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On zeros of functions satisfying certain differential-difference equations

by

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1. Introduction. A basic problem in the theory of sieves is to find good upper and lower bounds for the sifting function

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1.$$

Here $\mathcal{A}$ is a finite sequence of integers, $\mathcal{P}$ a sequence of primes, $z \geqslant 2$ (a real number) and

$$P(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} p.$$

Let

$$|\mathcal{A}_d| = |\{a \in \mathcal{A}:\ a \equiv 0\,(d)\}|.$$

We assume $|\mathcal{A}_d|$ to be written in the form

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + R(\mathcal{A}, d) \quad \text{for } d \mid P(z)$$

where $X > 1$ is independent of d and ω is a multiplicative function satisfying

$$0 < \omega(p) < p \quad \text{for } p \in \mathcal{P}.$$

Finally we define

$$V(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} (1 - \omega(p)/p).$$

Then the following theorem, due to Iwaniec ([7]), holds true.

THEOREM A. *Let* $y \geqslant z \geqslant 2$, $s = \log y/\log z$ *and suppose that there exist constants* $\varkappa > 0$, $K \geqslant 2$ *such that*

$$\frac{V(w_1)}{V(w_2)} < \left(\frac{\log w_2}{\log w_1}\right)^{\varkappa} \left(1 + \frac{K}{\log w_1}\right) \quad \textit{for } 2 \leqslant w_1 < w_2.\,(^1)$$

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(1) $\varkappa$ is called the *dimension* of the sieve.

Then

$$(1.1)\qquad S(\mathcal{A},\mathcal{P},z) < XV(z)\left(\tilde{F}_\varkappa(s)+O_{\varkappa,K}\left(\frac{e^{-s}}{(\log y)^{1/3}}\right)\right)+\sum_{\substack{d\mid P(z)\\ d<y}}|R(\mathcal{A},d)|,$$

and

$$(1.2)\qquad S(\mathcal{A},\mathcal{P},z) > XV(z)\left(\tilde{f}_\varkappa(s)+O_{\varkappa,K}\left(\frac{e^{-s}}{(\log y)^{1/3}}\right)\right)-\sum_{\substack{d\mid P(z)\\ d<y}}|R(\mathcal{A},d)|$$

holds true. $\tilde{F}_\varkappa(s)$ *and* $\tilde{f}_\varkappa(s)$ *are the continuous solutions of the following system of differential-difference equations:*

$$\begin{aligned} s^\varkappa \tilde{F}_\varkappa(s) &= A_\varkappa, & s &\leqslant \beta_\varkappa+1,\\ s^\varkappa \tilde{f}_\varkappa(s) &= B_\varkappa, & s &\leqslant \beta_\varkappa,\\ (s^\varkappa \tilde{F}_\varkappa(s))' &= \varkappa s^{\varkappa-1}\tilde{f}_\varkappa(s-1), & s &> \beta_\varkappa+1,\\ (s^\varkappa \tilde{f}_\varkappa(s))' &= \varkappa s^{\varkappa-1}\tilde{F}_\varkappa(s-1), & s &> \beta_\varkappa. \end{aligned}$$

The definitions of $A_\varkappa$, $B_\varkappa$, $\beta_\varkappa$ require some knowledge about the nontrivial solutions of

$$(sq_\varkappa(s))' = \varkappa q_\varkappa(s)+\varkappa q_\varkappa(s+1)$$

and

$$(sh_\varkappa(s))' = \varkappa h_\varkappa(s)-\varkappa h_\varkappa(s+1).$$

It is known for example that $\beta_\varkappa-1$ is the largest (real) zero of $q_\varkappa(s)$ if $\varkappa > 1/2$. Estimates for the largest zero of $q_\varkappa(s)$ will be proved in this paper.

If $\varkappa = 1$ Theorem A was already proved before by Jurkat–Richert ([6], [9]) via Selberg's sieve, whereas Iwaniec's proof uses Rosser's sieve. (2)

If $\varkappa > 1$ Iwaniec pointed out in [7] that an iteration of Selberg's sieve with Buchstab's identity would give better results than those in Theorem A. The first step of this iteration was already made by Ankeny–Onishi ([2]). A second step has been made by Porter ([10]). There are also numerical results due to Diamond–Jurkat (unpublished). Making a number of (plausible) assumptions, Iwaniec–van de Lune–te Riele ([8]) gave the limit of this iteration. They showed that instead of (1.1) and (1.2)

$$S(\mathcal{A},\mathcal{P},z) < XV(z)F_\varkappa(s)+\varepsilon(\varkappa,K,y,z,X),$$

$$S(\mathcal{A},\mathcal{P},z) > XV(z)f_\varkappa(s)-\varepsilon(\varkappa,K,y,z,X)$$

holds true, where $\varepsilon(\varkappa,K,y,z,X)$ is an upper bound for the error terms. (3)

(2) Indeed it is not exactly the same theorem, but the main terms are the same and the remainder terms are similar.

(3) See [3], [8].

$F_\varkappa(s)$ and $f_\varkappa(s)$ — superior to $\tilde{F}_\varkappa(s)$ and $\tilde{f}_\varkappa(s)$ of Theorem A — are the continuous solutions of the following system of differential-difference equations:

there exist $\alpha_\varkappa$ $(\geqslant 1)$, $\beta_\varkappa$ $(\geqslant 1)$ (4) such that

$$\begin{aligned} F_\varkappa(s) &= \frac{1}{\sigma_\varkappa(s)}, & s &\leqslant \alpha_\varkappa,\\ f_\varkappa(s) &= 0, & s &\leqslant \beta_\varkappa,\\ (s^\varkappa F_\varkappa(s))' &= \varkappa s^{\varkappa-1}f_\varkappa(s-1), & s &> \alpha_\varkappa,\\ (s^\varkappa f_\varkappa(s))' &= \varkappa s^{\varkappa-1}F_\varkappa(s-1), & s &> \beta_\varkappa, \end{aligned}$$

where the continuous function $\sigma_\varkappa(s)$ satisfies

$$\sigma_\varkappa(s) = \frac{1}{2^\varkappa e^{\gamma\varkappa}\Gamma(\varkappa+1)}s^\varkappa,\qquad 0\leqslant s\leqslant 2,$$

$$(s^{-\varkappa}\sigma_\varkappa(s))' = -\varkappa s^{-\varkappa-1}\sigma_\varkappa(s-2),\qquad s>2.$$

Numerical values for $\alpha_\varkappa$, $\beta_\varkappa$ can be found in [8] and [12].

Results concerning this Buchstab iteration were also proved independently by Rawsthorne ([11]). However, the question whether the iteration works at all, was not answered by these papers.

The work of Rawsthorne was picked up again by Diamond–Halberstam–Richert ([4]). They distinguish (according to Rawsthorne) the four cases

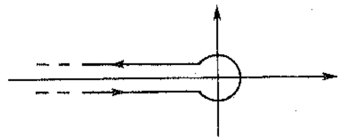
$$\alpha_\varkappa<\beta_\varkappa-1,\quad \beta_\varkappa-1\leqslant\alpha_\varkappa<\beta_\varkappa,\quad \beta_\varkappa\leqslant\alpha_\varkappa<\beta_\varkappa+1,\quad \beta_\varkappa+1\leqslant\alpha_\varkappa.$$

The cases one and two have been assumed to be impossible in connection with sieves ([8], [11]). This was meanwhile proved by Diamond–Halberstam–Richert ([4]). Their proof is based on a good lower bound for the largest real zero of the function $Q_\varkappa(s)$ (for a definition see (4.1)). Lower bounds and upper bounds for the largest real zero of $Q_\varkappa(s)$ are also proved in this paper. The results of Theorem 5 were needed in [4]. The cases $\beta_\varkappa\leqslant\alpha_\varkappa<\beta_\varkappa+1$, $\beta_\varkappa+1\leqslant\alpha_\varkappa$ are yet not completely solved. However, it turns out that also upper bounds for the largest zero of $q_\varkappa(s)$ are needed (for a definition see (2.1)). They are given in Theorem 6.

2. Some definitions and lemmata. Define

$$(2.1)\qquad q_\varkappa(s) = q(s) = \frac{\Gamma(2\varkappa)}{2\pi i}\int_{\mathscr{C}} z^{-2\varkappa}\exp\left(sz+\varkappa\int_0^z\frac{1-e^u}{u}du\right)dz\quad \text{for } s>0,\ \varkappa\geqslant 1,$$

where $\mathscr{C}$ is any curve of shape



(4) For details see [8].

and $z^{-2\varkappa} = \exp(-2\varkappa \operatorname{Log} z)$. Note that $q(s) \in C^\infty(0, \infty)$. It is easy to prove (see [7], 5.1) that $q(s)$ satisfies

$$(sq(s))' = \varkappa q(s) + \varkappa q(s+1).$$

Moreover let $q_\varkappa^{(\nu)}(s) = q^{(\nu)}(s)$ ([5]) — $\nu \in N_0$ — be the νth derivative of $q(s)$. Then it is obvious that

$$(2.2)\qquad (sq^{(\nu)}(s))' = (\varkappa - \nu) q^{(\nu)}(s) + \varkappa q^{(\nu)}(s+1) \quad \text{for } \nu \in N_0$$

holds true. ([6])

Furthermore we define inductively

$$(2.3)\quad q_\varkappa^{(\nu-1)}(s) = q^{(\nu-1)}(s)$$
$$= \frac{1}{2\varkappa - \nu}\Big(sq^{(\nu)}(s) - \varkappa \int_s^{s+1} q^{(\nu)}(t)\,dt\Big) \quad \text{for } -\nu \in N_0,\ s > 0\ (^5)$$

and finally we denote

$$z_\varkappa^{(\nu)} = z^{(\nu)} \quad \text{the largest zero of } q_\varkappa^{(\nu)}(s) \text{ for } \varkappa \geqslant 1\ (^5)$$

if $z^{(\nu)}$ exists (see Lemma 2).

LEMMA 1. *We have*

$$(2.4)\qquad (sq^{(\nu)}(s))' = (\varkappa - \nu) q^{(\nu)}(s) + \varkappa q^{(\nu)}(s+1) \quad \textit{for } \nu \in \mathbf{Z}.$$

Proof. For $\nu \in N_0$ (2.4) is (2.2). Now suppose that (2.4) holds true for some $-\nu \in N_0$. By (2.3) and induction hypothesis we obtain

$$(2.5)\qquad \frac{d}{ds} q^{(\nu-1)}(s) = \frac{1}{2\varkappa - \nu}\big((sq^{(\nu)}(s))' - \varkappa q^{(\nu)}(s+1) + \varkappa q^{(\nu)}(s)\big) = q^{(\nu)}(s).$$

On the other hand, again by (2.3) and induction hypothesis

$$(\varkappa - \nu) q^{(\nu-1)}(s) + \varkappa q^{(\nu-1)}(s+1) = \frac{1}{2\varkappa - \nu}\Big((\varkappa - \nu) sq^{(\nu)}(s) + \varkappa (s+1) q^{(\nu)}(s+1)$$
$$- \varkappa \int_s^{s+1} \big((\varkappa - \nu) q^{(\nu)}(t) + \varkappa q^{(\nu)}(t+1)\big)dt\Big) = sq^{(\nu)}(s)$$

holds true. Hence

$$(2.6)\qquad sq^{(\nu)}(s) = (\varkappa - \nu) q^{(\nu-1)}(s) + \varkappa q^{(\nu-1)}(s+1)$$

or, equivalently by (2.5),

$$(sq^{(\nu-1)}(s))' = (\varkappa - (\nu - 1)) q^{(\nu-1)}(s) + \varkappa q^{(\nu-1)}(s+1).$$

For the simplification of several expressions we put

$$\alpha_k = 2\varkappa - [2\varkappa].$$

([5]) $q^{(\nu)}(s)$ and $z^{(\nu)}$ depend on $\varkappa$, but for simplicity we omit $\varkappa$, if there is no confusion possible.

([6]) Further properties of q can be found in [7].

LEMMA 2.

$$(2.7)\qquad z_\varkappa^{(\nu)} \textit{ exists for } \nu \leqslant \varkappa_0 = \begin{cases} [2\varkappa] - 1, & \textit{if } 2\varkappa \notin N, \\ 2\varkappa - 2, & \textit{if } 2\varkappa \in N. \end{cases}$$

$$(2.8)\qquad z_\varkappa^{(\nu-1)} > z_\varkappa^{(\nu)} > 0 \quad \textit{for } \nu \leqslant \varkappa_0.$$

Proof. Let $\nu = \varkappa_0$. If $2\varkappa \in N$ we have (cf. [7], 5.1)

$$z^{(\varkappa_0)} = \varkappa.$$

Applying for $2\varkappa \notin N$ Lemma 2 from [7] with $a = \varkappa - \nu$, $\nu = [2\varkappa] - 1$, $b = \varkappa$, $N = 0$ gives, using (2.4),

$$(2.9)\quad q^{([2\varkappa]-1)}(s) = (2\varkappa - 1)(2\varkappa - 2) \cdot \ldots \cdot (\alpha_\varkappa + 1)$$
$$* \Big(s^{\alpha_\varkappa} + \frac{1}{\Gamma(-\alpha_\varkappa)} \int_0^\infty e^{-sz}\Big(\exp\Big(\varkappa \int_0^z \frac{1 - e^{-u}}{u}\,du\Big) - 1\Big)\frac{dz}{z^{\alpha_\varkappa + 1}}\Big).\ (^7)$$

Now $q^{([2\varkappa]-1)}(s) < 0$ for $s \to 0+$ and $q^{([2\varkappa]-1)}(s) \to \infty$ for $s \to \infty$. Hence

$$z^{(\varkappa_0)} \text{ exists and is positive.}$$

Suppose now that $z^{(\nu)}$ exists for some $\nu \leqslant \varkappa_0$. By (2.6) we have obviously

$$sq^{(\nu)}(s) = (2\varkappa - \nu) q^{(\nu-1)}(s) + \varkappa\big(q^{(\nu-1)}(s+1) - q^{(\nu-1)}(s)\big)$$

and the mean-value theorem now gives the existence of a

$$(2.10)\qquad \xi_s \in (s, s+1)$$

such that

$$(2.11)\qquad sq^{(\nu)}(s) = (2\varkappa - \nu) q^{(\nu-1)}(s) + \varkappa q^{(\nu)}(\xi_s).$$

Since $q^{(\varkappa_0)}(s) \to \infty$ as $s \to \infty$ and $\frac{d}{ds} q^{(\nu-1)}(s) = q^{(\nu)}(s)$ for $\nu \in \mathbf{Z}$, we have

$$(2.12)\qquad q^{(\nu)}(s) \to \infty \quad \text{if } s \to \infty \text{ and } \nu \leqslant \varkappa_0.$$

By the definition of $z^{(\nu)}$ we have therefore

$$(2.13)\qquad q^{(\nu)}(s) > 0 \quad \text{for } s > z^{(\nu)}.$$

Choosing now $s = z^{(\nu)}$ in (2.11) and using (2.10) and (2.13) gives

$$(2.14)\qquad q^{(\nu-1)}(z^{(\nu)}) < 0 \quad \text{for } \nu \leqslant \varkappa_0.$$

Hence, by (2.12), we see that $z^{(\nu-1)}$ exists with

$$z^{(\nu-1)} > z^{(\nu)}.$$

([7]) For the normalization of $q^{([2\varkappa]-1)}(s)$ see [7], 5.2, Remark of Lemma 4.

LEMMA 3. *We have*

(2.15) $$1+z^{(\nu)} \leqslant z^{(\nu-1)} \leqslant z^{(\nu)}+\frac{\varkappa}{2\varkappa-\nu} \quad \textit{for } \varkappa \leqslant \nu \leqslant \varkappa_0$$

and

(2.16) $$1+z^{(\nu)} > z^{(\nu-1)} > z^{(\nu)}+\frac{\varkappa}{2\varkappa-\nu} \quad \textit{for } \nu < \varkappa$$

In (2.15) *equality holds true, iff* $\nu = \varkappa$.

Proof. $s = z^{(\nu)}$ in (2.6) with $\nu \geqslant \varkappa$ gives, using (2.14),

$$q^{(\nu-1)}(z^{(\nu)}+1) \leqslant 0$$

and, by (2.12),

$$z^{(\nu)}+1 \leqslant z^{(\nu-1)} \quad \text{for } \nu \geqslant \varkappa.$$

Obviously equality holds true, iff $\nu = \varkappa$. With similar arguments we see that

$$z^{(\nu-1)} < z^{(\nu)}+1 \quad \text{for } \nu < \varkappa.$$

In order to prove the remaining inequalities in (2.15) and (2.16) we use the fact that

(2.17) $q^{(\nu-1)}(s)$ is strictly convex for $s \geqslant z^{(\nu)}$ if $\nu \leqslant \varkappa_0$.

As long as $z^{(\nu+1)}$ exists — by (2.7) this is the case for $\nu+1 \leqslant \varkappa_0$ — (2.17) is obvious by (2.8). Hence we may assume $\nu = \varkappa_0$. If $2\varkappa \in N$, $q^{(\varkappa_0-1)}(s)$ is a polynomial of degree 2 with positive leading coefficient (cf. [7], 5.1) and therefore trivially strictly convex. If $2\varkappa \notin N$, we have, by differentiating (2.9),

(2.18) $$q^{(\varkappa_0+1)}(s) = -\frac{(2\varkappa-1)\cdot\ldots\cdot(\alpha_\varkappa+1)}{\Gamma(-\alpha_\varkappa)}\int_0^\infty e^{-sz}\exp\left(\varkappa\int_0^z\frac{1-e^{-u}}{u}du\right)\frac{dz}{z^{\alpha_\varkappa}} > 0$$

which proves that $q^{(\varkappa_0-1)}(s)$ is strictly convex in this case.

For $0 < \eta < 1$ we have now by (2.17) and (2.6)

$$q^{(\nu-1)}(z^{(\nu)}+\eta) < (1-\eta)q^{(\nu-1)}(z^{(\nu)})+\eta q^{(\nu-1)}(z^{(\nu)}+1)$$
$$= \left(1-\eta+\eta\frac{\nu-\varkappa}{\varkappa}\right)q^{(\nu-1)}(z^{(\nu)}) = 0$$

choosing $\eta = \varkappa/(2\varkappa-\nu)$. This together with (2.12) proves the remaining part of (2.16).

Now let $\nu > \varkappa$. Again, by (2.17), we have

(2.19) $$q^{(\nu-1)}(s) > q^{(\nu-1)}(z^{(\nu)}+1)+(s-(z^{(\nu)}+1))(q^{(\nu-1)}(z^{(\nu)}+1)-q^{(\nu-1)}(z^{(\nu)})) \quad \text{for } s > z^{(\nu)}+1.$$

Using (2.6) — with $s = z^{(\nu)}$ — in (2.19) gives

$$q^{(\nu-1)}(s) > q^{(\nu-1)}(z^{(\nu)}+1)\left(1+(s-(z^{(\nu)}+1))\left(1-\frac{\varkappa}{\nu-\varkappa}\right)\right)$$

and this gives for $\nu > \varkappa$

$$q^{(\nu-1)}\left(z^{(\nu)}+\frac{\varkappa}{2\varkappa-\nu}\right) > 0.$$

Hence, by (2.13), the remaining part of (2.15) is proved.

3. Estimates for the zeros of $q^{(\nu)}(s)$.

LEMMA 4. *Let* $s > 0$, $r \in N_0$, $R = [2\varkappa]+r-1$. *Then we have*

(3.1) $$(-1)^r q^{(R+1)}(s) \geqslant 0,$$

(3.2) $$(-1)^r(sq^{(R+1)}(s)-(2\varkappa-R-1)q^{(R)}(s)) \geqslant 0,$$

(3.3) $$(-1)^r\left((s-\varkappa)q^{(\nu)}(s)-(2\varkappa-\nu)q^{(\nu-1)}(s)-\varkappa\sum_{\mu=\nu+1}^{R}\frac{q^{(\mu)}(s)}{(\mu+1-\nu)!}\right) \geqslant 0, \quad \nu \leqslant R.$$

In (3.1), (3.2) *and* (3.3) *equality holds true if* $2\varkappa \in N$.

Proof. If $2\varkappa \in N$, q is a polynomial of degree $2\varkappa-1$ (cf. [7]). Hence equality holds obviously true in (3.1) and (3.2) in this case.

If $2\varkappa \notin N$ (3.1) follows from (2.18). Applying Taylor's Theorem to $q^{(\nu-1)}(s+1)$ and using (3.1) gives

$$(-1)^r\left(q^{(\nu-1)}(s+1)-\sum_{\mu=\nu-1}^{R}\frac{q^{(\mu)}(s)}{(\mu+1-\nu)!}\right) \geqslant 0 \quad \text{for } \nu \leqslant R+1.$$

Now we insert this in (2.6). For $\nu = R+1$ this gives (3.2) and for $\nu \leqslant R$ this gives (3.3). Equality in (3.3) again is obvious if $2\varkappa \in N$.

The next theorem will be starting point in order to prove upper and lower bounds for $z_\varkappa^{(\nu)}$. We shall use the following definitions. Let

$$(a)_m = \prod_{\nu=0}^{m-1}(a+\nu) \quad \text{for } a \in \boldsymbol{R},\ m \in N_0.\text{[8]}$$

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[8] $\prod_{\nu=0}^{-1}(a+\nu) = 1.$

For fixed $r \in N_0$, $\varkappa \geqslant 1$ we define a sequence of polynomials in z (real) as follows:

$$\begin{aligned}
&P_0(z, r, \varkappa) = P_0(z) = 1,\\
(3.4)\quad &P_1(z, r, \varkappa) = P_1(z) = z,\\
&P_{n+1}(z, r, \varkappa) = P_{n+1}(z)\\
&\quad = zP_n(z, r, \varkappa) - \varkappa \sum_{\mu=0}^{n-1} \frac{(2\varkappa - R + \mu)_{n-\mu}}{(n+1-\mu)!} P_\mu(z, r, \varkappa), \text{[9]} \qquad n \in N,
\end{aligned}$$

where $R = [2\varkappa] + r - 1$ as before. We have the following

THEOREM 1. *Let* $r \in \{0, 1, 2, 3\}$, [10] $n \in N_0$, $R = [2\varkappa] + r - 1$. *If*

$$(3.5)\qquad s \geqslant (R+\varkappa)/3$$

and

$$(3.6)\qquad P_\nu(s-\varkappa, r, \varkappa) \geqslant 0 \quad \textit{for } 1 \leqslant \nu \leqslant n,$$

then we have

$$(3.7)\qquad (-1)^r \prod_{\mu=0}^{n-1} P_\mu(s-\varkappa, r, \varkappa)$$
$$* \big(P_{n+1}(s-\varkappa, r, \varkappa) q^{(R-n)}(s) - (2\varkappa - R + n) P_n(s-\varkappa, r, \varkappa) q^{(R-n-1)}(s)\big) \geqslant 0.$$

Proof. (3.3) with $\nu = R$ gives (3.7) for $n = 0$. Now suppose that

$$(3.8)\qquad (-1)^r \prod_{\mu=0}^{m-2} P_\mu(s-\varkappa, r, \varkappa)$$
$$* \big(P_m(s-\varkappa, r, \varkappa) q^{(R-m+1)}(s) - (2\varkappa - R + m - 1) P_{m-1}(s-\varkappa, r, \varkappa) q^{(R-m)}(s)\big) \geqslant 0$$

holds true for all $1 \leqslant m \leqslant M$ with $1 \leqslant M \leqslant n$. Then we have

$$\begin{aligned}
(3.9)\quad (-1)^r \prod_{\mu=0}^{l-1} P_\mu(s-\varkappa) \Big(& P_l(s-\varkappa) P_1(s-\varkappa) q^{(R-m)}(s)\\
& - \big(2\varkappa - (R-m)\big) P_l(s-\varkappa) q^{(R-(m+1))}(s)\\
& - \varkappa P_l(s-\varkappa) \sum_{\varrho=R-m+1}^{R-l} \frac{q^{(\varrho)}(s)}{(\varrho+1-R+m)!}\\
& - \varkappa q^{(R-l)}(s) \sum_{\lambda=0}^{l-1} \frac{(2\varkappa - (R-\lambda))_{l-\lambda}}{(m-\lambda+1)!} P_\lambda(s-\varkappa) \Big) \geqslant 0
\end{aligned}$$

for

$$0 \leqslant l \leqslant m \leqslant M.$$

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[9] $P_{n+1}(z, r, \varkappa)$ is a polynomial in z of degree $n+1$ with leading coefficient 1. We omit the arguments r, $\varkappa$ whenever there is no confusion possible.

[10] Adding further conditions, the theorem can also be formulated for $r \geqslant 4$.

This can be seen as follows:

(3.3) with $\nu = R - m$ is (3.9) with $l = 0$. Now suppose that (3.9) holds true for some l with $0 \leqslant l \leqslant m-1$. We multiply (3.9) with $P_{l+1}(s-\varkappa)$ ($\geqslant 0$) and use (3.8) with $m = l+1$. Since

$$\sum_{\lambda=0}^{l} \frac{(2\varkappa - (R-\lambda))_{l-\lambda}}{(m-\lambda+1)!} P_\lambda(s-\varkappa) \geqslant 0$$

by (3.5) and (3.6), we obtain

$$\begin{aligned}
(-1)^r \prod_{\mu=0}^{l} P_\mu(s-\varkappa) \Big(& P_{l+1}(s-\varkappa) P_1(s-\varkappa) q^{(R-m)}(s)\\
& - \big(2\varkappa - (R-m)\big) P_{l+1}(s-\varkappa) q^{(R-(m+1))}(s)\\
& - \varkappa P_{l+1}(s-\varkappa) \sum_{\varrho=R-m+1}^{R-l-1} \frac{q^{(\varrho)}(s)}{(\varrho+1-R+m)!}\\
& - \varkappa q^{(R-l-1)}(s) \sum_{\lambda=0}^{l} \frac{(2\varkappa - (R-\lambda))_{l-\lambda}}{(m-\lambda+1)!} \big(2\varkappa - (R-l)\big) P_\lambda(s-\varkappa) \Big) \geqslant 0.
\end{aligned}$$

This completes the proof of (3.9).

Taking now $l = m = M$ in (3.9) gives

$$\begin{aligned}
&(-1)^r \prod_{\mu=0}^{m-1} P_\mu(s-\varkappa)\\
&\quad \times \bigg(q^{(R-m)}(s) \Big(P_m(s-\varkappa) P_1(s-\varkappa) - \varkappa \sum_{\lambda=0}^{m-1} \frac{(2\varkappa - (R-\lambda))_{m-\lambda}}{(m-\lambda+1)!} P_\lambda(s-\varkappa) \Big)\\
&\qquad\qquad - \big(2\varkappa - (R-m)\big) P_m(s-\varkappa) q^{(R-(m+1))}(s) \bigg) \geqslant 0
\end{aligned}$$

and this proves, by (3.4), (3.8) for $m \leqslant n+1$.

Remark. The first polynomials read as follows ($\alpha_\varkappa = 2\varkappa - [2\varkappa]$):

$$P_2(z, r, \varkappa) = z^2 - \frac{\varkappa}{2}(2\varkappa - R)_1 = z^2 - \frac{\varkappa}{2}(\alpha_\varkappa + 1 - r),$$

$$\begin{aligned}
P_3(z, r, \varkappa) &= z^3 - \frac{\varkappa}{2} z \big((2\varkappa - R)_1 + (2\varkappa - R + 1)_1\big) - \frac{\varkappa}{3!}(2\varkappa - R)_2\\
&= z^3 - \frac{\varkappa}{2} z (2\alpha_\varkappa - 2r + 3) - \frac{\varkappa}{6}(\alpha_\varkappa - r + 1)(\alpha_\varkappa - r + 2),
\end{aligned}$$

$$\begin{aligned}
P_4(z, r, \varkappa) &= z^4 - \frac{\varkappa}{2} z^2 \big((2\varkappa - R)_1 + (2\varkappa - R + 1)_1 + (2\varkappa - R + 2)_1\big)\\
&\quad - \frac{\varkappa}{3!} z \big((2\varkappa - R)_2 + (2\varkappa - R + 1)_2\big)
\end{aligned}$$

$$-\frac{\varkappa}{4!}(2\varkappa-R)_3+\frac{\varkappa^2}{4}(2\varkappa-R)_1(2\varkappa-R+2)_1,$$

$$=z^4-\frac{3}{2}\varkappa z^2(\alpha_\varkappa-r+2)-\frac{\varkappa}{3}z(\alpha_\varkappa-r+2)^2$$

$$-\frac{\varkappa}{4}(\alpha_\varkappa-r+1)(\alpha_\varkappa-r+3)\left(\frac{\alpha_\varkappa-r+2}{6}-\varkappa\right),$$

$$P_5(z,r,\varkappa)=z^5-\frac{\varkappa}{2}z^3\big((2\varkappa-R)_1+(2\varkappa-R+1)_1+(2\varkappa-R+2)_1+(2\varkappa-R+3)_1\big)$$

$$-\frac{\varkappa}{3!}z^2\big((2\varkappa-R)_2+(2\varkappa-R+1)_2+(2\varkappa-R+2)_2\big)$$

$$-\frac{\varkappa}{4!}z\big((2\varkappa-R)_3+(2\varkappa-R+1)_3\big)$$

$$+\frac{\varkappa^2}{4}z\big((2\varkappa-R)(2\varkappa-R+2)+(2\varkappa-R)(2\varkappa-R+3)+(2\varkappa-R+1)(2\varkappa-R+3)\big)$$

$$-\frac{\varkappa}{5!}(2\varkappa-R)_4+\frac{\varkappa^2}{12}\big((2\varkappa-R)(2\varkappa-R+2)_2+(2\varkappa-R)_2(2\varkappa-R+3)\big).$$

We shall now show how Theorem 1 will be used to prove upper and lower bounds for the zeros $z_\varkappa^{(\nu)}$ of $q_\varkappa^{(\nu)}(s)$.

For $n\in N_0$, $r\in N_0$ let

$$\pi_n(r)=\pi_n(r,\varkappa) \text{ the largest (real) zero of } P_n(z,r,\varkappa)$$

if

$$P_n(z,r,\varkappa) \text{ has a real zero}$$

and

$$\pi_n(r)=\pi_n(r,\varkappa)=-\infty \quad \text{if } P_n(z,r,\varkappa) \text{ has no real zero.}$$

Then we have

LEMMA 5.

(3.10) $\pi_{n+1}(0,\varkappa)>\pi_n(0,\varkappa)$ *for* $n\geqslant 0$,

(3.11) $\pi_{n+1}(1,\varkappa)>\pi_n(1,\varkappa)$ *for* $n\geqslant 2$,

(3.12) $\pi_{n+1}(2,\varkappa)>\pi_n(2,\varkappa)$ *for* $n\geqslant 2$,

(3.13) $\pi_{n+1}(3,\varkappa)\geqslant\pi_n(3,\varkappa)$ *for* $n\geqslant 2$,

where equality holds true iff $n=3$ *and* $2\varkappa\in N$.

Proof. Obviously, by definition,

$$\pi_0(r)=-\infty,\qquad \pi_1(r)=0,$$

and, by (3.4),

$$P_{n+1}(\pi_n(r,\varkappa),r,\varkappa)=-\varkappa\sum_{\mu=0}^{n-1}\frac{(\alpha_\varkappa-r+\mu+1)_{n-\mu}}{(n+1-\mu)!}P_\mu(\pi_n(r,\varkappa),r,\varkappa) \tag{3.14}$$

for $n\in N$ if $\pi_n(r,\varkappa)\in R$.

If $r=0$, by induction on n, using (3.14), we have

$$P_{n+1}(\pi_n(0,\varkappa),0,\varkappa)<0 \quad \text{for } n\geqslant 1. \tag{3.15}$$

Since

$$P_{n+1}(z,r,k)\to\infty \quad \text{for } z\to\infty \tag{3.16}$$

(3.10) follows from (3.15) and (3.16).

If $r=1$, we have

$$\pi_1(1,\varkappa)\leqslant\pi_2(1,\varkappa)<\sqrt{\varkappa/2}\leqslant\pi_3(1,\varkappa) \tag{3.17}$$

and, again by induction on n using (3.14)

$$P_{n+1}(\pi_n(1,\varkappa),1,\varkappa)<0 \quad \text{for } n\geqslant 3.$$

This together with (3.16) and (3.17) proves (3.11).

If $r=2$ we have

$$-\infty=\pi_2(2,\varkappa)<\pi_3(2,\varkappa)<\sqrt{\varkappa/2}\leqslant\pi_4(2,\varkappa). \tag{3.18}$$

Since

$$\frac{(\alpha_\varkappa-1)_n}{(n+1)!}+\pi_n(2,\varkappa)\frac{(\alpha_\varkappa)_{n-1}}{n!}\geqslant 0, \quad \text{if } \pi_n(2,\varkappa)>\frac{1}{n+1},$$

we have again by induction on n

$$P_{n+1}(\pi_n(2,\varkappa),2,\varkappa)<0 \quad \text{for } n\geqslant 4. \tag{3.19}$$

This together with (3.16) and (3.18) proves (3.12).

If $r=3$ we have

$$-\infty=\pi_2(3,\varkappa)<\pi_3(3,\varkappa) \tag{3.20}$$

and for $n\geqslant 3$

$$\frac{(\alpha_\varkappa)_{n-2}}{(n-1)!}\left(\pi_n^2(3,\varkappa)+\frac{\varkappa}{2}(2-\alpha_\varkappa)\right)+\frac{(\alpha_\varkappa-1)_{n-1}}{n!}\pi_n(3,\varkappa)+\frac{(\alpha_\varkappa-2)_n}{(n+1)!}\geqslant 0$$

where equality holds true iff $\alpha_\varkappa=0$. Now again by induction on n, using (3.14), one proves

$$P_{n+1}(\pi_n(3,\varkappa),3,\varkappa)<0 \quad \text{for } n\geqslant 3 \text{ if } \alpha_\varkappa\neq 0,$$

$$P_{n+1}(\pi_n(3,\varkappa),3,\varkappa)<0 \quad \text{for } n\geqslant 5 \text{ if } \alpha_\varkappa=0.$$

This together with (3.16), (3.20) and $\pi_3(3,\varkappa)=\pi_4(3,\varkappa)<\pi_5(3,\varkappa)$ – if $2\varkappa\in N$ – proves (3.13).

THEOREM 2. *Let* $r \in \{0, 2\}$, $R = [2\varkappa]+r-1$ *and* $n \in N$. *If*

$$\pi_n(r, \varkappa) \geqslant (R-2\varkappa)/3\,(^{11})$$

and

$$P_n(s_0, r, \varkappa) < 0 \quad \textit{for some } s_0,$$

then

$$z_\varkappa^{(R-n)} > s_0+\varkappa.$$

Proof. It is sufficient to prove

$$\pi_n(0, \varkappa)+\varkappa \leqslant z_\varkappa^{([2\varkappa]-1-n)} \tag{3.21}$$

and

$$\pi_n(2, \varkappa)+\varkappa \leqslant z_\varkappa^{([2\varkappa]+1-n)}. \tag{3.22}$$

First we apply Theorem 1 with $r = 0$, $s = \pi_n(0, \varkappa)+\varkappa$. By (3.10) we have

$$P_\mu(\pi_n(0, \varkappa), 0, \varkappa) > 0 \quad \text{for } 1 \leqslant \mu < n.$$

Hence, by (3.7) and (3.15),

$$q^{([2\varkappa]-1-n)}(\pi_n(0, \varkappa)+\varkappa) \leqslant 0.$$

This together with (2.12) proves (3.21).

Next we apply Theorem 1 with $r = 2$, $s = \pi_n(2, \varkappa)+\varkappa$. By (3.12) and $\pi_n(2, \varkappa) \geqslant (R-2\varkappa)/3$ we have

$$P_\mu(\pi_n(2, \varkappa), 2, \varkappa) > 0 \quad \text{for } 1 \leqslant \mu < n.$$

Using (3.19) – note that (3.19) holds also true if $n = 3$ – (3.7) gives

$$q^{([2\varkappa]+1-n)}(\pi_n(2, \varkappa)+\varkappa) \leqslant 0.$$

This together with (2.12) completes the proof of (3.22).

THEOREM 3. *Let* $r \in \{1, 3\}$, $R = [2\varkappa]+r-1$ *and* $n \in N$. *If*

$$\pi_n(r, \varkappa) \geqslant (R-2\varkappa)/3\,(^{12})$$

and

$$P_n(s_1, r, \varkappa) > 0 \quad \textit{for all } s \geqslant s_1$$

(11) If $r = 0$ this condition is always satisfied by (3.10). If $r = 2$ this condition implies that $n \geqslant 3$; moreover it is always satisfied if $n \geqslant 4$, by (3.12) and (3.18).

(12) If $r = 1$ this condition is always satisfied. If $r = 3$ this condition is never satisfied for $n \leqslant 3$ and is always satisfied for $n \geqslant 5$, by (3.13) and $\pi_5(3, \varkappa) \geqslant \sqrt{\varkappa/2}$.

then

$$z_\varkappa^{(R-n)} < s_1+\varkappa.\,(^{13})$$

Proof. It is sufficient to prove

$$z_\varkappa^{([2\varkappa]-n)} \leqslant \pi_n(1, \varkappa)+\varkappa \quad \text{and} \quad z_\varkappa^{([2\varkappa]-n+2)} \leqslant \pi_n(3, \varkappa)+\varkappa.$$

Suppose first that

$$z_\varkappa^{([2\varkappa]-n)} > \pi_n(1, \varkappa)+\varkappa \tag{3.23}$$

holds true. By (3.11) we have

$$P_\nu(z_\varkappa^{([2\varkappa]-n)}-\varkappa, 1, \varkappa) > 0 \quad \text{for } 1 \leqslant \nu \leqslant n. \tag{3.24}$$

Applying Theorem 1 with $s = z_\varkappa^{([2\varkappa]-n)}$ $(> \varkappa)$ – cf. (3.23) – gives

$$P_n(z_\varkappa^{([2\varkappa]-n)}-\varkappa, 1, \varkappa)q^{([2\varkappa]-1-n)}(z_\varkappa^{([2\varkappa]-n)}) \geqslant 0$$

and by (3.24)

$$q^{([2\varkappa]-n-1)}(z_\varkappa^{([2\varkappa]-n)}) \geqslant 0.$$

But this is impossible by (2.14).

Suppose now that

$$z_\varkappa^{([2\varkappa]-n+2)} > \pi_n(3, \varkappa)+\varkappa$$

holds true. By (3.13) we have

$$P_\nu(z_\varkappa^{([2\varkappa]-n+2)}-\varkappa, 3, \varkappa) > 0 \quad \text{for } 1 \leqslant \nu \leqslant n. \tag{3.25}$$

Applying now Theorem 1 with $s = z_\varkappa^{([2\varkappa]-n+2)}$ gives, using (3.25),

$$q^{([2\varkappa]-n+1)}(z_\varkappa^{([2\varkappa]-n+2)}) \geqslant 0.$$

But this is impossible by (2.14).

Remarks. (i) If $2\varkappa \in N$ one proves easily (see (3.3) and (3.4)) that

$$P_{m+r}(z, r, \varkappa) = P_r(z, r, \varkappa)\frac{m!}{(2\varkappa-1)!}q^{(2\varkappa-1-m)}(z+\varkappa) \quad \text{for } m \geqslant 0,\ r \geqslant 0.$$

Especially, by Lemma 5,

$$z_\varkappa^{(0)} = \pi_R(r, \varkappa)+\varkappa \quad \text{for } \varkappa \geqslant 1,\ 0 \leqslant r \leqslant 2, \tag{3.26}$$

$$z_\varkappa^{(0)} = \pi_{2\varkappa+2}(3, \varkappa)+\varkappa \quad \text{for } \varkappa \geqslant 3/2. \tag{3.27}$$

Since it is known by Iwaniec ([7]) that

$$\lim_{\varkappa\to\infty} z_\varkappa^{(0)}/\varkappa = c,$$

(13) If $2\varkappa \in N$, $n = 1$, $r = 1$ we put $z_\varkappa^{(2\varkappa-1)} = -\infty$.

where c $(=3.59\ldots)$ is the unique solution of

$$c\log c - c = 1, \tag{3.28}$$

we have, by (3.26) and (3.27),

$$\lim_{\substack{\varkappa\to\infty\\ 2\varkappa\in N}} \pi_R(r,\varkappa)/\varkappa = c-1 \quad \text{for } 0\leqslant r\leqslant 3.$$

(ii) Choosing $R=n\geqslant \frac{3}{2}r+1$ in Theorem 2 gives for real $\varkappa\geqslant 1$

$$z_\varkappa^{(0)} \geqslant \pi_R(r,\varkappa)+\varkappa \quad \text{for } r\in\{0,2\}$$

and choosing $R=n\geqslant 2r-1$ in Theorem 3 gives

$$z_\varkappa^{(0)} \leqslant \pi_R(r,\varkappa)+\varkappa \quad \text{for } r\in\{1,3\}.$$

(iii) It is easy to prove (cf. (3.4); [13], pp. 64, 65) that for $1\leqslant r\leqslant 3$, $\varkappa\geqslant 1$

$$\pi_{[2\varkappa]+r-1}(r,\varkappa)\to\pi_{[2\varkappa]+r}\left(r,\frac{[2\varkappa]+1}{2}\right) \quad \text{for } \varkappa\to\frac{[2\varkappa]+1}{2}-.$$

Hence, by (3.26), (3.27),

$$\pi_{[2\varkappa]+r-1}(r,\varkappa)\to z^{(0)}_{([2\varkappa]+1)/2}-\frac{[2\varkappa]+1}{2} \quad \text{for } \varkappa\to\frac{[2\varkappa]+1}{2}-.$$

(iv) For $n=2$, $r=0$ Theorem 2 gives

$$z_\varkappa^{([2\varkappa]-3)} \geqslant \pi_2(0,\varkappa)+\varkappa = \varkappa+\sqrt{\varkappa(1+\alpha_\varkappa)/2}.$$

Hence for $1.5\leqslant\varkappa<2$

$$z_\varkappa^{(0)} \geqslant \varkappa+\sqrt{\varkappa(\varkappa-1)}$$

(see also [7]).

For $n=2$, $r=1$ Theorem 3 gives

$$z_\varkappa^{([2\varkappa]-2)} \leqslant \varkappa+\pi_2(1,\varkappa) = \varkappa+\sqrt{\varkappa\alpha_\varkappa/2},$$

especially

$$z_\varkappa^{(0)} \leqslant \varkappa+\sqrt{\varkappa(\varkappa-1)} \quad \text{for } 1\leqslant\varkappa<1.5 \tag{3.29}$$

(see also [7]).

We combine the results of Theorem 3 and Lemma 3 in

LEMMA 6. *Let* $R=[2\varkappa]+r-1$, $r\in\{1,3\}$, $m\leqslant[2\varkappa]-r$,

$$S_\varkappa(m,r)=\begin{cases}\pi_{R-m}(r,\varkappa)+\varkappa+m & \text{for } 0\leqslant m\leqslant[\varkappa],\\ \pi_{R-m}(r,\varkappa)+\varkappa+[\varkappa]+\sum\limits_{\nu=[\varkappa]+1}^{m}\dfrac{\varkappa}{2\varkappa-\nu} & \text{for } [\varkappa]<m\leqslant\varkappa_0.\end{cases}$$

Then

$$z_\varkappa^{(0)} \leqslant \min_r \min_{m\geqslant 0} S_\varkappa(m,r).$$

Proof. We have from Lemma 3

$$z_\varkappa^{(0)} \leqslant \begin{cases} z_\varkappa^{(m)}+m & \text{for } 0\leqslant m\leqslant[\varkappa],\\ z_\varkappa^{(m)}+[\varkappa]+\sum\limits_{\nu=[\varkappa]+1}^{m}\dfrac{\varkappa}{2\varkappa-\nu} & \text{for } [\varkappa]<m\leqslant\varkappa_0\end{cases}$$

and from Theorem 3 for $r\in\{1,3\}$, $m\leqslant[2\varkappa]-r$

$$z_\varkappa^{(m)} \leqslant \pi_{R-m}(r,\varkappa)+\varkappa.$$

Similarly we have from Lemma 3 and Theorem 2

LEMMA 7. *Let* $R=[2\varkappa]+r-1$, $r\in\{0,2\}$, $m\leqslant[2\varkappa]-2-r/2$,

$$\hat{S}_\varkappa(m,r)=\begin{cases}\pi_{R-m}(r,\varkappa)+\varkappa+\sum\limits_{\nu=1}^{m}\dfrac{\varkappa}{2\varkappa-\nu} & \text{for } 0\leqslant m\leqslant[\varkappa],\\ \pi_{R-m}(r,\varkappa)+m-[\varkappa]+\varkappa+\sum\limits_{\nu=1}^{[\varkappa]}\dfrac{\varkappa}{2\varkappa-\nu} & \text{for } [\varkappa]<m\leqslant\varkappa_0.\end{cases}$$

Then

$$z_\varkappa^{(0)} \geqslant \max_r \max_{m\geqslant 0} \hat{S}_\varkappa(m,r).$$

Moreover, it is possible to give another lower bound for $z_\varkappa^{(\nu)}$. Define for $m\leqslant n$, $\varkappa\geqslant 1$ a sequence of polynomials in z (real) as follows:

$$\begin{aligned} T_0(z) &= T_{0,n}(z,\varkappa)=1,\\ T_1(z) &= T_{1,n}(z,\varkappa)=z,\\ T_{m+1}(z) &= T_{m+1,n}(z,\varkappa)\\ &= zT_{m,n}(z,\varkappa)-\varkappa\sum_{\mu=0}^{m-1}\frac{(\alpha_\varkappa+n+1-m)_{m-\mu}}{(m+1-\mu)!}T_{\mu,n}(z,\varkappa)\\ &\qquad \text{for } 1\leqslant m\leqslant n.\end{aligned} \tag{3.30}$$

Let

$$\tau_m=\tau_{m,n}(\varkappa),\ 1\leqslant m\leqslant n+1, \text{ the largest zero of } T_{m,n}(z,\varkappa).$$

Then (cf. proof of Lemma 5)

$$0=\tau_1<\tau_2<\ldots<\tau_{n+1}$$

and

$$T_m(z)>0 \quad \text{for } z>\tau_m,\ m\in N_0.\text{(}^{14}\text{)} \tag{3.31}$$

(14) $\tau_0=-\infty$.

LEMMA 8. *Let* $n \in N_0$, $0 \leqslant m \leqslant n$. *If*

(3.32) $$z \geqslant \pi_{m-1}(0, \varkappa)\,(^{15})$$

and

(3.33) $$z \geqslant \tau_{n-m,n}(\varkappa),$$

then

(3.34) $$P_{n+1}(z, 0, \varkappa) \leqslant T_{n+1-m,n}(z, \varkappa) P_m(z, 0, \varkappa).$$

Proof. By (3.10) and (3.4) with $r = 0$ we have

(3.35) $$P_{n+1}(z) \leqslant zP_n(z) - \varkappa \sum_{\mu=m}^{n-1} \frac{(\alpha_\varkappa+1+\mu)_{n-\mu}}{(n+1-\mu)!} P_\mu(z) \quad \text{for } z \geqslant \pi_{m-1}(0).$$

Furthermore we have, if $z \geqslant \tau_{n-m}$ and $z \geqslant \pi_{m-1}(0)$,

(3.36) $$P_{n+1}(z) \leqslant T_{q+1}(z) P_{n-q}(z) - \varkappa \sum_{\mu=m}^{n-q-1} P_\mu(z) \sum_{\lambda=0}^{q} T_\lambda(z) \frac{(\alpha_\varkappa+1+\mu)_{n-\lambda-\mu}}{(n+1-\lambda-\mu)!}$$

for $0 \leqslant q \leqslant n-m$, which can be seen as follows:

For $q = 0$ (3.36) is (3.35). Now suppose (3.36) holds true for some q with $0 \leqslant q \leqslant n-m-1$. Then, by (3.35),

$$P_{n-q}(z) \leqslant zP_{n-q-1}(z) - \varkappa \sum_{\mu=m}^{n-q-2} \frac{(\alpha_\varkappa+1+\mu)_{n-q-1-\mu}}{(n-q-\mu)!} P_\mu(z)$$

and substituting this into (3.36) gives, if $z \geqslant \tau_{q+1}$,

$$\begin{aligned} P_{n+1}(z) &\leqslant zT_{q+1}(z) P_{n-1-q}(z) \\ &\quad - \varkappa P_{n-q-1}(z) \sum_{\lambda=0}^{q} T_\lambda(z) \frac{(\alpha_\varkappa+n-q)_{q-\lambda+1}}{(q+2-\lambda)!} \\ &\quad - \varkappa \sum_{\mu=m}^{n-q-2} P_\mu(z) \sum_{\lambda=0}^{q} T_\lambda(z) \frac{(\alpha_\varkappa+1+\mu)_{n-\lambda-\mu}}{(n+1-\lambda-\mu)!} \\ &\quad - \varkappa \sum_{\mu=m}^{n-q-2} P_\mu(z) T_{q+1}(z) \frac{(\alpha_\varkappa+1+\mu)_{n-q-1-\mu}}{(n-q-\mu)!} \\ &= P_{n-1-q}(z) \left(zT_{q+1}(z) - \varkappa \sum_{\lambda=0}^{q} T_\lambda(z) \frac{(\alpha_\varkappa+n-q)_{q-\lambda+1}}{(q+2-\lambda)!} \right) \\ &\quad - \varkappa \sum_{\mu=m}^{n-2-q} P_\mu(z) \sum_{\lambda=0}^{q+1} T_\lambda(z) \frac{(\alpha_\varkappa+1+\mu)_{n-\lambda-\mu}}{(n+1-\lambda-\mu)!}. \end{aligned}$$

Hence, by (3.30), (3.36) is proved. (3.36) with $q = n-m$ is (3.34).

(15) $\pi_{-1}(0, \varkappa) = -\infty$.

THEOREM 4. *Let* $1 \leqslant p \leqslant n+1$. *If*

$$T_{p,n}(s_2, \varkappa) < 0 \quad \textit{for some } s_2 > 0$$

then

$$z_\varkappa^{([2\varkappa]-2-n)} > s_2 + \varkappa.$$

Proof. Choosing $m = n$ in Lemma 8 gives

$$P_{n+1}(z) \leqslant T_1(z) P_n(z) \quad \text{if } z \geqslant \max(\pi_{n-1}(0), 0).$$

Hence, by (3.10),

$$T_1(z) > 0 \quad \text{for all } z > \pi_{n+1}(0),$$

especially, by (3.31),

$$\tau_1 \leqslant \pi_{n+1}(0).$$

Now suppose that we have already proved

$$P_{n+1}(z) \leqslant T_p(z) P_{n-p+1}(z) \quad \text{for } z \geqslant \pi_{n+1}(0) \geqslant \tau_p$$

for some $1 \leqslant p \leqslant n$. We now apply Lemma 8 with $m = n-p$. Then we obtain

(3.37) $$P_{n+1}(z) \leqslant T_{p+1}(z) P_{n-p}(z) \quad \text{for } z \geqslant \pi_{n+1}(0) \geqslant \tau_p.$$

Note that (3.32) and (3.33) are satisfied. (3.37) and (3.10) give

$$T_{p+1}(z) > 0 \quad \text{for } z > \pi_{n+1}(0).$$

Hence, by (3.31),

$$\tau_{p+1} \leqslant \pi_{n+1}(0)$$

and therefore

$$P_{n+1}(z) \leqslant T_p(z) P_{n-p+1}(z) \quad \text{for } z \geqslant \pi_{n+1}(0) \geqslant \tau_p,\ 1 \leqslant p \leqslant n+1.$$

If now

$$T_p(s_2) < 0 \quad \text{for some } s_2 > 0$$

then

$$s_2 < \pi_{n+1}(0)$$

and, by (3.21),

$$z_\varkappa^{([2\varkappa]-2-n)} > s_2 + \varkappa.$$

Remarks. (i) $T_{2,n}(z, \varkappa) = z^2 - (\alpha_\varkappa + n)\varkappa/2$, $n \geqslant 1$. We take $n = [2\varkappa]-2$ in Theorem 4. Then

$$z_\varkappa^{(0)} \geqslant \varkappa + \sqrt{\varkappa(\varkappa-1)} \quad \text{for } \varkappa \geqslant 3/2$$

(cf. [7]).

(ii) $T_{3,n}(z,\varkappa) = z^3 - \frac{\varkappa}{2}(2\alpha_\varkappa+2n-1)z - \frac{\varkappa}{6}(\alpha_\varkappa+n-1)(\alpha_\varkappa+n)$, $n \geqslant 2$. We take $n = [2\varkappa]-2$. Obviously $T_{3,[2\varkappa]-2}(\varkappa,\varkappa) < 0$ for $\varkappa \geqslant 2.24$. Hence, by Theorem 4

$$z_\varkappa^{(0)} > 2\varkappa \quad \text{for } \varkappa \geqslant 2.24. \tag{3.38}$$

Indeed, using Theorem 2 with $r = 2$, one can prove (3.38) for $2.17 \leqslant \varkappa \leqslant 2.24$, and using Theorem 3, one can prove

$$z_\varkappa^{(0)} < 2\varkappa \quad \text{for } \varkappa \leqslant 2.15.$$

4. Some explicit results. We define

$$Q_\varkappa(s) = Q(s) = \frac{sq(s)}{\sigma(s)} - \varkappa\int_{s-1}^{s}\frac{q(t+1)}{\sigma(t)}dt \quad \text{for } s > 1 \text{ and } \varkappa > 1, \tag{4.1}$$

where $q(s)$ is defined in (2.1) and the continuous function $\sigma(s) = \sigma_\varkappa(s)$ satisfies

$$\begin{aligned} \sigma(s) &= \frac{1}{2^\varkappa e^{\gamma\varkappa}\Gamma(\varkappa+1)}s^\varkappa && \text{for } 0 \leqslant s \leqslant 2,\\ (s^{-\varkappa}\sigma(s))' &= -\varkappa s^{-\varkappa-1}\sigma(s-2) && \text{for } s > 2.\ [16] \end{aligned} \tag{4.2}$$

We denote

$$\zeta_\varkappa = \zeta \quad \text{the largest zero of } Q_\varkappa(s) \text{ for } \varkappa > 1, \tag{4.3}$$

if $\zeta_\varkappa$ exists.

The following two theorems are needed in [4].

THEOREM 5. *$Q_\varkappa(s)$ has a unique zero in*

$$(\max\{2,\, z_\varkappa^{(0)}+1/2\},\, z_\varkappa^{(0)}+1).$$

This zero is $\zeta_\varkappa$. Moreover, we have $\zeta_\varkappa > v(\varkappa)$, where

$$v(\varkappa) = \begin{cases} 3\varkappa-1.4 & \text{for } 2.4 \leqslant \varkappa,\\ 3\varkappa-1.45 & \text{for } 1.5 \leqslant \varkappa < 2.4,\\ 3\varkappa-1.4 & \text{for } 1.44 \leqslant \varkappa < 1.5,\\ 2+9(\varkappa-1)/4 & \text{for } 1.05 \leqslant \varkappa < 1.44,\\ 2+2.48(\varkappa-1) & \text{for } 1 < \varkappa < 1.05. \end{cases} \tag{4.4}$$

THEOREM 6. *We have*

$$z_\varkappa^{(0)} \leqslant \begin{cases} 2 & \text{for } 1 \leqslant \varkappa \leqslant 1.34,\\ 2.6\varkappa-1.41 & \text{for } 1.34 \leqslant \varkappa \leqslant 1.85. \end{cases} \tag{4.5}$$

Remark. Using the results of Sections 2 and 3 it is possible to prove sharper bounds than those given in (4.4) and (4.5) (cf. proofs of Theorem 4 and Theorem 5). However, the results of (4.4) and (4.5) are sufficient in [4].

[16] Defining $\sigma(s) = 0$ for $s < 0$, this holds even true for $s > 0$.

Proof of Theorem 5. Since $\sigma(s)$ is strictly increasing in s (cf. [2], [6]) we have

$$Q(s) < \frac{1}{\sigma(s)}\Big(sq(s) - \varkappa\int_{s-1}^{s} q(t+1)dt\Big) = \frac{q^{(-1)}(s)}{\sigma(s)} \quad \text{for } s > z_\varkappa^{(0)}. \tag{4.6}$$

(Note that $z_\varkappa^{(0)} > \varkappa$ for $\varkappa > 1$ — see [7], 5.3.) From (2.4) we have

$$(s^{\nu-\varkappa+1}q^{(\nu)}(s))' = \varkappa s^{\nu-\varkappa}q^{(\nu)}(s+1). \tag{4.7}$$

Hence, by (4.1), (4.2) and (4.7) with $\nu = 0$, we have

$$Q'(s) = \varkappa q(s)\Big(\frac{\sigma(s-2)}{\sigma^2(s)} + \frac{1}{\sigma(s-1)}\Big) > 0 \quad \text{for } s > z_\varkappa^{(0)} \tag{4.8}$$

and, by (2.12) and $\sigma(s) \to 1$ for $s \to \infty$ (cf. [2], [6]),

$$Q(s) \to \infty \quad \text{for } s \to \infty. \tag{4.9}$$

Hence, by (4.6), (4.9) and (2.16) we obtain

$$\zeta_\varkappa > z_\varkappa^{(-1)} > z_\varkappa^{(0)} + 1/2. \tag{4.10}$$

For the proof of $1+z_\varkappa^{(0)} > \zeta_\varkappa$ we integrate (4.1) by parts, using (4.2) and (4.7). This gives

$$Q(s) = \frac{(s-1)q(s-1)}{\sigma(s-1)} + \varkappa\int_{s-1}^{s}\frac{q(t)\sigma(t-2)}{\sigma^2(t)}dt.$$

Hence $Q_\varkappa(z_\varkappa^{(0)}+1) > 0$, and, by (4.8),

$$\zeta_\varkappa < z_\varkappa^{(0)} + 1.$$

In order to complete the proof of Theorem 5, it is now sufficient to prove (4.4).

For $\varkappa \geqslant 1.44$ we shall prove Theorem 5 by applying Theorem 2 and Theorem 4.

(I). $\varkappa \geqslant 1.44$. We first apply Theorem 4 with $p = 5$, $n = [2\varkappa]-1$ and $\varkappa \geqslant 2.5$. It is easily checked that $T_{5,[2\varkappa]-1}(2\varkappa-1.39,\varkappa) < 0$ for $\varkappa \geqslant 2.5$. Hence, by Theorem 4,

$$z_\varkappa^{(-1)} > 3\varkappa - 1.39 \quad \text{for } \varkappa \geqslant 2.5. \tag{4.11}$$

If $2 \leqslant \varkappa < 2.5$ we apply Theorem 2 with $r = 2$ and $n = 6$. Since $P_6(2\varkappa-1.4, 2, \varkappa) < 0$ for $2.4 \leqslant \varkappa < 2.5$ and $P_6(2\varkappa-1.42, 2, \varkappa) < 0$ for $2 \leqslant \varkappa < 2.4$ we have

$$z_\varkappa^{(-1)} > 3\varkappa - 1.4 \quad \text{for } 2.4 \leqslant \varkappa < 2.5 \tag{4.12}$$

and

$$z_\varkappa^{(-1)} > 3\varkappa - 1.42 \quad \text{for } 2 \leqslant \varkappa < 2.4. \tag{4.13}$$

Next we use Theorem 2 with $r = 2$ and $n = 5$. This yields

(4.14) $$z_\varkappa^{(-1)} > 3\varkappa - 1.43 \quad \text{for } 1.5 \leqslant \varkappa < 2.$$

Finally we use Theorem 2 with $r = 2$ and $n = 4$ and obtain

(4.15) $$z_\varkappa^{(-1)} > 3\varkappa - 1.4 \quad \text{for } 1.44 \leqslant \varkappa < 1.5.$$

Hence, by (4.10) through (4.15) we have proved (4.4) for $\varkappa \geqslant 1.44$.

II. $1 < \varkappa < 1.44$. In this case we can no longer use (4.10), because we lose too much in this inequality.

We define $\varphi = \varphi_\varkappa$ by

(4.16) $$\sigma(s) = \frac{s^\varkappa}{2^\varkappa e^{\gamma\varkappa}\Gamma(\varkappa+1)}\varphi(s) \quad \text{for } s \geqslant 0.$$

Since $s^{-\varkappa}\sigma(s)$ is decreasing (cf. (4.2)) and positive for $s > 0$ (cf. [6]) we have

$$0 < \varphi_\varkappa(s) \leqslant 1.$$

Hence

$$Q_\varkappa(s) \leqslant (2e^\gamma)^\varkappa \Gamma(\varkappa+1)\left(\frac{sq(s)}{s^\varkappa \varphi(s)} - \varkappa \int_{s-1}^{s} \frac{q(t+1)}{t^\varkappa}dt\right) \quad \text{for } s > z_\varkappa^{(0)}$$

and, by (4.7) with $\nu = 0$ finally

$$Q_\varkappa(s) \leqslant (2e^\gamma)^\varkappa \Gamma(\varkappa+1)\hat{q}_\varkappa(s) \quad \text{for } s > z_\varkappa^{(0)},$$

where

$$\hat{q}_\varkappa(s) = \hat{q}(s) = s^{1-\varkappa}q(s)\left(\frac{1}{\varphi(s)} - 1\right) + (s-1)^{1-\varkappa}q(s-1).$$

We shall prove

(4.17) $$\hat{q}(2+2.25(\varkappa-1)) < 0 \quad \text{for } 1.05 < \varkappa \leqslant 1.44$$

and

(4.18) $$\hat{q}(2+2.48(\varkappa-1)) < 0 \quad \text{for } 1 < \varkappa \leqslant 1.05.$$

(Note that this is sufficient to complete the proof of Theorem 5.)

The proofs of (4.17) and (4.18) are based on the following two lemmata (for details see [5]).

LEMMA 9. *We have*

$$q(s) < s^{2\varkappa-1} - \varkappa(2\varkappa-1)s^{2\varkappa-2} + \tfrac{1}{2}(\varkappa-1)(2\varkappa-1)^2\varkappa^\varkappa s^{2\varkappa-3}$$
$$\textit{for } s > 0,\ 1 < \varkappa \leqslant 3/2.$$

Proof. We have (cf. [8])

$$q(s) = s^{2\varkappa-1} - \varkappa(2\varkappa-1)s^{2\varkappa-2} + \tfrac{1}{2}(\varkappa-1)(2\varkappa-1)^2\varkappa^\varkappa s^{2\varkappa-3} +$$
$$+ \frac{1}{\Gamma(1-2\varkappa)}\int_0^\infty e^{-sz}H_\varkappa(z)\frac{dz}{z^{2\varkappa}} \quad \text{for } \varkappa < \frac{3}{2},$$

where

$$H_\varkappa(z) = e^{\varkappa g(z)} - 1 - \varkappa z - \tfrac{1}{4}(2\varkappa-1)\varkappa^\varkappa z^2 \quad \text{for } z \geqslant 0,$$
$$g(z) = \int_0^z \frac{1-e^{-t}}{t}dt \quad \text{for } z \geqslant 0.$$

Since $H_\varkappa(0) = H'_\varkappa(0) = 0$ it is sufficient for (5.1) to prove

$$\frac{1}{\varkappa}H''_\varkappa(z) = \exp(\varkappa g(z) - z)\cdot a_\varkappa(z)/z^2 - d_\varkappa \leqslant 0 \quad \text{for } z \geqslant 0,$$

where

$$a_\varkappa(z) = z + 1 - 2\varkappa + (\varkappa-1)e^z + \varkappa e^{-z}, \qquad d_\varkappa = \frac{2\varkappa-1}{2}\varkappa^{\varkappa-1},$$

and this is easily done.

LEMMA 10. *Let* $2 \leqslant s \leqslant 4$. *Then*

(4.19) $$\varphi(s) \geqslant 1 - \varkappa\left(\frac{s-2}{s}\right)^{\varkappa+1}\left(\frac{1}{\varkappa+1} + \frac{(s-2)}{(\varkappa+2)s} + \frac{(s-2)^2}{(\varkappa+3)s^2} + \frac{(s-2)^3}{s^3}\right).$$

Proof. By (4.2) and (4.16) we have

$$\varphi(s) = 1 - \varkappa\int_2^s \left((t-2)^\varkappa/t^{\varkappa+1}\right)dt \quad \text{for } 2 \leqslant s \leqslant 4.$$

Partial integration and the monotonicity of $(t-2)^{\varkappa+3}/t^{\varkappa+4}$ give (4.19).

Proof of Theorem 6. For $\varkappa < \frac{4}{3}$ $z_\varkappa^{(0)} < 2$ follows from $z_\varkappa^{(0)} < \varkappa + \sqrt{\varkappa(\varkappa-1)}$ (cf. [6]).

If $\frac{4}{3} \leqslant \varkappa < 1.5$ we apply Theorem 3 with $r = 3$ and $n = 4$. (Note that $\pi_4(3, \varkappa) \geqslant \frac{4}{3}$ in this range.) We have

$$P_4(z, 3, \varkappa) > 0 \quad \text{for } z \geqslant 2 - \varkappa,\ \tfrac{4}{3} \leqslant \varkappa < 1.34$$

and

$$P_4(z, 3, \varkappa) > 0 \quad \text{for } z \geqslant 1.6\varkappa - 1.41,\ 1.34 \leqslant \varkappa < 1.5.$$

This proves Theorem 6 if $\varkappa < 1.5$.

For $1.5 \leqslant \varkappa \leqslant 1.85$ we apply Theorem 3 with $r = 3$ and $n = 4$. It is easily checked, that

$$P_5(z, 3, \varkappa) > 0 \quad \text{for } z \geqslant 1.6\varkappa - 1.41,\ 1.5 \leqslant \varkappa < 1.85$$

and this completes the proof.

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Equidistribution of Frobenius classes and the volumes of tubes

by

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1. Let G be a compact Lie group that fits in an exact sequence

$$1 \to \mathcal{T} \to G \xrightarrow{j} H \to 1, \tag{1}$$

where $\mathcal{T}$ is an n-dimensional real torus and H is a finite group. Given a countable index set $\mathcal{P}$ and a set of conjugacy classes $\{\sigma_p |\ p \in \mathcal{P}\}$ in G, we are interested in the following equidistribution problem. Let

$$|\cdot|\colon \mathcal{P} \to \boldsymbol{R}_+$$

be a map satisfying the asymptotic formula (8) below and let $\mathcal{A} \subseteq G$. For each x in $\boldsymbol{R}_+$, let

$$\mathcal{N}(\mathcal{A}, x) = \operatorname{card}\{p|\ p \in \mathcal{P},\ \sigma_p \cap \mathcal{A} \neq \emptyset,\ |p| < x\}.$$

One studies the asymptotics of $\mathcal{N}(\mathcal{A}, x)$ as $x \to \infty$. Without loss of generality we can assume that $\mathcal{A}$ is invariant under conjugation, i.e.

$$\tau^{-1}\mathcal{A}\tau = \mathcal{A} \quad \text{for } \tau \in G, \tag{2}$$

so that

$$\mathcal{N}(\mathcal{A}, x) = \operatorname{card}\{p|\ p \in \mathcal{P},\ \sigma_p \subseteq \mathcal{A},\ |p| \leqslant x\}. \tag{3}$$

The manifold G inherits the natural Riemannian structure from $\mathcal{T}$. Let μ be the Haar measure on G normalized by the condition $\mu(G) = 1$, and suppose that $\mathcal{A}$ satisfies the following condition:

$$\mu(\mathcal{U}_\delta(\partial\mathcal{A})) = O(C(\mathcal{A})\delta^\alpha) \quad \text{with } \alpha > 0, \tag{4}$$

where $\partial\mathcal{A}$ denotes the boundary of $\mathcal{A}$ and where $\mathcal{U}_\delta(\mathcal{A})$ denotes the δ-neighbourhood of $\mathcal{A}$, i.e. the subset

$$\{x|\ x \in G,\ \varrho(x, \mathcal{A}) < \delta\}; \tag{5}$$