References

[1] B. Brindza, *On some generalizations of the diophantine equation* $1^k+2^k+\ldots+x^k = y^z$, Acta Arith. 44 (1984), 99–107.
[2] — *On S-integral solutions of the equation* $f(x) = y^m$, Acta Math. Acad. Sci. Hungar. 44 (1984), 133–139.
[3] K. Dilcher, *Irreducibility and zeros of generalized Bernoulli polynomials*, C. R. Math. Rep. Acad. Sci. Canada 6 (1984), no. 5, 273–278.
[4] — *On a diophantine equation involving quadratic characters*, Comp. Math. 57 (1986), 383–403.
[5] K. Győry, R. Tijdeman and M. Voorhoeve, *On the equation* $1^k+2^k+\ldots+x^k = y^z$, Acta Arith. 37 (1980), 233–240.
[6] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, Heidelberg, New York 1976.
[7] W. J. LeVeque, *On the equation* $y^m = f(x)$, Acta Arith. 9 (1964), 209–219.
[8] J. J. Schäffer, *The equation* $1^p+2^p+\ldots+n^p = m^q$, Acta Math. 95 (1956), 155–189.
[9] A. Schinzel and R. Tijdeman, *On the equation* $y^m = P(x)$, Acta Arith. 31 (1976), 199–204.
[10] M. Voorhoeve, K. Győry and R. Tijdeman, *On the diophantine equation* $1^k+2^k+\ldots+x^k+R(x) = y^z$, Acta Math. 143 (1979), 1–8 and *Corrigendum*, Acta Math., to appear.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Warsaw, Poland

Received on 17.10.1986
and in revised form on 7.7.1987 (1682)

ACTA ARITHMETICA
LI (1988)

On the average number of direct factors of a finite Abelian group

by

EKKEHARD KRÄTZEL (Jena)

1. Introduction. Let G be a finite Abelian group. Let $\tau(G)$ denote the number of direct factors of G and

$$T(x) = \sum \tau(G),$$

where the summation is extended over all Abelian groups of order not exceeding x. E. Cohen [1] proved the representation

$$T(x) = \gamma_{1,1}x(\log x+2C-1)+\gamma_{1,2}x+\Delta(x),$$

where $\Delta(x)$ is estimated by

$$\Delta(x) \ll \sqrt{x}\log^2 x.$$

In this paper we improve this result by

$$\Delta(x) = \gamma_{2,1}\sqrt{x}(\tfrac{1}{2}\log x+2C-1)+\gamma_{2,2}\sqrt{x}+O(x^{5/12}\log^4 x).$$

In these formulas C denotes Euler's constant, and $\gamma_{1,1}, \ldots, \gamma_{2,2}$ are given by (22)–(25).

A similar situation takes place when we consider the unitary factors of G, that is, the total number of direct decompositions of G into 2 relatively prime factors. Let $t(G)$ denote the number of unitary factors of G and

$$T^*(x) = \sum t(G),$$

where again the summation is extended over all the Abelian groups of order not exceeding x. Here E. Cohen [1] proved that

$$T^*(x) = c_{1,1}x(\log x+2C-1)+c_{1,2}x+\Delta^*(x), \qquad \Delta^*(x) \ll \sqrt{x}\log x.$$

In this paper we prove

$$\Delta^*(x) = c_2\sqrt{x}+O(x^{11/29}\log^2 x),$$

where $c_{1,1}$, $c_{1,2}$, c_2 are defined by (13), (14).

It is not hard to prove this estimate for $\Delta^*(x)$. Therefore, the main point of this paper is the proof of the new result for $\Delta(x)$. Both problems are connected with some divisor problems.

In Section 2 we present some preliminary lemmas. In Section 3 we prove the representation for $\Delta^*(x)$, which is based on a result for a three-dimensional divisor problem. In Section 4 we prove the main result for $\Delta(x)$. For this purpose a new general result for a four-dimensional divisor problem is needed.

2. Preliminary lemmas.

LEMMA 1. *Let $d(1, 1, 2; n)$ denote the divisor function*

$$d(1, 1, 2; n) = \#\{(n_1, n_2, n_3):\ n_1, n_2, n_3 \in N,\ n_1 n_2 n_3^2 = n\},$$

and let $t_3(n)$ be defined by

$$\sum_{n=1}^{\infty} \frac{t_3(n)}{n^s} = \prod_{\nu=2}^{\infty} \zeta^2\big((2\nu-1)s\big)\zeta(2\nu s) \qquad (s > \tfrac{1}{3})$$

($\zeta(s)$ denotes Riemann's zeta-function). Then

$$T^*(x) = \sum_{mn \leqslant x} d(1, 1, 2; m) t_3(n). \tag{1}$$

Proof. It is known by Lemma 4.2 of [1] that

$$T^*(x) = \sum_{k \leqslant x} t_1(k),$$

where $t_1(k)$ is defined by

$$\sum_{k=1}^{\infty} \frac{t_1(k)}{k^s} = \prod_{\nu=1}^{\infty} \zeta^2\big((2\nu-1)s\big)\zeta(2\nu s) \qquad (s > 1).$$

Hence

$$t_1(k) = \sum_{mn=k} d(1, 1, 2; m) t_3(n)$$

and (1) follows at once.

LEMMA 2. *Let $d(1, 1, 2, 2; n)$ denote the divisor function*

$$d(1, 1, 2, 2; n) = \#\{(n_1, n_2, n_3, n_4):\ n_1, \ldots, n_4 \in N,\ n_1 n_2 n_3^2 n_4^2 = n\},$$

and let $\tau_3(n)$ be defined by

$$\sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s} = \prod_{\nu=3}^{\infty} \zeta^2(\nu s) \qquad (s > \tfrac{1}{3}).$$

Then

$$T(x) = \sum_{mn \leqslant x} d(1, 1, 2, 2; m) \tau_3(n). \tag{2}$$

Proof. Lemmas 2.2 and 2.17 of [1] show that

$$T(x) = \sum_{k \leqslant x} \tau_1(k),$$

where $\tau_1(k)$ is defined by

$$\sum_{k=1}^{\infty} \frac{\tau_1(k)}{k^s} = \prod_{\nu=1}^{\infty} \zeta^2(\nu s) \qquad (s > 1).$$

Hence

$$\tau_1(k) = \sum_{mn=k} d(1, 1, 2, 2; m) \tau_3(n)$$

and (2) follows at once.

The next lemma is a special case of Theorems 5 and 6 of the paper of M. Vogts [4], see the formulas (2), (3), (4) of his paper.

LEMMA 3. *Let $D(n; \beta; \alpha; x)$ be defined by*

$$D(n; \beta; \alpha; x) = \sum_{m_1^{b_1} \ldots m_n^{b_n} \leqslant x} m_1^{a_1} \ldots m_n^{a_n},$$

where $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n)$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are positive real numbers with $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$. Then

$$D(n; \beta; \alpha; x) = H(n; \beta; \alpha; x) + \Delta(n; \beta; \alpha; x) \tag{3}$$

with

$$H(n; \beta; \alpha; x) = \sum_{i=1}^{n} x^{(a_i+1)/b_i} \frac{1}{a_i+1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \zeta\left(\frac{(a_i+1)b_j}{b_i} - a_j\right) + \prod_{i=1}^{n} \zeta(-a_i), \tag{4}$$

$$\Delta(n; \beta; \alpha; x) = -\sum_{p \in \pi(n)} \Bigg\{ x^{a_{p_n}/b_{p_n}} \sum\nolimits_1 \left(\prod_{i=1}^{n-1} m_i^{a_{p_i} - a_{p_n} b_{p_i}/b_{p_n}} \right) \times \tag{5}$$
$$\times \psi\left(\left(\frac{x}{m_1^{b_{p_1}} \ldots m_{n-1}^{b_{p_{n-1}}}}\right)^{1/b_{p_n}}\right) + \sum_{i=1}^{n} O\big(x^{(a_{p_1}+\ldots+a_{p_i}+i-2)/(b_{p_1}+\ldots+b_{p_i})}\big) \Bigg\}.$$

$p \in \pi(n)$ means that the n-tuple $(p_1, \ldots, p_n)$ is a permutation of the numbers $1, \ldots, n$. Then the sum is extended over all permutations. The summation condition of $\sum_1$ is given by

$$\mathrm{SC}\Big(\sum\nolimits_1\Big):\ m_1^{b_{p_1}} \ldots m_{n-2}^{b_{p_{n-2}}} m_{n-1}^{b_{p_{n-1}}+b_{p_n}} \leqslant x, \qquad m_1 (\leqslant) \ldots (\leqslant) m_{n-1}.$$

$m_j (\leqslant) m_{j+1}$ *means that* $\quad m_j \leqslant m_{j+1}$ *for* $p_j < p_{j+1}$ *and*

$\qquad m_j < m_{j+1}$ *otherwise.*

The function $\psi(y)$ is defined by $\psi(y) = y - [y] - 1/2$.

Remark. We must suppose that $(a_i+1)b_j \neq (a_j+1)b_i$ in representation (4). However, in cases of some equalities we can take the limit values.

Lemma 4. *Let $0 < a < b \leqslant ua$, where $u > 1$ is a fixed number. Let $f(t)$ be a real algebraic function with continuous derivatives up to the third order in $[a, b]$. Suppose that*

$$|f''(t)| \asymp \lambda/a^2, \quad |f'''(t)| \asymp \lambda/a^3$$

throughout the interval. Let $\varphi(t)$ be defined by $f'(\varphi) = t$. Let $\alpha = \min f'(t)$ and $\beta = \max f'(t)$ in $[a, b]$. If $\lambda \gg a$, then

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = \varepsilon \sum_{\alpha\leqslant\nu\leqslant\beta} \frac{1}{\sqrt{|f''(\varphi(\nu))|}} e^{2\pi i (f(\varphi(\nu))-\nu\varphi(\nu))} + O\left(\frac{a}{\sqrt{\lambda}}\right) + O(\log(\lambda+1)), \tag{6}$$

where

$$\varepsilon = \begin{cases} e^{\pi i/4} & \text{for } f''(t) > 0, \\ e^{-\pi i/4} & \text{for } f''(t) < 0. \end{cases}$$

Proof. Lemma 4 is a special case of Hilfssatz 3 of [2]. If we put there $g(t) = 1$, $\lambda_2 = \lambda/a^2$, $\lambda_3 = \lambda/a^3$ and if we use there the trivial estimate $T(z) \ll a/\sqrt{\lambda}$, Lemma 4 follows immediately.

3. The estimate for $T^*(x)$. In order to prove an estimate for $T^*(x)$ it is seen from Lemma 1 that we must have an estimate for

$$D(1, 1, 2; x) = \sum_{n\leqslant x} d(1, 1, 2; n) = \sum_{n_1 n_2 n_3^2 \leqslant x} 1.$$

Therefore we need the following Lemma 5, which will be a special case of Theorem 6.3 of [3].

Lemma 5. *The representation*

$$D(1, 1, 2; x) = H(1, 1, 2; x) + \Delta(1, 1, 2; x) \tag{7}$$

holds with

$$H(1, 1, 2; x) = x\{\zeta(2)\log x + (2C-1)\zeta(2) - 2\zeta'(2)\} + \zeta^2(\tfrac{1}{2})\sqrt{x}, \tag{8}$$

$$\Delta(1, 1, 2; x) = -2S(1, 1, 2; x) - 2S(1, 2, 1; x) - 2S(2, 1, 1; x) + O(x^{1/4}), \tag{9}$$

where $S(a, b, c; x)$ is defined by

$$S(a, b, c; x) = \sum\nolimits_2 \psi\left(\left(\frac{x}{n^a m^b}\right)^{1/c}\right), \tag{10}$$

$$\mathrm{SC}(\textstyle\sum_2)\colon\ n^a m^{b+c} \leqslant x,\ n \leqslant m.$$

Moreover, we have

$$\Delta(1, 1, 2; x) \ll x^{11/29}\log^2 x. \tag{11}$$

Proof. We apply Lemma 3 with $n = 3$, $\alpha = (0, 0, 0)$, $\beta = (1, 1, 2)$. Then it is easily seen that

$$D(3; \beta; \alpha; x) = D(1, 1, 2; x),$$

$$H(3; \beta; \alpha; x) = H(1, 1, 2; x) + O(1),$$

$$\Delta(3; \beta; \alpha; x) = \Delta(1, 1, 2; x) + O(x^{1/4}).$$

Hence, if we use the notation (7), the representations (8), (9) follow from (4), (5). If we consider the part $n = m$ in (10) it is seen that

$$\sum_{n^4\leqslant x} \psi\left(\left(\frac{x}{n^{a+b}}\right)^{1/c}\right) \ll x^{1/4}.$$

Hence, we can always use the inequality $n \leqslant m$ in (10).

We now consider the sum

$$S(a, b, c; M, N; x) = \sum\nolimits_3 \psi\left(\left(\frac{x}{n^a m^b}\right)^{1/c}\right), \tag{12}$$

$$\mathrm{SC}(\textstyle\sum_3)\colon\ n^a m^{b+c} \leqslant x,\ n \leqslant m,\ N \leqslant n \leqslant 2N,\ M \leqslant m \leqslant 2M,$$

where $M, N \geqslant 1$. Then we apply Lemma 4 to the sum

$$\sum\nolimits_3 e^{2\pi i \mu f(m,n)}$$

with respect to m, where μ is a positive integer and $f(t, n)$ is defined by

$$f(t, n) = -\left(\frac{x}{n^a t^b}\right)^{1/c}.$$

The function $\varphi(t)$ in Lemma 4 is given by

$$\varphi(t) = \left(\frac{b^c \mu^c x}{c^c n^a t^c}\right)^{1/(b+c)}.$$

Further we have

$$\left|\frac{d^2}{dt^2}\mu f(t, n)\right| \asymp \frac{\lambda}{M^2}, \quad \left|\frac{d^3}{dt^3}\mu f(t, n)\right| \asymp \frac{\lambda}{M^3}$$

with

$$\lambda = \mu\left(\frac{x}{N^a M^b}\right)^{1/c}$$

such that $\lambda \gg M$. If

$$\alpha = \alpha(n) = \min \frac{b\mu}{c}\left(\frac{x}{n^a t^{b+c}}\right)^{1/c}, \quad \beta = \beta(n) = \max \frac{b\mu}{c}\left(\frac{x}{n^a t^{b+c}}\right)^{1/c},$$

we obtain from (6)

$$\sum\nolimits_3 e^{2\pi i\mu f(m,n)} = \sum_n \sum_m e^{2\pi i \mu f(m,n)}$$

$$= r\varepsilon \sum_n \sum_{\alpha \leqslant \nu \leqslant \beta} \left(\frac{\mu^c x}{n^a \nu^{b+2c}}\right)^{1/(2(b+c))} e^{-2\pi i s(\mu^c x n^{-a} \nu^b)^{1/(b+c)}}$$

$$+ O\left(\left(\frac{1}{\mu^c x} N^{a+2c} M^{b+2c}\right)^{1/(2c)}\right) + O(N \log(\mu x)),$$

where r and s are some positive constants. Because of $N^a M^{b+c} \ll x$, $N \ll M$ we have

$$\left(\frac{1}{x} N^{a+2c} M^{b+2c}\right)^{1/(2c)} \ll (N^2 M)^{1/2} \ll (N^a M^{b+c})^{\frac{3}{2}\frac{1}{a+b+c}} \ll x^{3/8}.$$

Hence

$$\sum\nolimits_3 e^{2\pi i \mu f(m,n)} = r\varepsilon \sum_\nu \sum_n \left(\frac{\mu^c x}{n^a \nu^{b+2c}}\right)^{1/(2(b+c))} e^{-2\pi i s(\mu^c x n^{-a} \nu^b)^{1/(b+c)}}$$

$$+ O\left(\frac{1}{\sqrt{\mu}} x^{3/8}\right) + O(x^{1/4} \log(\mu x)).$$

We now can apply van der Corput's method of exponent pairs to the sum over n. Let (k, l) be an exponent pair. Then it is easily seen that

$$\sum\nolimits_3 e^{2\pi i f(m,n)} \ll \sum_\nu \left(\frac{\mu^c x}{N^a \nu^{b+2c}}\right)^{1/(2(b+c))} \left(\frac{\mu^c x \nu^b}{N^{a+b+c}}\right)^{k/(b+c)} N^l$$

$$+ \frac{1}{\sqrt{\mu}} x^{3/8} + x^{1/4} \log(\mu x)$$

$$\ll \mu^{k+1/2} \left(\frac{x}{N^a M^b}\right)^{(2k+1)/(2c)} N^{l-k} + \frac{1}{\sqrt{\mu}} x^{3/8} + x^{1/4} \log(\mu x).$$

Now, it is well known that

$$\sum\nolimits_3 \psi(-f(m,n)) \ll \sum\nolimits_3 \frac{1}{z} + \sum_{\mu=1}^{\infty} \min\left(\frac{z^2}{\mu^3}, \frac{1}{\mu}\right) \left|\sum\nolimits_3 e^{2\pi i \mu f(m,n)}\right|.$$

Therefore, we obtain for the sum (12)

$$S(a, b, c; M, N; x)$$

$$\ll \frac{MN}{z} + z^{k+1/2} \left(\frac{x}{N^a M^b}\right)^{(2k+1)/(2c)} N^{l-k} + x^{3/8} + x^{1/4} \log^2(zx).$$

If we put

$$z = \left(\frac{1}{x^{2k+1}} M^{(2k+1)b+2c} N^{(2k+1)a+2(1-l+k)c}\right)^{1/(2k+3)c},$$

then

$$S(a, b, c; M, N; x) \ll (x^{2k+1} M^{(2k+1)(c-b)} N^{(2l+1)c-(2k+1)a})^{1/(2k+3)c} + x^{3/8}$$

provided that $z > 1$. But otherwise the estimate is trivial. It will be seen that $(\frac{2}{18}, \frac{13}{18})$ is a useful exponent pair. Thus, because of $a+b+c = 4$,

$$S(a, b, c; M, N; x) \ll (x M^{c-b} N^{2c-a})^{11/29c} + x^{3/8}$$

$$= \left(x^4 (N^a M^{b+c})^{3c-a-b} \left(\frac{N}{M}\right)^{2c((b+c)-a)}\right)^{11/116c} + x^{3/8} \ll x^{11/29}.$$

We can divide the sum $\sum_2$ into $O(\log^2 x)$ subsums of type $\sum_3$. Then (10) and (12) show that

$$S(a, b, c; x) \ll x^{11/29} \log^2 x.$$

Now estimate (11) follows from (9).

THEOREM 1. *Let* $c_{1,1}$, $c_{1,2}$, c_2 *be defined by*

$$c_{1,1} = \zeta(2) \sum_{n=1}^{\infty} \frac{t_3(n)}{n}, \quad c_2 = \zeta^2(\tfrac{1}{2}) \sum_{n=1}^{\infty} \frac{t_3(n)}{\sqrt{n}}, \tag{13}$$

$$c_{1,2} = -\sum_{n=1}^{\infty} \frac{t_3(n)}{n} (\zeta(2) \log n - 2\zeta'(2)). \tag{14}$$

Then

$$T^*(x) = c_{1,1} x(\log x + 2C - 1) + c_{1,2} x + c_2 \sqrt{x} + O(x^{11/29} \log^2 x). \tag{15}$$

Proof. We apply Lemmas 1 and 5 and obtain from (1), (8), (11)

$$T^*(x) = \sum_{n \leqslant x} t_3(n) D(1, 1, 2; x/n)$$

$$= \sum_{n \leqslant x} \frac{t_3(n)}{n} x \left\{\zeta(2) \log \frac{x}{n} + (2C-1)\zeta(2) + 2\zeta'(2)\right\}$$

$$+ \zeta^2\left(\frac{1}{2}\right) \sum_{n \leqslant x} t_3(n) \left(\frac{x}{n}\right)^{1/2} + O\left(\sum_{n \leqslant x} t_3(n) \left(\frac{x}{n}\right)^{11/29} \log^2 \frac{x}{n}\right).$$

Since the Dirichlet series for $t_3(n)$ is absolutely convergent for $s > 1/3$, result (15) follows immediately.

4. The estimate for $T(x)$. In order to obtain an estimate for $T(x)$ it is seen from Lemma 2 that we must have an estimate for

$$D(1, 1, 2, 2; x) = \sum_{n \leqslant x} d(1, 1, 2, 2; n).$$

This will be an immediate consequence from the following new general theorem.

THEOREM 2. *Let a_1, a_2, a_3, a_4 be real numbers with $1 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Put*

$$a = (a_1, a_2, a_3, a_4), \quad A_\nu = a_1 + \ldots + a_\nu \quad \text{for } \nu = 1, 2, 3, 4.$$

Let

$$D(a; x) = \# \{(n_1, n_2, n_3, n_4)\colon n_1, \ldots, n_4 \in N,\ n_1^{a_1} n_2^{a_2} n_3^{a_3} n_4^{a_4} \leqslant x\}.$$

Then the representation

$$D(a; x) = H(a; x) + \Delta(a; x) \tag{16}$$

holds with

$$H(a; x) = \sum_{\nu=1}^{4} \alpha_\nu x^{1/a_\nu}, \quad \alpha_\nu = \prod_{\substack{\mu=1\\ \mu \neq \nu}}^{4} \zeta(a_\mu / a_\nu), \tag{17}$$

$$\Delta(a; x) = -\sum_u S(u; x) + O(x^{2/A_4}), \tag{18}$$

provided that $2A_3 > A_4$. Here $u = (u_1, u_2, u_3, u_4)$ runs over all permutations of the numbers a_1, a_2, a_3, a_4. $S(u; x)$ is defined by

$$S(u; x) = \sum\nolimits_4 \psi\left(\left(\frac{x}{n_1^{u_1} n_2^{u_2} n_3^{u_3}}\right)^{1/u_4}\right), \tag{19}$$

where the summation condition is described by

$$\mathrm{SC}(\textstyle\sum_4)\colon \quad n_1^{u_1} n_2^{u_2} n_3^{u_3+u_4} \leqslant x, \quad 1 \leqslant n_1 (\leqslant) n_2 \leqslant n_3.$$

$n_1 (\leqslant) n_2$ means that $n_1 \leqslant n_2$ for $u_1 = a_i$, $u_2 = a_j$ and $i < j$ and $n_1 < n_2$ otherwise.

Moreover, the estimation

$$\Delta(a; x) \ll x^{5/2A_4} \log^4 x \tag{20}$$

holds under the conditions

$$15A_1 \geqslant 2A_4, \quad 3A_2 \geqslant A_4, \quad 5A_3 \geqslant 3A_4.$$

Remark. In the representation (17) of the main term we must suppose that $a_1 < a_2 < a_3 < a_4$. However, in cases of some equalities we can take the limit values.

Proof. We apply Lemma 3 with $n = 4$, $\alpha = (0, 0, 0, 0)$, $\beta = (a_1, a_2, a_3, a_4)$. Then

$$D(4; \beta; \alpha; x) = D(a; x),$$
$$H(4; \beta; \alpha; x) = H(a; x) + O(1),$$
$$\Delta(4; \beta; \alpha; x) = \Delta(a; x) + O(x^{2/A_4}).$$

Because of $2A_3 > A_4$ we have on the one hand the single error term $O(x^{2/A_4})$, and on the other hand it is one and the same whether $n_2 = n_3$ or not. Namely, if we consider the part $n_2 = n_3$ in (19) it is seen that

$$\sum\nolimits_5 \psi\left(\left(\frac{x}{n_1^{u_1} n_2^{u_2+u_3}}\right)^{1/u_4}\right) \ll \sum_{n_1^{A_4} \leqslant x} (x n_1^{-u_1})^{1/(u_2+u_3+u_4)} \ll x^{2/A_4},$$

$$\mathrm{SC}(\textstyle\sum_5)\colon \quad n_1^{u_1} n_2^{u_2+u_3+u_4} \leqslant x, \quad 1 \leqslant n_1 (\leqslant) n_2.$$

Thus (16) holds with (17), (18), (19).

Let $N = (N_1, N_2, N_3)$. We now consider the sum

$$S(u; N; x) = \sum\nolimits_6 \psi\left(\left(\frac{x}{n_1^{u_1} n_2^{u_2} n_3^{u_3}}\right)^{1/u_4}\right),$$

$$\mathrm{SC}(\textstyle\sum_6)\colon \quad n_1^{u_1} n_2^{u_2} n_3^{u_3+u_4} \leqslant x, \quad 1 \leqslant n_1 (\leqslant) n_2 \leqslant n_3,$$

$$N_\nu \leqslant n_\nu \leqslant 2N_\nu \quad (\nu = 1, 2, 3).$$

We now apply the inequality (27), Hilfssatz 6 [2] or, what is the same, inequality (6.17), Lemma 6.4 [3] to the sum over n_2, n_3. Then

$$\begin{aligned} S(u; N; x) &\ll \sum_{N_1 \leqslant n_1 \leqslant 2N_1} (x^2 n_1^{-2u_1} N_2^{4u_4 - 2u_2} N_3^{3u_4 - 2u_3})^{1/6u_4} \log x \\ &\ll (x^2 N_1^{6u_4 - 2u_1} N_2^{4u_4 - 2u_2} N_3^{3u_4 - 2u_3})^{1/6u_4} \log x \\ &\ll \left(x^2 (N_1^{u_1} N_2^{u_2} N_3^{u_3+u_4})^{y_1} \left(\frac{N_1}{N_2}\right)^{y_2} \left(\frac{N_2}{N_3}\right)^{y_3}\right)^{1/6u_4} \log x, \end{aligned} \tag{21}$$

y_1, y_2, y_3 are given by

$$A_4 y_1 = 15u_4 - 2A_4,$$
$$A_4 y_2 = 3u_4(2A_4 - 5u_1),$$
$$A_4 y_3 = 5u_4(3(u_3 + u_4) - A_4),$$

since $u_1 + u_2 + u_3 + u_4 = A_4$. We have $y_1, y_2, y_3 \geqslant 0$ for all combinations of u_1, u_2, u_3, u_4 if

$$15a_1 \geqslant 2A_4, \quad 2A_4 \geqslant 5a_4, \quad 3(a_1 + a_2) \geqslant A_4.$$

Because of the conditions of the theorem these inequalities are satisfied. Therefore, we can use the inequalities

$$N_1^{u_1} N_2^{u_2} N_3^{u_3+u_4} \ll x, \qquad N_1 \ll N_2 \ll N_3$$

in (21). Then

$$S(u;\, N;\, x) \ll x^{5/2A_4} \log x,$$

$$S(u;\, x) \ll x^{5/2A_4} \log^4 x.$$

Now (20) follows from (18).

We are now in a position to prove the estimate for $T(x)$.

THEOREM 3. *Let* $\gamma_{1,1}$, $\gamma_{1,2}$, $\gamma_{2,1}$, $\gamma_{2,2}$ *be defined by*

(22) $$\gamma_{1,1} = \zeta^2(2) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n},$$

(23) $$\gamma_{1,2} = -\sum_{n=1}^{\infty} \frac{\tau_3(n)}{n} (\zeta^2(2) \log n - 4\zeta(2)\zeta'(2)),$$

(24) $$\gamma_{2,1} = \zeta^2(\tfrac{1}{2}) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{\sqrt{n}},$$

(25) $$\gamma_{2,2} = -\sum_{n=1}^{\infty} \frac{\tau_3(n)}{\sqrt{n}} (\tfrac{1}{2}\zeta^2(\tfrac{1}{2}) \log n - \zeta(\tfrac{1}{2})\zeta'(\tfrac{1}{2})).$$

Then

(26) $$T(x) = \gamma_{1,1} x(\log x + 2C - 1) + \gamma_{1,2} x + \gamma_{2,1} \sqrt{x}(\tfrac{1}{2}\log x + 2C - 1) + \gamma_{2,2}\sqrt{x} + O(x^{5/12} \log^4 x).$$

Proof. We use Theorem 2 with $a_1 = a_2 = 1$, $a_3 = a_4 = 2$. We have $A_1 = 1$, $A_2 = 2$, $A_3 = 4$, $A_4 = 6$, and the conditions of the theorem are satisfied. We obtain by simple calculations for the main term

$$H(1, 1, 2, 2;\, x) = \zeta^2(2) x(\log x + 2C - 1) + 4\zeta(2)\zeta'(2) x + \zeta^2(\tfrac{1}{2})\sqrt{x}(\tfrac{1}{2}\log x + 2C - 1) + \zeta(\tfrac{1}{2})\zeta'(\tfrac{1}{2})\sqrt{x}.$$

It is seen from (20) that

$$\Delta(1, 1, 2, 2;\, x) \ll x^{5/12} \log^4 x.$$

We now apply equation (2) and obtain

$$T(x) = \sum_{n \leqslant x} \tau_3(n) D(1, 1, 2, 2;\, x/n)$$
$$= \sum_{n \leqslant x} \frac{\tau_3(n)}{n} x \left\{ \zeta^2(2) \log\frac{x}{n} + (2C-1)\zeta^2(2) + 4\zeta(2)\zeta'(2) \right\}$$
$$+ \sum_{n \leqslant x} \tau_3(n) \sqrt{\frac{x}{n}} \left\{ \tfrac{1}{2}\zeta^2(\tfrac{1}{2}) \log\frac{x}{n} + (2C-1)\zeta^2(\tfrac{1}{2}) + \zeta(\tfrac{1}{2})\zeta'(\tfrac{1}{2}) \right\}.$$

Since the Dirichlet series for $\tau_3(n)$ is absolutely convergent for $s > 1/3$, result (26) follows immediately.

References

[1] E. Cohen, *On the average number of direct factors of a finite abelian group*, Acta Arith. 6 (1960), 159–173.

[2] E. Krätzel, *Zweifache Exponentialsummen und dreidimensionale Gitterpunktprobleme*; in: *Elementary and Analytic Theory of Numbers*, Banach Center Publications 17, PWN—Polish Scientific Publishers, Warsaw 1985, 337–369.

[3] — *Lattice Points*, Berlin, to appear.

[4] M. Vogts, *Many-dimensional generalized divisor problems*, Math. Nachr. 124 (1985), 103–121.

Received on 1.12.1986
and in revised form on 17.3.1987 (1694)