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On some expansions of p-adic functions

by

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Introduction. Throughout this note, $\mathbf{Z}_p$, $\mathbf{Q}_p$, Ω_p and $C(\mathbf{Z}_p, \Omega_p)$ denote the ring of p-adic integers, the field of p-adic numbers, the completion of the algebraic closure of $\mathbf{Q}_p$ and the algebra of continuous functions $f: \mathbf{Z}_p \to \Omega_p$ respectively (p prime). We also put $N_0 = N \cup \{0\}$ and $E = \{0, 1, \ldots, p-1\}$.

For an arbitrary prime number p, we define a system $(\varphi_m)_{m\in N_0}$ of elements of $C(\mathbf{Z}_p, \Omega_p)$. The definition is similar to the known definition of Walsh ($p = 2$) and Chrestenson ($p > 2$) systems in real analysis (see [5] and [1]). We investigate the main properties of (φ_m) and consider some questions concerning the expansions of functions belonging to $C(\mathbf{Z}_p, \Omega_p)$ with respect to (φ_m). Finally, we give some examples of such expansions.

2. Definition and main properties of the system $(\varphi_m)_{m\in N_0}$. Let p be an arbitrary fixed prime number. We denote by ζ an arbitrary fixed primitive pth root of 1 in Ω_p. If m is an arbitrary nonnegative integer and $m = c_0 + c_1 p + \ldots + c_s p^s$ where $c_j \in E$ for $j = 0, 1, \ldots, s$ then we put

$$\varphi_m(a_0 + a_1 p + a_2 p^2 + \ldots) = \zeta^{a_0 c_0 + a_1 c_1 + \ldots + a_s c_s}, \tag{1}$$

where $a_0 + a_1 p + \ldots$ is a p-adic number written in the Hensel form (i.e. $a_i \in E$).

From the above definition one easily obtains:

THEOREM 1. (i) *For every $m \in N_0$ the function φ_m is continuous.*

(ii) *For all $x \in \mathbf{Z}_p$ we have $|\varphi_m(x)|_p = 1$, where $|\ |_p$ denotes the p-adic norm.*

(iii) *The set $\{\varphi_m\}_{m\in N_0}$ forms an abelian group under the ordinary multiplication of functions.*

(iv) *If $l, m \in N_0$ and $n \in N$ then*

$$p^{-n} \sum_{i=0}^{p^n-1} \varphi_m(i)\varphi_l^{-1}(i) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{p^n}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(v) *If $x, y \in \mathbf{Z}_p$ and $n \in N$ then*

$$p^{-n} \sum_{m=0}^{p^n-1} \varphi_m(x)\varphi_m^{-1}(y) = \begin{cases} 1 & \text{if } x \equiv y \pmod{p^n}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(vi) *The sequence* $\varphi_0, \varphi_1, \ldots, \varphi_{p^n-1}$ *is a basis of the* Ω_p*-vector space*

$$V_n = \{f \in C(\mathbf{Z}_p, \Omega_p): \bigwedge_{x,y \in \mathbf{Z}_p} x \equiv y \ (\mathrm{mod}\ p^n) \Rightarrow f(x) = f(y)\}.$$

Proof. Properties (i)-(iii) are obvious. For the proof of (iv) we put $m = c_0 + c_1 p + c_2 p^2 + \ldots$ and $l = g_0 + g_1 p + g_2 p^2 + \ldots$ where $c_j, g_j \in E$ for all $j \in N_0$ and $c_j = g_j = 0$ for almost all $j \in N_0$. Then (2) follows from the equalities

$$\sum_{i=0}^{p^n-1} \varphi_m(i)\varphi_l^{-1}(i) = \sum_{a_0=0}^{p-1} \ldots \sum_{a_{n-1}=0}^{p-1} \varphi_m(a_0 + a_1 p + \ldots + a_{n-1}p^{n-1}) \times \varphi_l^{-1}(a_0 + a_1 p + \ldots + a_{n-1}p^{n-1})$$

$$= \sum_{a_0=0}^{p-1} \ldots \sum_{a_{n-1}=0}^{p-1} \varphi_{c_0}(a_0)\varphi_{c_1}(a_1) \ldots \varphi_{c_{n-1}}(a_{n-1}) \times \varphi_{g_0}^{-1}(a_0)\varphi_{g_1}^{-1}(a_1) \ldots \varphi_{g_{n-1}}^{-1}(a_{n-1})$$

$$= \prod_{k=0}^{n-1} \sum_{a_k=0}^{p-1} \zeta^{a_k(c_k - g_k)}.$$

For the proof of (v) we put $x = a_0 + a_1 p + \ldots$ and $y = b_0 + b_1 p + \ldots$ where $a_j, b_j \in E$ for all $j \in N_0$. Now, it suffices to notice that

$$\sum_{m=0}^{p^n-1} \varphi_m(x)\varphi_m^{-1}(y) = \sum_{c_0=0}^{p-1} \ldots \sum_{c_{n-1}=0}^{p-1} \varphi_{c_0}(a_0)\varphi_{c_1}(a_1) \ldots \varphi_{c_{n-1}}(a_{n-1}) \times \varphi_{c_0}^{-1}(b_0)\varphi_{c_1}^{-1}(b_1) \ldots \varphi_{c_{n-1}}^{-1}(b_{n-1})$$

$$= \prod_{k=0}^{n-1} \sum_{c_k=0}^{p-1} \zeta^{(a_k - b_k)c_k}.$$

We now prove (vi). The relation $x \equiv y \ (\mathrm{mod}\ p^n)$ divides $\mathbf{Z}_p$ into p^n disjoint classes, hence the vector space V_n is p^n-dimensional. Therefore it is sufficient to show that every $f \in V_n$ can be written as a linear combination (over Ω_p) of $\varphi_0, \varphi_1, \ldots, \varphi_{p^n-1}$. Let $x \in \mathbf{Z}_p$ and $a \in N_0$ satisfy $x \equiv a \ (\mathrm{mod}\ p^n)$, $0 \leqslant a < p^n$. Using (3) we obtain

$$f(x) = f(a) = \sum_{i=0}^{p^n-1} f(i)\Big(p^{-n} \sum_{m=0}^{p^n-1} \varphi_m(a)\varphi_m^{-1}(i)\Big)$$

$$= \sum_{m=0}^{p^n-1} \Big(p^{-n} \sum_{i=0}^{p^n-1} f(i)\varphi_m^{-1}(i)\Big)\varphi_m(x).$$

Hence

$$f = \sum_{m=0}^{p^n-1} \Big(p^{-n} \sum_{i=0}^{p^n-1} f(i)\varphi_m^{-1}(i)\Big)\varphi_m. \tag{4}$$

The proof of Theorem 1 is complete.

Taking into account property (ii) we shall also write $\bar{\varphi}_m$ instead of φ_m^{-1}.

3. Expansions of functions from $C(\mathbf{Z}_p, \Omega_p)$ with respect to $(\varphi_m)_{m \in N_0}$. Let $b_m \in \Omega_p$ for $m = 0, 1, 2, \ldots$ From Theorem 1 (ii) it follows that the series $\sum_{m=0}^{\infty} b_m \varphi_m$ is convergent if and only if $\lim_{m\to\infty} b_m = 0$ (in Ω_p). If $b_m \to 0$ then $\sum b_m \varphi_m$ is a continuous function as the sum of a uniformly convergent series of continuous functions. For this reason we shall expand only continuous functions with respect to the system (φ_m).

Let $f \in C(\mathbf{Z}_p, \Omega_p)$. It is clear that $f = \lim_{n\to\infty} f_n$, where $f_n(x) = f(x^{(n)})$ $(n = 1, 2, \ldots)$ and

$$(a_0 + a_1 p + a_2 p^2 + \ldots)^{(n)} = a_0 + a_1 p + \ldots + a_{n-1}p^{n-1}. \tag{5}$$

According to Theorem 1 (vi) we have

$$f_n(x) = \sum_{m=0}^{p^n-1} b_m^{(n)} \varphi_m(x), \tag{6}$$

where the numbers $b_m^{(n)}$ are defined as follows:

$$b_m^{(n)} = p^{-n} \sum_{i=0}^{p^n-1} f(i)\varphi_m^{-1}(i). \tag{7}$$

Basing on the above considerations we introduce the following definition.

Definition. Let $f \in C(\mathbf{Z}_p, \Omega_p)$. We say that a function f has an *expansion with respect to the system* (φ_m) if:

$$\bigwedge_{m \in N_0} \bigvee_{b_m \in \Omega_p} b_m = \lim_{n\to\infty} b_m^{(n)}, \tag{I}$$

$$\lim_{m\to\infty} b_m = 0. \tag{II}$$

In that case the series $\sum b_m \varphi_m$ is called the *expansion of f with respect to the system* $(\varphi_m)_{m \in N_0}$, and we write

$$f \sim \sum_{m=0}^{\infty} b_m \varphi_m.$$

Remark 1. The limit $\lim_{n\to\infty} p^{-n} \sum_{i=0}^{p^n-1} f(i)$ (if it exists) plays very important role in p-adic analysis (see e.g. [4]) and can be used to define the p-adic integral over $\mathbf{Z}_p$. Therefore we can write the coefficient b_m as

$$b_m = \int_{\mathbf{Z}_p} f \bar{\varphi}_m d\mu$$

or in the form of "inner product": $b_m = \langle f, \varphi_m \rangle$.

The system (φ_m) is *orthonormal*, i.e. $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker symbol.

We now prove the following

THEOREM 2. *If* $f = \sum_{m=0}^{\infty} d_m \varphi_m$ *then* $f \sim \sum_{m=0}^{\infty} d_m \varphi_m$.

Proof. From Theorem 1 (ii) and the condition $d_k \to 0$ we get

$$b_m^{(n)} = p^{-n} \sum_{i=0}^{p^n-1} \Big(\sum_{k=0}^{\infty} d_k \varphi_k(i) \Big) \bar{\varphi}_m(i) = \sum_{k=0}^{\infty} d_k \sum_{i=0}^{p^n-1} p^{-n} \varphi_k(i) \bar{\varphi}_m(i).$$

Applying (2) and the condition $d_k \to 0$ we obtain $\lim_{n\to\infty} b_m^{(n)} = d_m$. For every $m \in N_0$ we have $b_m = d_m$, and so condition (I) is satisfied. Condition (II) is also obviously satisfied.

COROLLARY 1. *Suppose that f and g are continuous functions having expansions which converge to f and g respectively. Then the product fg has an expansion converging to fg.*

Proof. We can multiply term by term the series representing f and g and rearrange the products so as to obtain a series of the form $\sum h_m \varphi_m$. This series represents fg. Application of Theorem 2 finishes the argument.

COROLLARY 2. *The set of functions f having expansions converging to f forms a subalgebra of $C(\boldsymbol{Z}_p, \Omega_p)$.*

COROLLARY 3. *If the series $\sum b'_m \varphi_m$, $\sum b''_m \varphi_m$ converge to the same function then $b'_m = b''_m$ for $m \in N_0$.*

Even if f has an expansion, it need not converge to f. In fact, we have

THEOREM 3. *There exists a function $f \in C(\boldsymbol{Z}_p, \Omega_p)$ such that $f \neq 0$ and $f \sim 0$.*

Proof. We first define inductively the values of f on N_0; these values will be integers. Put $f(0) = \ldots = f(p-2) = 1$ and $f(p-1) = 1-p$. For any $n \in N$, $r \in \{0, \ldots, n-1\}$ and $a_0, \ldots, a_r \in E$ we write

$$S_n(a_0, \ldots, a_r) = \sum_{\substack{0 \le i < p^n \\ i \equiv a_0 + a_1 p + \ldots + a_r p^r (\operatorname{mod} p^{r+1})}} f(i)$$

$$\Big(= \sum_{a_{r+1}=0}^{p-1} \ldots \sum_{a_{n-1}=0}^{p-1} f(a_0 + a_1 p + \ldots + a_{n-1} p^{n-1}) \Big).$$

Assuming that $f(i)$ is already defined for $0 \le i < p^{2l+1}$, where $l \in N_0$, we put

$$f(a_0 + \ldots + a_{2l+2} p^{2l+2}) = \begin{cases} f(a_0 + \ldots + a_{2l} p^{2l}) & \text{if } a_i \neq p-1 \text{ for some } i \in \{l+1, \ldots, 2l+2\}, \\ f(a_0 + \ldots + a_{2l} p^{2l}) - p^2 S_{2l+1}(a_0, \ldots, a_l) & \text{otherwise.} \end{cases}$$

We shall prove that for any $l \in N$

$$(\mathrm{A}_l) \qquad \bigwedge_{l \le r \le 2l} \ \bigwedge_{a_0, \ldots, a_r \in E} p^{2l-r} \mid S_{2l+1}(a_0, \ldots, a_r),$$

$$(\mathrm{B}_l) \qquad \bigwedge_{0 \le r < l} \ \bigwedge_{a_0, \ldots, a_r \in E} S_{2l+1}(a_0, \ldots, a_r) = 0.$$

Notice that the (A_l) conditions with $r = l$ imply the continuity of f on N_0. So f can be extended uniquely to a continuous function on $\boldsymbol{Z}_p$, since N_0 is dense in $\boldsymbol{Z}_p$ and $\boldsymbol{Z}_p$ is a compact metric space. At the same time, the (B_l) conditions imply that for any $n \in N_0$ we have $b_m = \lim_{n\to\infty} b_m^{(n)} = 0$. In fact, $b_m^{(n)} = 0$ if $p^{(n-1)/2} \ge m+1$. Therefore $f \sim 0$.

We first prove the (B_l) conditions. (B_1) reduces to $S_3(a_0) = 0$, which follows at once from the equality

$$f(a_0 + a_1 p + a_2 p^2) = \begin{cases} f(a_0) & \text{if } a_1 \neq p-1 \text{ or } a_2 \neq p-1, \\ (1-p^2) f(a_0) & \text{otherwise,} \end{cases}$$

and the definition of $S_3(a_0)$. If $l > 1$ then it is sufficient to prove that $S_{2l+1}(a_0, \ldots, a_{l-1}) = 0$. We have

$$\begin{aligned} S_{2l+1}&(a_0, \ldots, a_{l-1}) \\ &= \sum_{a_l, \ldots, a_{2l-2} \in E} f(a_0 + \ldots + a_{2l-2} p^{2l-2} + (p-1) p^{2l-1} + (p-1) p^{2l}) \\ &\qquad + (p^2-1) \sum_{a_l, \ldots, a_{2l-2} \in E} f(a_0 + \ldots + a_{2l-2} p^{2l-2}) \\ &= \sum_{\substack{a_l, \ldots, a_{2l-2} \in E \\ a_t \neq p-1 \text{ for some } t \in \{l, \ldots, 2l-2\}}} f(a_0 + \ldots + a_{2l-2} p^{2l-2}) \\ &\qquad + f(a_0 + \ldots + a_{l-1} p^{l-1} + (p-1) p^l + \ldots + (p-1) p^{2l}) \\ &\qquad - p^2 S_{2l-1}(a_0, \ldots, a_{l-1}) + (p^2-1) S_{2l-1}(a_0, \ldots, a_{l-1}) \\ &= S_{2l-1}(a_0, \ldots, a_{l-1}) - p^2 S_{2l-1}(a_0, \ldots, a_{l-1}) \\ &\qquad + (p^2-1) S_{2l-1}(a_0, \ldots, a_{l-1}) = 0. \end{aligned}$$

Next, we prove the (A_l) conditions by induction on l.

(A_1) follows from the equality

$$S_3(a_0, a_1) = \begin{cases} p f(a_0) & \text{if } a_1 \neq p-1, \\ (p-p^2) f(a_0) & \text{otherwise.} \end{cases}$$

If $l \ge 1$ and $l+1 \le r \le 2l$ then we have

$$\begin{aligned} S_{2l+3}&(a_0, \ldots, a_r) \\ &= \begin{cases} p^2 S_{2l+1}(a_0, \ldots, a_r) & \text{if } a_t \neq p-1 \text{ for some } t \in \{l+1, \ldots, r\}, \\ p^2 S_{2l+1}(a_0, \ldots, a_r) - p^2 S_{2l+1}(a_0, \ldots, a_l) & \text{otherwise,} \end{cases} \end{aligned}$$

moreover,

$$S_{2l+3}(a_0, \ldots, a_{2l+1}) = \begin{cases} pf(a_0+ \ldots +a_{2l}p^{2l}) & \text{if } a_t \neq p-1 \text{ for some } t \in \{l+1, \ldots, 2l+1\}, \\ pf(a_0+ \ldots +a_{2l}p^{2l})-p^2 S_{2l+1}(a_0, \ldots, a_l) & \text{otherwise.} \end{cases}$$

Now, it is sufficient to apply the inductive assumption to the sums $S_{2l+1}(\ldots)$.

The proof of the theorem is complete.

Let us now formulate a sufficient condition for a function $f \in C(\boldsymbol{Z}_p, \Omega_p)$ to have an expansion converging to f.

THEOREM 4. *Every function f belonging to $C(\boldsymbol{Z}_p, \Omega_p)$ and satisfying the conditions*

(A) $$\bigwedge_{\varepsilon>0} \bigvee_{N(\varepsilon)} \bigwedge_{n>N} \bigwedge_{0 \leqslant m < p^n} |b_m^{(n+1)} - b_m^{(n)}|_p < \varepsilon,$$

(B) $$\bigwedge_{\varepsilon>0} \bigvee_{N(\varepsilon)} \bigwedge_{n>N} \bigwedge_{p^n \leqslant m < p^{n+1}} |b_m^{(n+1)}|_p < \varepsilon$$

has an expansion with respect to $(\varphi_m)_{m \in N_0}$, and the expansion converges to f.

Proof. Let $f \in C(\boldsymbol{Z}_p, \Omega_p)$ satisfy (A) and (B). Condition (A) implies that for any fixed m the sequence $b_m^{(n)}$ converges to some $b_m \in \Omega_p$, so that (I) is fulfilled. Condition (B) and the equality

$$b_m = b_m^{(n+1)} + \sum_{k=n+1}^{\infty} (b_m^{(k+1)} - b_m^{(k)})$$

imply (II). Hence f has an expansion with respect to (φ_m).

We now prove that the expansion converges to f. For this purpose note that (A) implies

$$\bigwedge_{\varepsilon>0} \bigvee_{N(\varepsilon)} \bigwedge_{n>N} \bigwedge_{0 \leqslant m < p^n} |b_m^{(n)} - b_m|_p < \varepsilon.$$

Thus for any $\varepsilon > 0$ and sufficiently large n we have

$$\begin{aligned} \Big|f - \sum_{m=0}^{p^n-1} b_m \varphi_m\Big|_p &= \Big|f - \sum_{m=0}^{p^n-1} b_m^{(n)} \varphi_m + \sum_{m=0}^{p^n-1} (b_m^{(n)} - b_m)\varphi_m\Big|_p \\ &\leqslant \max\Big(|f-f_n|_p, \Big|\sum_{m=0}^{p^n-1} (b_m^{(n)} - b_m)\varphi_m\Big|_p\Big) \\ &\leqslant \max\Big(|f-f_n|_p, \Big|\max_{0 \leqslant m < p^n} (b_m^{(n)} - b_m)\Big|_p\Big) < \varepsilon. \end{aligned}$$

The proof of the theorem is complete.

Let us now introduce the following definition:

DEFINITION. We say that a differentiable function $f \in C(\boldsymbol{Z}_p, \Omega_p)$ is *uniformly differentiable* if

$$\bigwedge_{\varepsilon>0} \bigvee_{\delta(\varepsilon)>0} \bigwedge_{x \in \boldsymbol{Z}_p} \bigwedge_{t \in \boldsymbol{Z}_p} 0 < |t|_p < \delta \Rightarrow \left|\frac{f(x+t)-f(x)}{t} - f'(x)\right|_p < \varepsilon.$$

Using Theorem 4 we shall prove that every uniformly differentiable function $f \in C(\boldsymbol{Z}_p, \Omega_p)$ has an expansion which converges to f. It follows at once from the definition that the derivative f' of a uniformly differentiable function f is continuous. Moreover, every analytic function on $\boldsymbol{Z}_p$ (in particular, every polynomial) is uniformly differentiable.

THEOREM 5. *Every uniformly differentiable function f has an expansion converging to f.*

Proof. Let $f \in C(\boldsymbol{Z}_p, \Omega_p)$ be uniformly differentiable. We show that then f satisfies condition (A). Fix $n \in N_0$. Then for any m such that $0 \leqslant m < p^n$ we have

$$\begin{aligned} b_m^{(n+1)} - b_m^{(n)} &= p^{-n-1} \sum_{k=0}^{p^n-1} \sum_{l=0}^{p-1} f(k+p^n l)\bar{\varphi}_m(k+p^n l) - p^{-n-1} \sum_{k=0}^{p^n-1} \sum_{l=0}^{p-1} f(k)\bar{\varphi}_m(k) \\ &= p^{-n-1} \sum_{k=0}^{p^n-1} \sum_{l=0}^{p-1} [f(k+p^n l) - f(k)]\bar{\varphi}_m(k) \\ &= p^{-1} \sum_{l=1}^{p-1} l \sum_{k=0}^{p^n-1} \frac{f(k+p^n l)-f(k)}{p^n l} \bar{\varphi}_m(k) \\ &= p^{-1} \sum_{l=1}^{p-1} l \sum_{k=0}^{p^n-1} \left[\frac{f(k+p^n l)-f(k)}{p^n l} - f'(k)\right] \bar{\varphi}_m(k) \\ &\quad + p^{-1} \sum_{l=1}^{p-1} \sum_{k=0}^{p^n-1} f'(k)\bar{\varphi}_m(k) \\ &= S_1 + S_2. \end{aligned}$$

The uniform differentiability of f and Theorem 1 (ii) imply that the p-adic norm of S_1 is arbitrarily small for n sufficiently large. We show that the same is true of S_2. The continuity of f' mentioned above and the compactness of $\boldsymbol{Z}_p$ imply the uniform continuity of f'. Hence for any $\varepsilon > 0$ we can choose $r \in N$ such that for all $k, l \in N$

(7) $$|f'(k+p^r l) - f'(k)|_p < \varepsilon.$$

If $n > r$ then

$$\sum_{k=0}^{p^n-1} f'(k)\bar{\varphi}_m(k) = \sum_{k=0}^{p^r-1}\sum_{l=0}^{p^{n-r}-1} f'(k+p^r l)\bar{\varphi}_m(k+p^r l)$$
$$= \sum_{k=0}^{p^r-1}\sum_{l=0}^{p^{n-r}-1} [f'(k+p^r l)-f'(k)]\bar{\varphi}_m(k+p^r l)$$
$$+\sum_{k=0}^{p^r-1} f'(k)\sum_{l=0}^{p^{n-r}-1} \bar{\varphi}_m(k+p^r l) = S_2'+S_2''.$$

By virtue of Theorem 1 (ii) and (7) the norm of S_2' is less than ε. As to S_2'', it is sufficient to note that $|f'|_p$ is bounded on $\boldsymbol{Z}_p$ and for any k satisfying $0 \leqslant k < p^r$ the sum

$$\sum_{l=0}^{p^{n-r}-1} \bar{\varphi}_m(k+p^r l)$$

is either 0 or p^{n-r}. Hence for n sufficiently large the p-adic norm of S_2 is arbitrarily small. This proves that f satisfies condition (A).

Now, we show that f satisfies (B). Take natural numbers n, m such that $p^n \leqslant m < p^{n+1}$. Let $m = c_0+c_1p+\ldots+c_np^n$ be the p-adic expansion of m. Then $c_n \neq 0$. Denoting $m^{(n)} = c_0+\ldots+c_{n-1}p^{n-1}$ by m_1 we have

$$b_m^{(n+1)} = p^{-n-1}\sum_{i=0}^{p^{n+1}-1} f(i)\bar{\varphi}_m(i) = p^{-n-1}\sum_{k=0}^{p^n-1}\sum_{l=0}^{p-1} f(k+p^n l)\bar{\varphi}_m(k+p^n l)$$
$$= p^{-n-1}\sum_{k=0}^{p^n-1}\sum_{l=0}^{p-1} f(k+p^n l)\bar{\varphi}_{m_1}(k)\bar{\varphi}_{c_n}(l)$$
$$= p^{-n-1}\sum_{k=0}^{p^n-1}\sum_{l=0}^{p-1} [f(k+p^n l)-f(k)]\bar{\varphi}_{m_1}(k)\bar{\varphi}_{c_n}(l)$$
$$= p^{-1}\sum_{l=1}^{p-1} l\bar{\varphi}_{c_n}(l)\sum_{k=0}^{p^n-1}\frac{f(k+p^n l)-f(k)}{p^n l}\bar{\varphi}_{m_1}(k).$$

Proceeding now as in the proof of (A) we arrive at (B). The proof of the theorem is finished.

Remark. There are differentiable functions which do not have expansions with respect to (φ_m). As an example, we can take the function

$$f(x) = \begin{cases} p^{2n_1}+a_2p^{n_2-n_1}+a_3p^{n_3-n_1}+\ldots & \text{if } x = p^{n_1} \\ & \text{or } x = p^{n_1}+a_2p^{n_2}+\ldots+a_kp^{n_k}+\ldots, \\ & \text{with } a_2, a_3, \ldots \neq 0, \\ & n_1 < 3n_1 < n_2 < n_3 < \ldots < n_k < \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

In [3] J. Mináč shows that f is differentiable on $\boldsymbol{Z}_p$ and f' is unbounded. Hence f is not uniformly differentiable. For the proof that f does not have an expansion with respect to (φ_m) it is sufficient to show that the sequence $(b_0^{(3n+1)})$ is not convergent in Ω_p. We have

$$b_0^{(3n+1)} = p^{-3n-1}\sum_{k=1}^{3n} s_k$$

where

$$s_k = \sum_{i=0}^{p^{3n+1}-1}{}^{*} f(i),$$

the symbol $*$ indicating that the sum is over all i for which $n_1 = k$. It is easy to calculate that

$$s_k = \begin{cases} (1-(p/2))p^{3n-k}+\frac{1}{2}p^{6n-4k+1} & \text{if } 1 \leqslant k < n, \\ p^{2k} & \text{if } n \leqslant k \leqslant 3n. \end{cases}$$

Hence $|s_n|_p = p^{-2n}$ and $|s_k|_p < p^{-2n}$ for $k \neq n$. Therefore

$$|b_0^{(3n+1)}|_p = p^{n+1}.$$

Thus the sequence $b_0^{(3n+1)}$ is not convergent.

4. Examples of expansions

(a) *The function "s-th digit"*. Let $s \in \boldsymbol{N}_0$. The function $i_s\colon \boldsymbol{Z}_p \to E$ is defined as follows: if $x = a_0+a_1p+a_2p^2+\ldots$, where $a_i \in E$ for $i \in \boldsymbol{N}_0$, then $i_s(x) = a_s$. For an arbitrary $k \in E$ we have

$$p^{-1}\sum_{j=0}^{p-1}\zeta^{-jk}\varphi_{jp^s}(x) = \begin{cases} 1 & \text{if } k = a_s, \\ 0 & \text{if } k \neq a_s. \end{cases} \tag{8}$$

Therefore

$$i_s(x) = \sum_{j=0}^{p-1}\Big(p^{-1}\sum_{k=0}^{p-1} k\zeta^{-jk}\Big)\varphi_{jp^s}(x).$$

(b) *The characteristic function of a residue class*. Let $t = t_0+t_1p+\ldots+t_{n-1}p^{n-1}$, where $t_j \in E$ for $j = 0, 1, \ldots, n-1$, let

$$A = \{a \in \boldsymbol{Z}_p\colon a \equiv t \pmod{p^n}\},$$

and let $\chi_A\colon \boldsymbol{Z}_p \to \{0, 1\}$ be the characteristic function of A. It is clear that

$$\chi_A(x) = \prod_{s=0}^{n-1}\Big(p^{-1}\sum_{j=0}^{p-1}\zeta^{-jt_s}\varphi_{jp^s}(x)\Big).$$

Therefore

$$\chi_A(x) = \sum_{m=0}^{p^n-1} b_m\varphi_m(x)$$

where for $m = c_0+c_1p+\ldots+c_{n-1}p^{n-1}$ $(c_j \in E)$,

$$b_m = p^{-n}\zeta^{-\sum_{s=0}^{n-1} c_s t_s}.$$

(c) *Digit functions.*

DEFINITION. A function $f\colon \mathbf{Z}_p \to \Omega_p$ is called a *digit function* if there exists a function $t\colon E \to \Omega_p$ such that for every $x = a_0+a_1p+\ldots$ $(a_j \in E, j = 0, 1, 2, \ldots)$,

(9) $$f(a_0+a_1p+a_2p^2+\ldots) = t(a_0)+t(a_1)p+t(a_2)p^2+\ldots$$

For example, the function $f(a_0+a_1p+a_2p^2+\ldots) = a_0^2+a_1^2p+a_2^2p^2+\ldots$ is a digit function for which $t(a) = a^2$. In [2] J. Dieudonné showed that this f is not differentiable if $p > 2$.

We have the following obvious theorem:

THEOREM 6. *The set of all digit functions forms a linear subspace of $C(\mathbf{Z}_p, \Omega_p)$. Every constant function on $\mathbf{Z}_p$ is a digit function.*

If f is a digit function then we can write

$$f = \sum_{s=0}^{\infty}\Big(\sum_{k=0}^{p-1} t(k)p^{-1}\sum_{j=0}^{p-1}\zeta^{-jk}\varphi_{jp^s}\Big)p^s.$$

Therefore

(10) $$f = [p(1-p)]^{-1}\sum_{k=0}^{p-1} t(k)\varphi_0+\sum_{s=0}^{\infty}\sum_{j=1}^{p-1}\Big(p^{s-1}\sum_{k=0}^{p-1} t(k)\zeta^{-jk}\Big)\varphi_{jp^s}.$$

It is clear that if $t(i) = \chi(i)$, where χ is a Dirichlet character modulo p then p-adic Gauss sums appear on the right side of (10).

Putting $t(i) = i$ in (10) we get the expansion

(11) $$x = -\tfrac{1}{2}+\sum_{s=0}^{\infty}\sum_{j=1}^{p-1}\Big(p^{s-1}\sum_{k=0}^{p-1} k\zeta^{-jk}\Big)\varphi_{jp^s}(x),$$

and so

(12) $$x = -\tfrac{1}{2}+\sum_{s=0}^{\infty}\sum_{j=1}^{p-1} p^s(\zeta^{-j}-1)^{-1}\varphi_{jp^s}(x).$$

If $p = 2$ then the expansion (12) reduces to

(13) $$x = -\tfrac{1}{2}-\tfrac{1}{2}\sum_{s=0}^{\infty} 2^s\varphi_{2^s}(x).$$

(d) *Powers of digit functions.* Since every digit function f has an expansion with respect to (φ_m), and the expansion converges to f, it follows from Corollary 1 that the same is true for every natural power of f. In particular, for each $n \in \mathbf{N}_0$ the limit

$$b_0(f^n) = \lim_{k\to\infty} p^{-k}\sum_{i=0}^{p^k-1} f^n(i) \quad (0^0 = 1)$$

exists. Moreover, applying Theorem 6 we see that for every $n \in \mathbf{N}_0$ the limit

$$b_0(g^n) = \lim_{k\to\infty} p^{-k}\sum_{i=0}^{p^k-1} g^n(i) \quad (0^0 = 1),$$

also exists, where $g(x) = f(x)-f(0)$ for $x \in \mathbf{Z}_p$. Evidently, $f(0) = t(0)/(1-p)$.

Introducing the notation

(14) $$B_{n,f} = b_0(f^n) \quad (n \in \mathbf{N}_0)$$

we deduce that for every $m = c_0+c_1p+\ldots+c_sp^s$ $(c_j \in E, j = 0, 1, \ldots, s)$ we can write

$$\begin{aligned} b_m(f^n) &= \lim_{k\to\infty} p^{-k}\sum_{i=0}^{p^k-1} f^n(i)\bar{\varphi}_m(i) \\ &= \lim_{k\to\infty} p^{-k}\sum_{i=0}^{p^{s+1}-1}\sum_{j=0}^{p^{k-s-1}-1}[f(i)+p^{s+1}g(j)]^n\bar{\varphi}_m(i) \\ &= \lim_{k\to\infty} p^{-s-1}\sum_{i=0}^{p^{s+1}-1}\sum_{l=0}^{n}\binom{n}{l}p^{l(s+1)}\Big(p^{-k+s+1}\sum_{j=0}^{p^{k-s-1}-1} g^l(j)\Big)f^{n-l}(i)\bar{\varphi}_m(i) \\ &= \sum_{l=0}^{n}\binom{n}{l}p^{(l-1)(s+1)}B_{l,g}\sum_{i=0}^{p^{s+1}-1} f^{n-l}(i)\bar{\varphi}_m(i). \end{aligned}$$

Note that we do not suppose that $c_s \neq 0$. Putting $m = 0$, we obtain the recurrence formula

$$B_{n,f} = \sum_{l=0}^{n}\binom{n}{l}p^{(l-1)(s+1)}B_{l,g}\sum_{i=0}^{p^{s+1}-1} f^{n-l}(i).$$

If $s = 0$, we get the relation

(15) $$B_{n,f} = \sum_{l=0}^{n}\binom{n}{l}p^{l-1}B_{l,g}\sum_{i=0}^{p-1} f^{n-l}(i).$$

The formula (15) together with the obvious equalities

$$B_{n,g} = \sum_{l=0}^{n}\binom{n}{l}B_{l,f}(-f(0))^{n-l} \quad (n \in \mathbf{N},\ 0^0 = 1)$$

enables us to determine successively the coefficients $B_{n,g}$ and $B_{n,f}$. If $m \neq 0$ then

(16) $$b_m(f^n) = \sum_{l=0}^{n}\binom{n}{l}p^{(l-1)(s+1)}B_{l,g}\sum_{i=0}^{p^{s+1}-1} f^{n-l}(i)\bar{\varphi}_m(i).$$

(e) *The function* a^x. If $a \in 1+p\boldsymbol{Z}_p$ then the function $a^x = \exp(x\log a)$ has an expansion with respect to (φ_m), converging to a^x, because a^x is analytic on $\boldsymbol{Z}_p$ (see Theorem 5). If $a = 1$ then $a^x = \varphi_0(x)$. Now suppose that $a \in 1+p\boldsymbol{Z}_p$ and $a \neq 1$. If $m = c_0+c_1p+ \ldots +c_sp^s$ ($c_j \in E$ for $j = 0, 1, \ldots, s$) and $a^{p^{s+1}} \neq 1$, then

$$\begin{aligned} b_m &= \lim_{k\to\infty} p^{-k} \sum_{i=0}^{p^{s+1}-1} \sum_{j=0}^{p^{k-s-1}-1} a^{i+jp^{s+1}} \bar{\varphi}_m(i) \\ &= \lim_{k\to\infty} p^{-k} \frac{a^{p^k}-1}{a^{p^{s+1}}-1} \sum_{i=0}^{p^{s+1}-1} a^i \bar{\varphi}_m(i) \\ &= \frac{\log a}{a^{p^{s+1}}-1} \sum_{d_0=0}^{p-1} \ldots \sum_{d_s=0}^{p-1} a^{d_0+pd_1+\ldots+p^sd_s}\zeta^{-(d_0c_0+\ldots+d_sc_s)} \\ &= \frac{\log a}{a^{p^{s+1}}-1} \prod_{j=0}^{s} \frac{a^{p^{j+1}}-1}{a^{p^j}\zeta^{-c_j}-1}. \end{aligned}$$

In particular, $b_0 = \log a/(a-1)$.

(f) *The functions* $\exp(ax)$, $\sin(ax)$, $\cos(ax)$. Let

$$D = \{x \in \Omega_p\colon |x|_p < p^{-1/(p-1)}\}$$

and let a be a fixed p-adic number in D. Then the series

$$\sum_{n=0}^{\infty} \frac{(ax)^n}{n!}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n(ax)^{2n+1}}{(2n+1)!}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n(ax)^{2n}}{(2n)!}$$

define functions, analytic on $\boldsymbol{Z}_p$: $\exp(ax)$, $\sin(ax)$, $\cos(ax)$ respectively (see [4]). Therefore for every $a \in D$ each of the above functions has an expansion converging to it. It is easy to calculate that if $m = c_0+c_1p+ \ldots +c_{s-1}p^{s-1}$ ($c_0, c_1, \ldots, c_{s-1} \in E$) then the coefficients of the expansions are the following:

$\exp(ax)$:

$$b_0 = a(\exp a-1)^{-1}, \quad b_m = a[\exp(ap^s)-1]^{-1} \sum_{i=0}^{p^s-1} \exp(ai)\bar{\varphi}_m(i);$$

$\sin(ax)$:

$$b_0 = -a/2, \quad b_m = \frac{a}{2}\sin^{-1}\frac{ap^s}{2} \sum_{i=0}^{p^s-1} [\sin a(i-p^s/2)]\,\bar{\varphi}_m(i);$$

$\cos(ax)$:

$$b_0 = \frac{a}{2}\cot\frac{a}{2}, \quad b_m = \frac{a}{2}\sin^{-1}\frac{ap^s}{2} \sum_{i=0}^{p^s-1} [\cos a(i-p^s/2)]\,\bar{\varphi}_m(i).$$

(If $p = 2$ and $a/2 \notin D$ then by $\cot(a/2)$ we mean the quotient $(1+\cos a)/\sin a$.)

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