H^3\colon\ w = x_1 + x_2 i + yj,\ y > 1/(2k)\}$$

for $k > 0$ and define

$$F_k(g(\infty)) = g(F_k(\infty))$$

for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. Since

$$g = \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1/c \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} \tag{3}$$

for $c \neq 0$, we see the following:

Lemma 2.1. *$F_k(g(\infty))$ is the open ball in H^3 with the (Euclidean) radius $r(g)^2 \cdot k$ and tangent to C at a/c provided $c \neq 0$.*

Proof. Since $r(g) = 1/|c|$, the assertion of the lemma follows from (3) easily.

Since ∞ is a cusp of Γ, it turns out that there exists $k_0 > 0$ such that for $0 < k < k_0$ we have

$$F_k(g(\infty)) \cap F_k(g'(\infty)) = \emptyset$$

provided $g(\infty) \neq g'(\infty)$ and $g, g' \in \Gamma$. We call

$$F_k(g(\infty))\colon\ g \in \Gamma,\ 0 < k < k_0$$

the *k-Ford balls of* Γ and the supremum of such k_0 the *Ford radius of* Γ.

Proposition 2.2 (Ford [6]). *For $\Gamma = \mathrm{PSL}(2, o_d)$ with a square free negative integer d, the Ford radius is equal to $1/2$.*

Now we introduce a diophantine approximation with respect to cusps of Γ. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $c \neq 0$, and $k > 0$, we consider the inequality

$$|z - g(\infty)| < r(g)^2 \cdot k \quad \text{for } z \in C, \tag{4}$$

which corresponds to a problem of Lehner ([7], pp. 334–336) in the case of zonal Fuchsian groups. Since the set of k-Ford balls are invariant under the Γ-action, we see the next lemma.

Lemma 2.3. *The following are equivalent for $z \in C$:*
(i) $|z - g(\infty)| < r(g)^2 \cdot k$,
(ii) $F_k(g(\infty)) \cap \overrightarrow{(\infty, z)} \neq \emptyset$,
(iii) $F_k((g' \cdot g)(\infty)) \cap \overrightarrow{(g'(\infty), g'(z))} \neq \emptyset$ *for* $g' \in \Gamma$,

where $\overrightarrow{(z_1, z_2)}$ denotes the geodesic curve in H^3 with the initial point z_1 and the terminal point z_2.

If we take $g' = g^{-1}$ in (iii) of the above lemma, then we have the next lemma.

Lemma 2.4. *The inequality* (4) *holds if and only if*

$$|g^{-1}(\infty) - g^{-1}(z)| > 1/k.$$

From this lemma and the construction of D in Lemma 1.2, we have the following:

Proposition 2.5. *Let $\{\hat{w}_t\colon t \in R\} \subset \mathrm{UT}(D)$, $\mathrm{UT}(D)$ denotes the unit tangent space of D, be the geodesic path congruent to $\{(\infty, z, t)\colon t \in R\}$ of $\mathrm{UT}(H^3)$ with $\hat{w}_0$ to be congruent to $(\infty, z, 0)$. Then it follows that if the base point of $\hat{w}_t$ goes into $F_k(\infty)$ at $t_0 > 0$, then there exists $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying* (4) *such that*

$$k \cdot e^{t_0} < r(g)^{-2} < 2 \cdot k \cdot e^{t_0}. \tag{5}$$

Proof. The existence of g follows from Lemma 2.3. Since $\hat{w}_{t_0}$ is congruent to (∞, z, t_0) and the base point of (∞, z, t_0) is $z + e^{-t_0} j \in H^3$, we have the inequality (5), see Fig. 1.

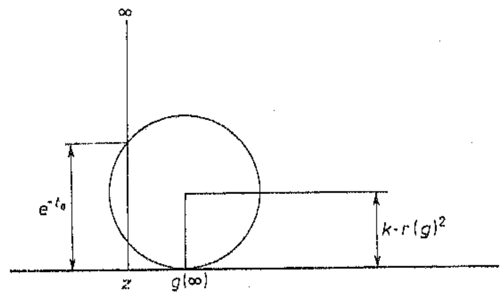

Fig. 1

3. Metrical theory of Γ-approximation. It is possible to count the entrance times of a geodesic path into Ford balls. In the sequel, we identify the unit tangent space $\mathrm{UT}(D)$ with a subset of

$$[P^1(C) \times P^1(C) \setminus \{\text{diagonal}\}] \times R.$$

For $\hat{w} \in \mathrm{UT}(D)$, we denote by $G_s(\hat{w})$ the geodesic flow on $\mathrm{UT}(D)$, that is, for $\hat{w} = (z_1, z_2, u)$, $G_t(\hat{w})$ is the point in $\mathrm{UT}(D)$ congruent to $(z_1, z_2, u+t)$. We put

$$D_k = D \cap \{w = x_1 + x_2 i + yj \colon y = 1/(2k)\}$$

and

$$\hat{D}_k = \{\hat{w} = (z_1, z_2, u) \colon u < 0 \text{ and } w \in D_k, \text{ where } w \text{ is the base point of } \hat{w}\}.$$

THEOREM 3.1. *For μ-almost all $\hat{w} \in \mathrm{UT}(D)$,*

$$\lim_{t\to\infty} \frac{\#\{s\colon 0<s<t,\ G_s(\hat{w}) \in \hat{D}_k\}}{t} = \frac{\lambda(D_*)\cdot\pi}{\bar{\mu}(\mathrm{UT}(D))}\cdot k^2 \tag{6}$$

for any positive k less than the Ford radius k_0, where λ denotes Lebesgue measure of $\boldsymbol{C}$.

Proof. Since $k_0 = \min \frac{1}{2}\cdot r(g)$, where g runs over all elements in $\Gamma \backslash \Gamma_\infty$, we see that

$$D_* = \left\{x_1 + x_2 i \colon x_1 + x_2 i + \frac{1}{2k} j \in D_k\right\}.$$

For any fixed $k < k_0$, we can choose a small positive number ε such that

$$\{G_s(\hat{w})\colon -\varepsilon < s < \varepsilon\} \cap \hat{D}_k = \{\hat{w}\}$$

for any $\hat{w} \in \hat{D}_k$. For this ε, we define

$$D_{k,\varepsilon} = \{\hat{w} \in \mathrm{UT}(D)\colon \hat{w} \notin F_k(\infty), \text{ the base point of } G_s(\hat{w}) \text{ is in } F_k(\infty) \text{ for some } s,\ 0<s<\varepsilon\}$$

and

$$\varphi_k(w) = \begin{cases} 1/\varepsilon & \text{if } \hat{w} \in \hat{D}_{k,\varepsilon}, \\ 0 & \text{otherwise.}\end{cases}$$

From this definition of φ_k, it follows that

$$\int_0^t \varphi_k(G_s(\hat{w}))ds - 1 \leqslant \int_0^{t-\varepsilon} \varphi_k(G_s(\hat{w}))ds \leqslant \#\{s\colon 0<s<t,\ G_s(\hat{w}) \in \hat{D}_k\} \tag{7}$$

$$\leqslant \int_{-\varepsilon}^{t} \varphi_k(G_s(\hat{w}))ds \leqslant \int_0^t \varphi_k(G_s(\hat{w}))ds + 1.$$

From (7) and the individual ergodic theorem, we have

$$\lim_{t\to\infty} \frac{\#\{s\colon 0<s<t,\ G_s(\hat{w}) \in \hat{D}_k\}}{t} = \frac{\int \varphi_k d\bar{\mu}}{\bar{\mu}(\mathrm{UT}(D))} \tag{8}$$

for almost all $\hat{w} \in \mathrm{UT}(D)$, since the geodesic flow G_t on $\mathrm{UT}(D)$ is ergodic (e.g., see [3]). Furthermore, it follows from (2) that

$$\int \varphi_k d\bar{\mu} = \iiiint_{\{(z_1,z_2)\colon |z_1-z_2|>1/k,(z_1+z_2)/2\in D_*\}} \frac{4dx_1\,dx_2\,dx'_1\,dx'_2}{|z_1-z_2|^4} = 4\cdot\pi\cdot\lambda(D_*)\cdot k^2, \tag{9}$$

with $z_1 = x_1 + x_2 i$ and $z_2 = x'_1 + x'_2 i$. We put

$$E_k = \{\hat{w} \in \mathrm{UT}(D)\colon (8) \text{ holds}\}$$

and choose $\{k_n\}$ to be a countable dense subset of the open interval $(0, k_0)$. Then it turns out that

$$\bar{\mu}\big((\bigcap_n E_{k_n})^C\big) = 0$$

and (6) holds for any $\hat{w} \in \bigcap_n E_{k_n}$. This completes the proof of the theorem.

From this theorem, we compute the number of solutions of (4) for almost all $z \in \boldsymbol{C}$.

THEOREM 3.2. *For almost all $z \in \boldsymbol{C}$,*

$$\lim_{N\to\infty} \frac{\#\{g(\infty)\colon g\in\Gamma,\ (4) \text{ holds},\ r(g) \geqslant 1/N\}}{\log N} = \frac{8\cdot\lambda(D_*)\cdot\pi}{\bar{\mu}(\mathrm{UT}(D))}\cdot k^2$$

for any positive k less than the Ford radius of Γ.

Proof. For every geodesic curve $\overrightarrow{(z', z'')}$ in $\boldsymbol{H}^3$, there exists $g \in \Gamma$ such that $g\overrightarrow{(z', z'')} \cap D \neq \emptyset$. Moreover, if we put

$$J = \{z_2 \in \boldsymbol{C}\colon \text{there exist } z_1 \in \boldsymbol{P}^1(\boldsymbol{C}) \text{ and } s \in \boldsymbol{R} \text{ such that } (z_1, z_2, s) \in \mathrm{UT}(D)\},$$

then for almost all $z_2 \in J$, there exist $z_1 \in \boldsymbol{P}^1(\boldsymbol{C})$ and $u \in \boldsymbol{R}$ such that (6) holds for $\hat{w} = (z_1, z_2, u) \in \mathrm{UT}(D)$. For such a complex number z_2, it turns out that the point $\hat{v} \in \mathrm{UT}(D)$ congruent to $(\infty, z_2\bar{u})$ satisfies (6) by Lemma 1.1. Hence it is clear that $G_t(\hat{v})$ also satisfies (6) for any $t \in \boldsymbol{R}$. Now by Lemma 2.4 and Proposition 2.5, we have

$$\#\{s\colon 0<s<\log(N^2/2\cdot k),\ G_s G_{-\bar{u}}(\hat{v}) \in \hat{D}_k\}$$
$$\leqslant \#\{g(\infty)\colon g\in\Gamma,\ (4) \text{ holds with } z = z_2,\ r(g) \geqslant 1/N\}$$
$$\leqslant \#\{s\colon 0<s<\log(N^2/k),\ G_s G_{-\bar{u}}(\hat{v}) \in \hat{D}_k\}.$$

Consequently, we get for z_2,

$$\lim_{N\to\infty} \frac{\#\{g(\infty)\colon g\in\Gamma,\ (4) \text{ holds},\ r(g) \geqslant 1/N\}}{2\cdot\log N} = \frac{4\cdot\lambda(D_*)\cdot\pi}{\bar{\mu}(\mathrm{UT}(D))}\cdot k^2. \tag{10}$$

If z'' is congruent to z_2, in Γ, then we can prove (10) by the same way. This implies the assertion of the theorem.

Finally, we consider a cusp α of Γ that is not congruent to ∞. For such a cusp α, there exists $h \in \mathrm{PSL}(2, \boldsymbol{C})$ such that $\alpha = h^{-1}(\infty)$ (clearly $h \notin \Gamma$). In this case, we put $\Gamma_\alpha = h\Gamma h^{-1}$ so that Γ_α has a cusp at ∞. Applying the above discussion, we have the following:

THEOREM 3.3. *For almost all $z \in \mathbf{C}$,*

$$\lim_{N\to\infty} \frac{\#\{g(\infty):\ g\in\Gamma,\ |z-g(\infty)| < r(gh^{-1})^2\cdot k,\ r(gh^{-1}) \geqslant 1/N\}}{\log N} = \frac{8\cdot\lambda((D_\alpha)_*)\cdot\pi}{\bar{\mu}(\mathrm{UT}(D_\alpha))}\cdot k^2$$

for any k, $0<k<k_0$, where k_0 is the Ford radius of Γ_α, D_α and $(D_\alpha)_*$ are the fundamental regions (in $\mathbf{H}^3$ and $\mathbf{C}$, resp.) of Γ_α and $(\Gamma_\alpha)_\infty$, respectively.

4. Approximation over imaginary quadratic field. We apply Theorems 3.2 and 3.3 for $\Gamma = \mathrm{PSL}(2, o_d)$ and its subgroups. In this case, the following facts are known (see [2] and its references):

(i) ∞ is a cusp of Γ for any square free negative integer d and $\Gamma(\infty)$ is identical with

$$\{p/q:\ p, q\in o_d,\ (p, q) = 1\}\cup\{\infty\}.$$

In general, this is not identical with $\mathbf{Q}(\sqrt{d})\cup\{\infty\}$ (= the set of cusps of Γ). Let K_d be the set of cusps of Γ, then there are finite number of cusps $z_1, z_2, \ldots, z_{L-1}\in\mathbf{Q}(\sqrt{d})$ such that

$$K_d = \Gamma(\infty)\cup\Gamma(z_1)\cup\ldots\cup\Gamma(z_{L-1}) \quad \text{(disjoint union)},$$

here L is the class number of $\mathbf{Q}(\sqrt{d})$.

(ii) We recall that the discriminant and the zeta-function of $\mathbf{Q}(\sqrt{d})$ are defined by

$$\hat{d} = \begin{cases} 4d & \text{if } d\equiv 2, 3 \pmod 4,\\ d & \text{if } d\equiv 1 \pmod 4\end{cases} \tag{11}$$

and

$$\zeta_d(s) = \sum 1/N(I)^s, \quad \text{respectively},$$

where I runs over all ideals in $\mathbf{Q}(\sqrt{d})$ and $N(I)$ denotes its norm. We have

$$\mu(D) = \frac{|\hat{d}|^{3/2}}{4\cdot\pi^2}\cdot\zeta_d(2) \tag{12}$$

for any square free negative integer d.

THEOREM 4.1. *For almost all $z\in\mathbf{C}$,*

$$\lim_{N\to\infty}\frac{\#\left\{\frac{p}{q}\in\mathbf{Q}(\sqrt{d}):\ \frac{p}{q} \text{ satisfies (1)},\ |q|\leqslant N\right\}}{\log N} = C_d\cdot k^2 \tag{13}$$

for any $k>0$ with

$$C_d = \begin{cases} \dfrac{\pi^2}{2\cdot\zeta_d(2)} & \text{if } d=-1,\\ \dfrac{4\cdot\pi^2}{9\cdot\zeta_d(2)} & \text{if } d=-3,\\ \dfrac{\pi^2}{|d|\cdot\zeta_d(2)} & \text{if } d\neq -1,\ d\equiv 2, 3 \pmod 4,\\ \dfrac{4\cdot\pi^2}{|d|\cdot\zeta_d(2)} & \text{if } d\neq -3,\ d\equiv 1 \pmod 4.\end{cases} \tag{14}$$

Proof. Since $\bar{\mu}(\mathrm{UT}(D)) = \mu(D)\times 4\pi$ and

$$\lambda(D_*) = \begin{cases} 1/2 & \text{if } d=-1,\\ 1/2\sqrt{3} & \text{if } d=-3,\\ \sqrt{|d|} & \text{if } d\neq -1,\ d\equiv 2, 3 \pmod 4,\\ \sqrt{|d|}/2 & \text{if } d\neq -3,\ d\equiv 1 \pmod 4,\end{cases}$$

it follows from Proposition 2.2 and Theorem 3.2 that for almost all $z\in\mathbf{C}$, (13) holds with (14) for $0<k<1/2$. (For $g = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$, $c\neq 0$, we note that $g(\infty) = a/c$ and $r(g) = 1/|c|$.)

Next we consider the case of $k>1/2$. To do this, we consider the principal congruence subgroup Γ_m of $\Gamma = \mathrm{PSL}(2, o_d)$ of level m, $m\neq 0, \in o_d$:

$$\Gamma_m = \left\{g\in\Gamma:\ g\equiv\begin{bmatrix}\pm 1 & 0\\ 0 & \pm 1\end{bmatrix} \pmod m\right\}.$$

Every principal congruence subgroup is normal and of finite index. Moreover, there exist $g_1, g_2, \ldots, g_{l-1}\in\Gamma$ such that

$$\Gamma(\infty) = \Gamma_m(\infty)\cup g_1(\Gamma_m(\infty))\cup\ldots\cup g_{l-1}(\Gamma_m(\infty)) \quad \text{(disjoint union)}. \tag{15}$$

It is easy to show that the Ford radius of Γ_m is equal to $m/2$ for positive rational integer m. In the sequel, we always assume that m is a positive integer. If we apply Theorem 3.2 to Γ_m, then we have for almost all $z\in\mathbf{C}$,

$$\lim_{N\to\infty}\frac{\#\{p/q\in\mathbf{Q}(\sqrt{d}):\ p/q \text{ satisfies (1)},\ |q|\leqslant N \text{ and } p/q\in\Gamma_m(\infty)\}}{\log N} = C_{d,m}\cdot k^2 \tag{16}$$

for k, $0 < k < m/2$, where $C_{d,m}$ is a constant depending on d and m. On the other hand, since

$$g_n \Gamma_m g_n^{-1} = \Gamma_m \quad \text{for } 1 \leqslant n \leqslant l-1,$$

we can apply Theorem 3.3 to Γ_m and get the following: for almost all $z \in C$

(17)
$$\lim_{N\to\infty} \frac{\# \left\{\frac{p}{q} \in Q(\sqrt{d}) \colon \frac{p}{q} \text{ satisfies (1)}, \ |q| \leqslant N \text{ and } \frac{p}{q} \in g_n \Gamma_m(\infty)\right\}}{\log N} = C_{d,m} \cdot k^2$$

for k, $0 < k < m/2$, $1 \leqslant n \leqslant l-1$. From (13), (15), (16) and (17), we see that

$$C_{d,m} = C_d/l$$

and (13) holds for $0 < k < m/2$. Since we can choose m arbitrarily large, we have the assertion of the theorem.

Remarks. (i) If it is possible to construct a normal subgroup of Γ with finite index whose Ford radius is greater than that of Γ, then it turns out that the assertions of Theorems 3.1, 3.2 and 3.3 hold with "for any $k > 0$".

(ii) If the class number of $Q(\sqrt{d})$ is not equal to one, then we can discuss the similar property for each congruence class of cusps by Theorem 3.3.

(iii) It is possible to establish a theorem similar to a result by Moeckel [9]. For example, it is easy to show that

$$\lim_{N\to\infty} \frac{\# \left\{\frac{p}{q} \in Q(\sqrt{d}) \colon \frac{p}{q} \text{ satisfies (1)}, \ |q| \leqslant N \text{ and } \frac{p}{q} \in g_n \Gamma_m(\infty)\right\}}{\# \left\{\frac{p}{q} \in Q(\sqrt{d}) \colon \frac{p}{q} \text{ satisfies (1)}, \ |q| \leqslant N\right\}} = \frac{1}{l}$$

for almost all $z \in C$ and any $k > 0$.

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