ACTA ARITHMETICA
LI (1988)

On the distribution of s-dimensional Kronecker-sequences

by

GERHARD LARCHER (Salzburg)

0. Introduction. It is well known that the sequence $w := (\{k\alpha_1\}, \ldots, \{k\alpha_s\})$; $k \in N$ (we will call it *s-dimensional Kronecker-sequence*) is uniformly distributed in the s-dimensional unit-cube if and only if $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over the rationals.

A measure for the quality of the uniform distribution is the sequence of discrepancies of such a sequence. In higher dimensions it is possible to define different forms of discrepancies. The two most important are the *usual discrepancy*

$$D_N := \sup_R \left|\frac{A_N(R)}{N} - \mu(R)\right|$$

(where the supremum is taken over all s-dimensional intervals

$$R = \prod_{i=1}^{s} [a_i, b_i) \quad \text{with } 0 \leqslant a_i \leqslant b_i \leqslant 1,$$

where by $A_N(R)$ we denote the number of the first N sequence-elements which lie in R, and where μ is the s-dimensional Lebesgue-measure), and the so-called *isotropic discrepancy*

$$J_N := \sup_C \left|\frac{A_N(C)}{N} - \mu(C)\right|$$

(where the supremum now is taken over all convex subsets of the unit-cube I^s).

For the definition of discrepancy and for general results for J_N see for example [6], [9], [11], [12], [8].

The sequence w is *uniformly distributed* if and only if $\lim_{N\to\infty} D_N = 0$, and because of the inequality $D_N \leqslant J_N \leqslant c_s D_N^{1/s}$ this is the case if and only if $\lim_{N\to\infty} J_N = 0$.

The usual discrepancy D_N of Kronecker-sequences was very well studied. For example Hlawka [5] and Ostrowski [10] gave relations of D_N to the approximability of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ with respect to the maximum norm. For example it was shown that if α is badly approximable with respect to this norm, then

$$\limsup_{N\to\infty} N^{1/s} D_N < \infty .$$

But it turned out, that for studying the usual discrepancy another "multiplicative norm" is much more important in this connection. From results of Niederreiter and Schmidt ([9] and [13]) for example it follows, that there even exist α not badly approximable with respect to the maximum norm and with

$$\limsup_{N\to\infty} N^{1-\varepsilon} D_N < \infty \quad \text{for every } \varepsilon > 0.$$

It is the aim of this work to show that the maximum norm is most accurate if we consider now the isotropic discrepancy, and that the extremal results, we will obtain now for J_N, are for badly approximable α essentially the same as for the smaller D_N and (in difference to D_N) now in some sense best possible. So for example we will show in Theorem 1 that

$$\limsup_{N\to\infty} N^{1/s} J_N < \infty$$

if and only if α is badly approximable with respect to the maximum norm.

From Theorem 2 for example it will even follow that for $s = 2$:

$$0 < \limsup_{N\to\infty} N^{1/2} J_N < \infty$$

if and only if α is badly approximable with respect to the maximum norm.

This can be proved with the help of a sharp result of Davenport and Mahler on diophantine approximation.

In Theorem 3 we give a metric result, thereby improving a result in [8], by showing:

$$J_N = O\left(N^{-1/s} (\log N)^{(s-1)/s+\varepsilon}\right)$$

for every $\varepsilon > 0$ and for almost all $\alpha \in \mathbf{R}^s$.

Further, in Lemma 6, we will give a general upper bound for J_N. (For the exact statement of results see Chapter 2.)

1. Notation and definitions. We use the following notation:

For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbf{R}^s$ we denote by $1 = q_0 < q_1 < q_2 < \ldots$ the best simultaneous approximation denominators to α with respect to the maximum norm. For such $q_i := q$, α_j shall be of the form

$$\alpha_j = \frac{p_i}{q} + \frac{\theta_j}{q^{1/s}} \quad \text{for } j = 1, 2, \ldots, s \text{ with } |\theta_j| \leqslant 1.$$

a_i is defined to be $\left[\frac{q_{i+1}}{q_i}\right]$ and $\|x\|$ always denotes the distance of x to the nearest integer.

By w_N we define the sequence $(\{k\alpha_1\}, \ldots, \{k\alpha_s\})$; $k = 1, 2, \ldots, N$ and for given $q_i := q$ by $\bar{w}_q$ the sequence

$$\left(\left\{k\frac{p_1}{q}\right\}, \ldots, \left\{k\frac{p_s}{q}\right\}\right), \quad k = 1, 2, \ldots, q.$$

If we have any finite sequence of length N: w, $w(1)$, $\tilde{w}$ etc. then we denote the according isotropic discrepancy by J_N, $J_N^{(1)}$, $\tilde{J}_N$ etc., and for a convex set C, the number of sequence elements in C by $A(C)$, $A^{(1)}(C)$, $\tilde{A}(C)$ etc.

μ always is the s-dimensional Lebesgue-measure and I^s the s-dimensional unit-cube.

For given $\bar{w}_q$ and for $k \in N$ we define

$$M_k := \max_{j=1,\ldots,s} (\|k\alpha_1\|, \ldots, \|k\alpha_s\|)$$

and

$$\bar{M}_k := \max_{j=1,\ldots,s} \left(\left\|k\frac{p_1}{q}\right\|, \ldots, \left\|k\frac{p_s}{q}\right\|\right).$$

For given i we define the lattice Γ_i to be spanned by the vectors

$$x_i := \left(\frac{p_1}{q}, \ldots, \frac{p_s}{q}\right), \quad e_2 := (0, 1, 0, \ldots, 0), \ldots, e_s := (0, \ldots, 0, 1).$$

Then the set of points of $\bar{w}_q$ is equal to $\Gamma_i \cap I^s$.

By $\lambda_1, \lambda_2, \ldots, \lambda_s$ we denote the successive minima of Γ_i with respect to the euclidean norm.

Constants c_i which have the same form in different lemmata in general are not equal. c_i always denotes a constant depending at most on the dimension s.

2. Results and their proofs. General upper bound

LEMMA 1. *Let*

$$w(1) := (x_1, \ldots, x_N) \quad \textit{and} \quad w(2) := (y_1, \ldots, y_N)$$

be two finite sequences in $\mathbf{R}^s$ *with* $d := \max_{i=1,\ldots,N} \|x_i - y_i\|$.

Then for the corresponding isotropic discrepancies $J_N^{(1)}$ *and* $J_N^{(2)}$ *we have:*

$$J_N^{(1)} \leqslant 3^s J_N^{(2)} + 2sd(1+2d)^{s-1}.$$

Proof. Let P be any convex subset of I^s and P_O and P_I outer and inner parallel regions to P in distance d in $\mathbf{R}^s$. (P_I of course can be empty.)

We have $\mu(P \setminus P_I) \leqslant 2sd$ and $\mu(P_O \setminus P) \leqslant S(P_O)d$ where we denote by $S(P_O)$ the volume of the $(s-1)$-dimensional surface of P_O. P_O is convex and contained in the cube $[-d, 1+d]^s$.

Therefore $S(P_O) \leqslant 2s(1+2d)^{s-1}$ and $\mu(P_O \setminus P) \leqslant 2sd(1+2d)^{s-1}$. Let $\tilde{w}$ be the set of points $y_i + g$ with $i = 1, \ldots, N$ and $g \in \mathbf{Z}^s$; then:

$$A^{(2)}(P_I)-N\cdot\mu(P_I)-N\cdot\mu(P\backslash P_I)\leqslant A^{(1)}(P)-N\cdot\mu(P)$$
$$\leqslant \tilde{A}(P_O)-N\cdot\mu(P_O)+N\cdot\mu(P_O\backslash P)$$

and so

$$|A^{(1)}(P)-N\cdot\mu(P)|$$
$$\leqslant \max\big(NJ_N^{(2)}+2sdN,\ |A(P_O)-N\cdot\mu(P_O)|+2sdN(1+2d)^{s-1}\big).$$

Because of $d<1$, P_O is contained in $W:=\,]-1,2[^s$. W is divided in 3^s unit-cubes W_i; $i=1,\ldots,3^s$ and $P_O\cap W_i$ is convex for every i. If we denote by C_i the convex part $P_O\cap W_i$ translated modulo Z^s into the unit cube, then we have:

$$|A(P_O)-N\cdot\mu(P_O)| = \Big|\sum_{i=1}^{3^s}\big(A(P_O\cap W_i)-N\cdot\mu(P_O\cap W_i)\big)\Big|$$
$$\leqslant \sum_{i=1}^{3^s}|A^{(2)}(C_i)-N\cdot\mu(C_i)|\leqslant 3^s\cdot N\cdot J_N^{(2)}$$

and the result follows.

LEMMA 2. *For a given $i\in N$ let q' be the largest best approximation denominator less than $q:=q_i$ to $\left(\frac{p_1}{q},\ldots,\frac{p_s}{q}\right)$; then, with certain constants c_1, c_2, we have*:

$$\frac{c_1}{q\cdot(\bar{M}_{q'})^{s-1}}\prod_{j=2}^{s-1}\left(\frac{\lambda_1}{\lambda_j}\right)\leqslant \bar{J}_q\leqslant \frac{c_2}{q\cdot(\bar{M}_{q'})^{s-1}}\prod_{j=2}^{s-1}\left(\frac{\lambda_1}{\lambda_j}\right).$$

Proof. Without restriction of generality let $(p_1,q)=1$ and $tp_1\equiv 1\ (\mathrm{mod}\ q)$, then the set

$$\left(k\frac{1}{q},\left\{k\frac{p_2t}{q}\right\},\ldots,\left\{k\frac{p_st}{q}\right\}\right),\qquad k=1,2,\ldots,q$$

is equal to the set of the sequence values of $\bar{w}_q$.

We have $\bar{M}_{q'}\leqslant\lambda_1$ and $\lambda_1\leqslant\sqrt{s}\bar{M}_{q'}$ and so by Beispiel c) in [8] the result follows.

Remark. For $s=2$ the estimate reduces to the simple form

$$\frac{c_1}{q\bar{M}_{q'}}\leqslant\bar{J}_q\leqslant\frac{c_2}{q\bar{M}_{q'}}.$$

LEMMA 3. *For given $i\in N$ with $q:=q_i>2^s$ we write $\bar{q}$ instead of q_{i-1}; then*

$$\tfrac{1}{2}M_{\bar{q}}\leqslant\bar{M}_{q'}\leqslant 2M_{\bar{q}}.$$

Proof. $\alpha_j=p_j/q+r_j$ with $r_j:=\theta_j/q\cdot q_{i+1}^{1/s}$. We have

(a) $$\bar{M}_{q'}=\max_j\left(\left\|q'\frac{p_j}{q}\right\|\right)\leqslant\max_j\left(\left\|\bar{q}\frac{p_j}{q}\right\|\right)$$
$$=\max_j(\|\bar{q}\alpha_j-\bar{q}r_j\|)\leqslant\max_j(\|\bar{q}\alpha_j\|)+\max_j(|\bar{q}r_j|)$$
$$\leqslant M_{\bar{q}}+\max_j(|qr_j|)=M_{\bar{q}}+\max_j(\|qr_j\|)$$

(because $|qr_j|=\left|q\frac{\theta_j}{q\cdot q_{i+1}^{1/s}}\right|\leqslant q^{-1/s}<\frac{1}{2}$) and the last is equal to $M_{\bar{q}}+M_q\leqslant 2M_{\bar{q}}$.

(b) $$M_{\bar{q}}\leqslant M_{q'}=\max_j\left(\left\|q'\frac{p_j}{q}+q'r_j\right\|\right)$$
$$\leqslant\max_j\left(\left\|q'\frac{p_j}{q}\right\|\right)+\max_j(|q'r_j|)=\bar{M}_{q'}+\frac{q'}{q}\max_j(|qr_j|)$$
$$=\bar{M}_{q'}+\frac{q'}{q}M_q\leqslant\bar{M}_{q'}+\frac{q'}{q}M_{\bar{q}}\qquad\text{for } q>2^s.$$

If $q'\leqslant q/2$, then $\frac{1}{2}M_{\bar{q}}\leqslant\bar{M}_{q'}$.

If $q'>q/2$, then $0<q-q'<q/2$, $\bar{M}_{q'}=\bar{M}_{q-q'}$ and analogously we get $\frac{1}{2}M_{\bar{q}}\leqslant\bar{M}_{q-q'}=\bar{M}_{q'}$ and the proof is finished.

We need two further general lemmata:

LEMMA 4. *Let $w:=x_1,x_2,\ldots,x_N$ be a sequence in I^s and for a $x\in\mathbf{R}^s$ let $\bar{w}$ be the sequence $\{x_i+x\}$, $i=1,2,\ldots,N$, in I^s. Then*

$$\bar{J}_N\leqslant 2^sJ_N.$$

Proof. Let C be a convex subset of I^s and $C(x)\subseteq I^s$ shall be the set C translated in $\mathbf{R}^s$ by $-x$ and taken modulo I^s. $C(x)$ is the union of at most 2^s convex pairwise disjunct subsets $C_i(x)$ of I^s. We have

$$|\bar{A}(C)-N\cdot\mu(C)|=|A(C(x))-N\cdot\mu(C(x))|$$
$$=\Big|\sum_i\big(A(C_i(x))-N\cdot\mu(C_i(x))\big)\Big|$$
$$\leqslant\sum_i|A(C_i(x))-N\cdot\mu(C_i(x))|\leqslant 2^sJ_N.$$

LEMMA 5. *Let $N=b_rq_r+b_{r-1}q_{r-1}+\ldots+b_1q_1+b_0$ with $b_i\leqslant a_i$ and let now J_n be the isotropic discrepancy of $w_n:=\{k\boldsymbol{\alpha}\}$, $k=1,\ldots,n$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_s)$ in I^s, then*:

$$N\cdot J_N\leqslant 2^s\sum_{i=0}^{r}b_iq_iJ_{q_i}.$$

Proof. Let $w(i, b)$ be the sequence $\{l\alpha+k\alpha\}$, $k=1, \ldots, q_i$, with $l=b_r q_r+ \ldots +b_{i+1}q_{i+1}+(b-1)q_i$.

Further let C be a convex subset of I^s, then we have:

$$|A(C)-N\cdot\mu(C)| = \Big|\sum_{i=0}^{r}\sum_{b=1}^{b_i}(A^{(i,b)}(C)-q_i\mu(C))\Big|$$
$$\leqslant \sum_{i=0}^{r}\sum_{b=1}^{b_i}|A^{(i,b)}(C)-q_i\mu(C)|$$
$$\leqslant 2^s\sum_{i=0}^{r} b_i q_i J_{q_i}$$

by Lemma 4.

Now we are able to prove the following upper bound:

LEMMA 6. *For the N of before and with constants c_6, c_7 we have:*

$$N\cdot J_N \leqslant c_6\sum_{i=0}^{r}\frac{b_i}{(M_{q_{i-1}})^{s-1}}\prod_{j=2}^{s}\left(\frac{\lambda_1(i)}{\lambda_j(i)}\right)+c_7\sum_{i=0}^{r}\frac{b_i q_i^{(s-1)/s}}{a_i^{1/s}}.$$

Proof. By Lemma 1 we have $J_{q_i}\leqslant 3^s\bar{J}_{q_i}+\frac{2^s s}{q_{i+1}^{1/s}}$. This is by Lemma 2 less than

$$\frac{c_3}{q_i\cdot(\bar{M}_{q'})^{s-1}}\prod_{j=2}^{s}\left(\frac{\lambda_1(i)}{\lambda_j(i)}\right)+\frac{c_4}{q_i^{1/s}a_i^{1/s}}$$

where by $\lambda_j(i)$ we note, that this is the jth minimum of the ith lattice. The last expression, by Lemma 3, is less than:

$$\frac{c_5}{q_i\cdot(M_{q_{i-1}})^{s-1}}\prod_{j=2}^{s}\left(\frac{\lambda_1(i)}{\lambda_j(i)}\right)+\frac{c_4}{q_i^{1/s}a_i^{1/s}}.$$

(Here $M_{q_{-1}}$ stands for 1.) Finally by Lemma 5 we get the result.

Remark. Of course we can estimate $\prod_{j=2}^{s}\lambda_1(i)/\lambda_j(i)$ to get the simpler estimate:

$$N\cdot J_N\leqslant c_6\sum_{i=0}^{r}\frac{b_i}{(M_{q_{i-1}})^{s-1}}+c_7\sum_{i=0}^{r}\frac{b_i q_i^{(s-1)/s}}{a_i^{1/s}}.$$

It is difficult to use information contained in the original formula, for still very little is known on the behaviour of the successive minima of such lattices. (See for example [14].)

Lower bounds

LEMMA 7. *Let $i\in N$ and $q:=q_i$ be such that $q>Q$ for a certain fixed Q, and such that $4\sqrt{sqM_q}\lambda_1\ldots\lambda_{s-1}\leqslant 1$. Then there is a N with $q_i\leqslant N\leqslant q_{i+1}$ and*

$$N\cdot J_N\geqslant\frac{c_2}{\lambda_1\lambda_2\ldots\lambda_{s-1}}\cdot\min\left(a_i, \frac{1}{q_1 M_{q_1}\lambda_1\ldots\lambda_{s-1}}\right).$$

Proof. We take $N:=Bq_i$ with $B\in N$ and choose B later. First we consider the points of the sequence $\bar{w}_q:=x_1, \ldots, x_q$. Let E be the hyperplane spanned by $s-1$ linearly independent vectors of the corresponding lattice with length $\lambda_1, \ldots, \lambda_{s-1}$. If we take E' parallel to E in distance $1/q\lambda_1\ldots\lambda_{s-1}$, then for q large enough it is clearly always possible to find a point P such that: If we place E such that $P\in E$, then for every hyperplane E'' parallel to E and lying between E and E', $E''\cap I^s$ has $s-1$-dimensional volume larger than $c_1/q\lambda_1\ldots\lambda_{s-1}$ with an absolute constant c_1 depending at most on s. Between E and E' there is no point of $\bar{w}_q$. Now we consider $w_{Bq}:=y_1, \ldots, y_{Bq}$. Let d denote the euclidean distance on the s-dimensional torus, then we have for $b\leqslant B$:

$$d(x_l, y_{l+bq})\leqslant\sqrt{s}\max_j\left(\left\|l\frac{p_j}{q}-(l+bq)\left(\frac{p_j}{q}+\frac{\theta_j}{q\cdot q_{i+1}^{1/s}}\right)\right\|\right)$$
$$=\sqrt{s}\max_j\left(\left\|(l+bq)\frac{\theta_j}{q\cdot q_{i+1}^{1/s}}\right\|\right)\quad(\text{if } q>2^s)$$
$$\leqslant\sqrt{s}(l+bq)\max_j\left(\left\|\frac{\theta_j}{q\cdot q_{i+1}^{1/s}}\right\|\right)=\sqrt{s}\frac{l+bq}{q}M_q\leqslant\sqrt{s}BM_q.$$

Since $4\sqrt{sqM_q}\lambda_1\ldots\lambda_{s-1}\leqslant 1$, we may choose

$$B:=\min\left(a_i, \left[\frac{1}{4\sqrt{sqM_q}\lambda_1\ldots\lambda_{s-1}}\right]\right);$$

then we consider the convex set C in I^s, which is formed by the inner parallel region of the set of points lying between E and E' with distance $\sqrt{s}BM_q$, dissected with I^s. In C there is no point of w_{Bq} and so we have:

$$N\cdot J_N\geqslant Bqc_1\left|\frac{1}{q\lambda_1\ldots\lambda_{s-1}}-2\sqrt{s}BM_q\right|\geqslant\frac{Bc_1}{2\lambda_1\ldots\lambda_{s-1}}$$
$$\geqslant\frac{c_1}{2\lambda_1\ldots\lambda_{s-1}}\min\left(a_i, \frac{1}{8\sqrt{sqM_q}\lambda_1\ldots\lambda_{s-1}}\right).$$

Remark. We have

$$\frac{1}{4\sqrt{sq_iM_{q_i}}\lambda_1\ldots\lambda_{s-1}}\geqslant\frac{a_i^{1/s}}{4\sqrt{sc_3}}$$

with c_3 depending only on s, by the theorem of Minkowski on successive minima (see [2]). So the condition in the lemma is for example fulfilled whenever $a_i \geqslant (4\sqrt{sc_3})^s$.

Now we consider more carefully the case $s = 2$, to avoid the restriction of Lemma 7:

LEMMA 8. *For all* (α_1, α_2) *and infinitely many* N *we have*:

$$N^{1/2}\cdot J_N \geqslant \frac{1}{2}\left(1-\frac{2}{\sqrt[4]{23}}\right) = 0.0433\ldots$$

Proof. Since the assertion is clearly true if $1, \alpha_1, \alpha_2$ are linearly dependent over $\mathbf{Q}$, we may assume in the following, that $1, \alpha_1$ and α_2 are linearly independent over $\mathbf{Q}$.

Davenport and Mahler [3] have shown: There are infinitely many $q, p_1, p_2 \in \mathbf{N}$ with

$$\alpha_i = \frac{p_i}{q}+\frac{\theta_i}{q^{3/2}}, \quad i = 1, 2$$

and

$$\theta_1^2+\theta_2^2 \leqslant 2/\sqrt{23}.$$

Let p_1, p_2, q be fixed with the above property, and let q' be the largest best approximation denominator less than q to $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ and $\frac{p_i}{q} = \frac{p_i'}{q'}+\frac{\xi_i}{q'^{3/2}}$, $i = 1, 2$.

By the theorem of Dirichlet on simultaneous approximation we have:

$$\frac{1}{q'^{1/2}}\max_i (|\xi_i|) \leqslant \frac{1}{q^{1/2}},$$

that is:

$$\left(\frac{q'}{q}\right)^{1/2} \frac{1}{\max(|\xi_1|, |\xi_2|)} \geqslant 1.$$

Without restriction of generality let $\xi_2 \leqslant \xi_1$. We consider the case $\xi_1, \xi_2 \geqslant 0$; the other cases can be treated quite analogously. Let

$$x_k := \left(\left\{k\frac{p_1}{q}\right\}, \left\{k\frac{p_2}{q}\right\}\right).$$

On the line through $x_{q'}$ and the edge of the unit square lying next to $x_{q'}$ there are at least $L := q'^{1/2}/\xi_1$ of the points x_i; $i = 1, \ldots, q$. We denote these points by $x_{l_1}, \ldots, x_{l_L}$. Let

$$y_k := (\{k\alpha_1\}, \{k\alpha_2\}) = \left(\left\{k\frac{p_1}{q}+k\frac{\theta_1}{q^{3/2}}\right\}, \left\{k\frac{p_2}{q}+k\frac{\theta_2}{q^{3/2}}\right\}\right)$$
$$= \left(\left(x_k+k\cdot\left(\frac{\theta_1}{q^{3/2}}, \frac{\theta_2}{q^{3/2}}\right)\right) \bmod 1\right).$$

We consider the following stripe P with boundary, modulo I^2: $P := P_1 \cup P_2$ (see Figure 1) τ shall be minimal, such that $y_{l_1}, \ldots, y_{l_L}$ are in P. Since $k \leqslant q$ we see, that

$$\tau \leqslant \frac{|\theta_2|}{q^{1/2}}+\frac{\xi_2}{\xi_1}\cdot\frac{|\theta_1|}{q^{1/2}} \leqslant \frac{|\theta_1|+|\theta_2|}{q^{1/2}} \leqslant \frac{\sqrt{2}}{q^{1/2}}\sqrt{\theta_1^2+\theta_2^2} \leqslant \frac{2}{\sqrt[4]{23}}\cdot\frac{1}{q^{1/2}}.$$

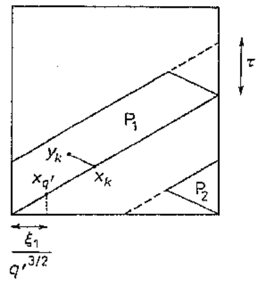

Fig. 1

Therefore for P we have:

$$\frac{A(P)}{q^{1/2}}-q^{1/2}\mu(P) \geqslant \left(\frac{q'}{q}\right)^{1/2}\frac{1}{\xi_1}-\frac{2}{\sqrt[4]{23}} \geqslant 1-\frac{2}{\sqrt[4]{23}}.$$

So for at least one of the two parts P_1, P_2 we have:

$$\frac{A(P_i)}{q^{1/2}}-q^{1/2}\mu(P_i) \geqslant \frac{1}{2}\left(1-\frac{2}{\sqrt[4]{23}}\right) = 0.0433\ldots$$

and from this the result follows.

Conclusions and special bounds.

LEMMA 9.

$$A_i := \frac{a_i^{1/s}}{\lambda_1\lambda_2\ldots\lambda_{s-1}q_i^{(s-1)/s}},$$

$$B_i := \frac{\min\left(a_i, \frac{1}{q_i M_{q_i}\lambda_1\ldots\lambda_{s-1}}\right)^{1/s}}{\lambda_1\ldots\lambda_{s-1}q_i^{(s-1)/s}},$$

$$C_i := \frac{a_i^{1/s}}{q_i^{(s-1)/s}\cdot(M_{q_{i-1}})^{s-1}}, \quad i \in N,$$

are bounded if and only if the form $L := (\sum_{j=1}^{s} m_j\alpha_j) - m$ *is extremal.*

Proof. L extremal means, that there is a constant $c_1 > 0$ such that for all $N \in N$ and all $m_1, \ldots, m_s, m$ with $|m_j| \leqslant N$ for all j, we have $N^s|L| \geqslant c_1$.

By the transference principle of Khintchine (see [1]), this is equivalent to: $q_i^{1/s}M_{q_i} > c_2$ for a fixed $c_2 > 0$ and all $i \in N$, and this is equivalent to:

$$D_i := \frac{1}{q_i^{1/s}M_{q_i}} \text{ is bounded.}$$

Since $\lambda_1\lambda_2 \ldots \lambda_{s-1}q_i^{(s-1)/s}$ is always less than a fixed constant c_3, and since

$$A_i \leqslant \frac{1}{q_iM_{q_i}\lambda_1 \ldots \lambda_{s-1}},$$

it is easy to see that A_i is bounded if and only if B_i is bounded. We show now:

(a) C_i bounded $\leftrightarrow$ D_i bounded.

Let C_i be bounded. Since $q_i^{1/s}M_{q_{i-1}} \leqslant 1$, we get that a_i and $1/q_i^{1/s}M_{q_{i-1}}$ must be bounded. Therefore

$$\frac{(a_{i-1})^{1/s}}{q_i^{1/s}M_{q_{i-1}}} = \frac{1}{(q_i/a_{i-1})^{1/s}M_{q_{i-1}}} \geqslant \frac{1}{2^{1/s}q_{i-1}^{1/s}M_{q_{i-1}}}$$

is bounded, and therefore D_i is bounded.

If D_i is bounded, then because of

$$c_2 < q_i^{1/s}M_{q_i} \leqslant \left(\frac{q_i}{q_{i+1}}\right)^{1/s} \leqslant \left(\frac{1}{a_i}\right)^{1/s}$$

we have that a_i is bounded, and further, because of

$$\frac{1}{q_i^{1/s}M_{q_{i-1}}} \leqslant \frac{1}{q_{i-1}^{1/s}M_{q_{i-1}}}$$

we see, that C_i is bounded.

(b) A_i bounded $\leftrightarrow$ C_i bounded.

Because of $M_{q_{i-1}} \leqslant 2\lambda_1$ we have $C_i \geqslant \frac{1}{2^{s-1}}A_i$ so one direction is clear.

Let now A_i be bounded. From the theorem of Minkowski on successive minima it follows that $q_i^{(s-1)/s}\lambda_1 \ldots \lambda_{s-1} < c_4$, and so a_i and $1/q_i^{(s-1)/s}\lambda_1 \ldots \lambda_{s-1}$ are bounded, and because of $\lambda_1 \leqslant 2\sqrt{s}M_{q_{i-1}}$ we only have to show, that $1/q_i^{1/s}\lambda_1$ is bounded.

We have $q_i^{(s-1)/s}\lambda_1 \ldots \lambda_{s-1} > c_5$ and so by Minkowski $\lambda_s q_i^{1/s} \leqslant c_6$ and because of $q_i\lambda_1 \ldots \lambda_s \geqslant 1$, it follows that

$$q_i^{1/s}\lambda_1 \geqslant 1/c_6^{s-1} > 0 \quad \text{for all } i$$

and the proof is finished.

Remark. We also have the relation $B_i \geqslant \min(A_i, A_i^{1/s}/c_3)$.

Theorem 1. *We have*

$$\limsup_{N\to\infty} N^{1/s}\cdot J_N < \infty$$

if and only if $L := (\sum_{j=1}^{s} m_j\alpha_j) - m$ *is an extremal form.*

Proof. If L is extremal, then by Lemma 9 and by the proof of Lemma 9:

$$C_i := \frac{a_i^{1/s}}{q_i^{(s-1)/s}\cdot(M_{q_{i-1}})^{s-1}}$$

and a_i are bounded.

By Lemma 6 and by the remark after this lemma, and by noting that $N \geqslant b_rq_r$, we get:

$$\begin{aligned} N^{1/s}\cdot J_N &\leqslant c_6 \sum_{i=0}^{r} \frac{b_i^{1/s}}{(q_i^{1/s}M_{q_{i-1}})^{s-1}}\left(\frac{b_i}{b_r}\right)^{(s-1)/s}\left(\frac{q_i}{q_r}\right)^{(s-1)/s} \\ &\quad + c_7\sum_{i=0}^{r}\left(\frac{b_i}{a_i}\right)^{1/s}\left(\frac{b_i}{b_r}\right)^{(s-1)/s}\left(\frac{q_i}{q_r}\right)^{(s-1)/s} \\ &\leqslant (c_6B_ia_i^{(s-1)/s} + c_7a_i^{(s-1)/s})\sum_{i=0}^{r}\left(\frac{q_i}{q_r}\right)^{(s-1)/s}. \end{aligned}$$

By Lagarias [7] we have for the best simultaneous approximation denominators with respect to the maximum norm:

$$\frac{q_{i+2^{s+1}}}{q_i} \geqslant 3 \quad \text{for all } i$$

and so

$$\sum_{i=0}^{r}\left(\frac{q_i}{q_r}\right)^{(s-1)/s} \leqslant 2^{s+1}\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^{i(s-1)/s} = c_8 < \infty,$$

and so $N^{1/s}\cdot J_N$ is bounded.

If L is not extremal, then by Lemma 9 A_i and B_i are not bounded. And because of

$$\frac{1}{4\sqrt{s}q_iM_{q_i}\lambda_1 \ldots \lambda_{s-1}} \geqslant A_i > 1 \quad \text{for infinitely many } i,$$

we get by Lemma 7, that $N^{1/s}\cdot J_N \geqslant B_i$ and this is, by the remark after Lemma 9, larger or equal to $\min(A_i, A_i^{1/s}/c_3)$ and the result easily follows.

THEOREM 2. *For the case $s = 2$ moreover we get: For $N \in \mathbf{N}$ let $i(N)$ be such that $q_{i(N)} \leqslant N \leqslant q_{i(N)+1}$, then*

(a) *If for a $c_1 > 0$ and a $\sigma \geqslant 1/2$ we have*

$$q_{i+1}^{\sigma} M_{q_i} \geqslant c_1 \quad \text{for all } i,$$

then

$$N^{\sigma}\cdot J_N \leqslant c_2 (\max_{i \leqslant i(N)} a_i)^{1-\sigma}.$$

(b) *If for a $c_3 > 0$ and a $\sigma \geqslant 1/2$ we have*

$$q_{i+1}^{\sigma} M_{q_i} \leqslant c_3 \quad \text{for infinitely many } i,$$

then for every J large enough, there is a $N \leqslant q_{J+1}$ with

$$N^{\sigma}\cdot J_N \geqslant c_4 \max\big(1, \max_{i \in A_J} \min(a_i, q_i^{\sigma-1/2} a_i^{1/2})\big)^{1-\sigma}$$

where

$$A_J := \{i \leqslant J |\ q_i^{1/2} M_{q_{i-1}} \leqslant \tfrac{1}{16} \text{ and } q_i^{\sigma} M_{q_{i-1}} \leqslant c_3\}.$$

Proof. (a) By Lemma 6 we have, like in Theorem 1:

$$N^{1-\sigma}\cdot J_N \leqslant c_6 \sum_{i=1}^{r} \frac{b_i/b_r^{\sigma}}{c_1}\left(\frac{q_i}{q_r}\right)^{\sigma} + c_7 \sum_{i=0}^{r} \frac{b_i/b_r^{\sigma}}{a_i^{1/2}}\left(\frac{q_i}{q_r}\right)^{\sigma}$$
$$\leqslant c_5 \sum_{i=0}^{r} \frac{b_i}{b_r^{\sigma}}\left(\frac{q_i}{q_r}\right)^{\sigma} \leqslant c_8\left(b_r^{1-\sigma} + b_{r-1}^{1-\sigma} + b_{r-2}^{1-\sigma}\left(\frac{q_{r-1}}{q_r}\right)^{\sigma} + \dots\right)$$
$$\leqslant c_9 (\max_{i \leqslant i(N)} a_i)^{1-\sigma}.$$

(b) If A_J is empty for all J, then $\sigma = 1/2$ and the result follows by Lemma 8.

Otherwise let J be so large that A_J is not empty and let $i \in A_J$ be such that $\min(a_i, q_i^{\sigma-1/2} a_i^{1/2})$ is maximal.

We take $N = Bq_i$ like in Lemma 7, and from this lemma the result easily follows, if we remark that

$$\frac{1}{4\sqrt{2}\, q_i M_{q_i} \lambda_1} \geqslant \frac{1}{16 q_i^{1/2} M_{q_{i-1}}} \geqslant 1,$$

and that

$$\frac{1}{q_i M_{q_i} \lambda_1} \geqslant \frac{1}{q_i^{\sigma} M_{q_{i-1}} 2\sqrt{2}} \cdot \frac{1}{q_i^{1-\sigma} M_{q_i}} \geqslant \frac{1}{2\sqrt{2}\, c_3} q_i^{\sigma-1/2} a_i^{1/2}.$$

Remark 1. In Theorem 2(b) σ always is less or equal 1.

Remark 2. Theorem 2(a) of course can be shown quite analogously for general $s \geqslant 2$.

THEOREM 3. *For $s \geqslant 2$ and for almost all $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_s)$ in $\mathbf{R}^s$ in the sense of Lebesgue-measure, we have for every $\varepsilon > 0$:*

$$J_N = O\big(N^{-1/s} (\log N)^{(s-1)/s+\varepsilon}\big).$$

Proof. For example from Davenport–Schmidt [4] it follows, that for almost all $\boldsymbol{\alpha}$ we can choose σ in the s-dimensional form of Theorem 2(a) equal to $1/s$. Further (see for example [8]) we have $a_i = O\big((\log q_i)^{1+\varepsilon}\big)$ for every $\varepsilon > 0$ and almost all $\boldsymbol{\alpha}$. So by the general form of Theorem 2(a) the result follows.

References

[1] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge 1959.
[2] — *An Introduction to the Geometry of Numbers*, Grundlehren 99, Springer 1953.
[3] H. Davenport and K. Mahler, *On simultaneous Diophantine Approximation*, Duke Math. J. 13 (1946), 105–111.
[4] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on Diophantine Approximation II*, Acta Arith. 16 (1970), 413–424.
[5] E. Hlawka, *Über eine Methode von E. Hecke in der Theorie der Gleichverteilung*, ibid. 24 (1973), 11–31.
[6] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, New York 1974.
[7] J. C. Lagarias, *Best Simultaneous Diophantine Approximation I*, Trans. Amer. Math. Soc. (2) 272 (1982), 545–554.
[8] G. Larcher, *Über die isotrope Diskrepanz von Folgen*, Arch. Math. 46 (1986), 240–249.
[9] H. Niederreiter, *Methods for estimating discrepancy*; In: *Applications of Number Theory to Numerical Analysis* (S. K. Zaremba ed.), New York 1972, 203–236.
[10] A. M. Ostrowski, *Error term in Multidimensional Diophantine Approximation*, Acta Arith. 41 (1982), 163–183.
[11] W. M. Schmidt, *Lectures on Irregularities of Distribution*, Tata Institute, Bombay 1977.
[12] — *Irregularities of Distribution IX*, Acta Arith. 27 (1975), 385–396.
[13] — *Metrical theorems on fractional parts of sequences*, Trans. Amer. Math. Soc. 110 (1964), 493–518.
[14] — *Open problems in Diophantine Approximation*; In: *Approximations Diophantiennes et Nombres Transcendants* (D. Bertrand, M. Waldschmidt, ed.), Birkhäuser 1983, 271–288.

INSTITUT FÜR MATHEMATIK
UNIVERSITÄT SALZBURG
Hellbrunnerstraße 34
A-5020 Salzburg
Austria

Received on 7.10.1986 (1678)