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On certain infinite products III

by

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1. Introduction. Let $\eta(z)$ be the Dedekind eta function defined by

$$\eta(z) = \exp(\pi iz/12)\prod_{n=1}^{\infty}(1-\exp(2\pi inz)),$$

where z is the standard variable on the upper half-plane. Then it is famous that

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z).$$

In 1968, Weil [5] obtained a very natural proof of the above formula by using the functional equation for the Riemann zeta function. The author [3] gave a characterization of $\eta(z)$ by developing the ideas of Hecke [1] and Weil [5]. (Along with [5] and [1] see also [2] and [6].) Later, in [4], he determined infinite products satisfying a certain functional equation on the upper half-plane. In the present paper, we will discuss a slight generalization of the previous result given in [4].

Let $a_1(1), a_1(2), a_1(3), \ldots$ and $a_2(1), a_2(2), a_2(3), \ldots$ be two sequences of complex numbers such that $a_1(n), a_2(n) = O(n^c)$ for some $c > 0$, and form

$$\varphi_1(s) = \sum_{n=1}^{\infty} a_1(n)n^{-s} \quad\text{and}\quad \varphi_2(s) = \sum_{n=1}^{\infty} a_2(n)n^{-s}.$$

Then $\varphi_1(s)$ and $\varphi_2(s)$ are convergent Dirichlet series. For any real number λ, we put $q(\lambda) = \exp(2\pi iz/\lambda)$. Further we define, for $\mathrm{Im}(z) > 0$,

$$f_1(z) = \exp(2\pi i\delta_1 z)\prod_{n=1}^{\infty}(1-q(\lambda_1)^n)^{a_1(n)} \tag{1}$$

and

$$f_2(z) = \exp(2\pi i\delta_2 z+B)\prod_{n=1}^{\infty}(1-q(\lambda_2)^n)^{a_2(n)}, \tag{2}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$ and δ_1, δ_2, B are real numbers. Throughout this paper, for complex numbers x and w with $x \neq 0$, $x^w = \exp(w\log x)$ and the principal branch is taken for the logarithm. Then these infinite products converge absolutely and uniformly in every compact subset of the upper half-plane. Hence $f_1(z)$ and $f_2(z)$ are holomorphic in the upper half-plane.

THEOREM. *Let N be a positive number, and put $M = \lambda_1\lambda_2 N$. Assume that $\varphi_1(s)$ and $\varphi_2(s)$ can be continued through the whole s-plane as non-zero meromorphic functions with a finite number of poles and that there exists a real number k such that*

$$f_1(-1/Nz) = (\sqrt{N}z/i)^k f_2(z). \tag{3}$$

Then M is a positive integer, $f_1(z)$ and $f_2(z)$ can be expressed in the form

$$f_1(z) = \prod_{m|M} \eta(mz/\lambda_1)^{c(m)} \tag{4}$$

and

$$f_2(z) = \exp(B) \prod_{m|M} \eta(mz/\lambda_2)^{c(M/m)}, \tag{5}$$

where $c(m)$, defined for m dividing M, are complex numbers,

$$\delta_1 = \frac{1}{24\lambda_1} \sum_{m|M} mc(m), \tag{6}$$

$$\delta_2 = \frac{1}{24\lambda_2} \sum_{m|M} mc(M/m), \tag{7}$$

$$k = \tfrac{1}{2} \sum_{m|M} c(m), \tag{8}$$

and

$$B = \tfrac{1}{2}(\log\lambda_1\sqrt{N}) \sum_{m|M} c(m) - \tfrac{1}{2} \sum_{m|M} c(m)\log m. \tag{9}$$

Conversely, let M be a positive integer and let $c(m)$, for integers m dividing M, be arbitrary complex numbers such that four numbers

$$\sum_{m|M} c(m), \quad \sum_{m|M} mc(m), \quad \sum_{m|M} mc(M/m), \quad \text{and} \quad \sum_{m|M} c(m)\log m$$

are real numbers. Further, define $f_1(z)$, $f_2(z)$, k and B by (4), (5), (8) *and* (9), *respectively. Then $f_1(z)$ and $f_2(z)$ satisfy the functional equation* (3).

From the theorem, we can easily obtain the following corollaries.

COROLLARY 1. *Under the same assumptions as in the theorem, if we take $\lambda_1 = \lambda_2 = N = 1$ and $\delta_1 = 1/24$, then $f_1(z) = f_2(z) = \eta(z)$.*

This is a characterization of $\eta(z)$.

COROLLARY 2. *Under the same assumptions as in the theorem, if we take $\lambda_1 = \lambda_2 = \lambda$, $M = \lambda^2 N$ and $a_1(n) = a_2(n)$ for all n, then M is a positive integer and*

$$f_1(z) = f_2(z) = \prod_{m|M} \eta(mz/\lambda)^{c(m)}, \tag{10}$$

where $c(m)$, defined for m dividing M, are complex numbers such that $c(m) = c(M/m)$ for any divisor m of M,

$$\delta_1 = \delta_2 = \frac{1}{24\lambda} \sum_{m|M} mc(m) \tag{11}$$

and

$$k = \tfrac{1}{2} \sum_{m|M} c(m). \tag{12}$$

Conversely, let M be a positive integer and let $c(m)$, for m dividing M, be arbitrary complex numbers such that $c(m) = c(M/m)$ for any divisor m of M and that two numbers

$$\sum_{m|M} c(m) \quad \text{and} \quad \sum_{m|M} mc(m)$$

are real numbers. Further, define $f(z)$ and k by the right-hand side of (10) *and* (12), *respectively. Then $f(z)$ satisfies the functional equation*

$$f(-1/Nz) = (\sqrt{N}z/i)^k f(z).$$

Now we consider the case when $\lambda_1 = \lambda_2 = \lambda$, $\delta_1 = \delta_2 = \delta$, $M = \lambda^2$,

$$\varphi_1(s) = \varphi_2(s) = \varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

and

$$f(z) = \exp(2\pi i\delta z) \prod_{n=1}^{\infty} (1-q(\lambda)^n)^{a(n)}.$$

Taking $N = 1$ in Corollary 2, we have the following.

COROLLARY 3. *Assume that $\varphi(s)$ can be continued through the whole s-plane as a non-zero meromorphic function with a finite number of poles and that there exists a real number k such that*

$$f(-1/z) = (z/i)^k f(z). \tag{13}$$

Then λ^2 is a positive integer and $f(z)$ can be expressed in the form

$$f(z) = \prod_{m|\lambda^2} \eta(mz/\lambda)^{c(m)}, \tag{14}$$

where $c(m)$, defined for m dividing λ^2, are complex numbers such that $c(m) = c(\lambda^2/m)$ for any divisor m of λ^2,

$$\delta = \frac{1}{24\lambda} \sum_{m|\lambda^2} mc(m) \tag{15}$$

and

$$k = \tfrac{1}{2} \sum_{m|\lambda^2} c(m). \tag{16}$$

Conversely, let λ^2 be a positive integer and let $c(m)$, for m dividing λ^2, be arbitrary complex numbers such that $c(m) = c(\lambda^2/m)$ for any divisor m of λ^2 and that two numbers

$$\sum_{m|\lambda^2} c(m) \quad \text{and} \quad \sum_{m|\lambda^2} mc(m)$$

are real numbers. Further, define $f(z)$ and k by (14) and (16), respectively. Then $f(z)$ satisfies the functional equation (13).

As is easily seen, this is equivalent to the theorem given in [4], since

$$\prod_{m|\lambda^2} \eta(mz/\lambda)^{c(m)} = \exp(2\pi i\delta z) \prod_{m|\lambda^2} \prod_{n=1}^{\infty} (1-q(\lambda)^{mn})^{c(m)},$$

where δ is defined by (15). It was not until quite recently that the author found the above equality.

Remark 1. In the theorem and its corollaries, $a_1(1), a_1(2), a_1(3), \ldots$ and $a_2(1), a_2(2), a_2(3), \ldots$ are sequences of rational integers, if, and only if, the numbers $c(m)$ $(m|M)$ are rational integers.

Remark 2. The theta function

$$\vartheta(z) = \prod_{n=1}^{\infty} (1-q(2)^{2n})(1+q(2)^{2n-1})^2$$

is a modular form of weight 1/2 and satisfies

$$\vartheta(-1/z) = \sqrt{z/i}\,\vartheta(z).$$

Then from Corollary 3 we see that

$$\vartheta(z) = \frac{\eta(z)^5}{\eta(z/2)^2\eta(2z)^2}.$$

2. Some lemmas. We shall use the same notation as in the previous section. For $y > 0$, we set

$$G_1(y) = -\{\log f_1(iy/\sqrt{N}) + (2\pi\delta_1 y/\sqrt{N})\}$$

and

$$G_2(y) = -\{\log f_2(iy/\sqrt{N}) + (2\pi\delta_2 y/\sqrt{N}) - B\}.$$

Then from (1) and (2), we have

$$(17) \qquad G_j(y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_j(n)}{m}\exp(-2nm\pi y/\lambda_j\sqrt{N}) \quad (j = 1, 2).$$

LEMMA 1. *If $y \geqslant 1$, then*

$$G_j(y) \ll \exp(-\pi y/\lambda_j\sqrt{N}) \qquad (j = 1, 2).$$

Proof. By the assumption, $a_j(n) = O(n^c)$, and the inequality $2nm \geqslant n+m$, we have

$$G_j(y) \ll \sum_{n=1}^{\infty} n^c \exp(-n\pi y/\lambda_j\sqrt{N}) \sum_{m=1}^{\infty} \frac{1}{m}\exp(-m\pi y/\lambda_j\sqrt{N})$$

$$\ll \sum_{m=1}^{\infty} \exp(-m\pi y/\lambda_j\sqrt{N}) \ll \exp(-\pi y/\lambda_j\sqrt{N}) \quad (j = 1, 2).$$

as required.

Next, let

$$\xi_j(s) = (2\pi/\lambda_j\sqrt{N})^{-s}\Gamma(s)\zeta(s+1)\varphi_j(s) \qquad (j = 1, 2),$$

where, as usual, $\Gamma(s)$ and $\zeta(s)$ denote the gamma function and the Riemann zeta function, respectively.

LEMMA 2. *Let $N > 0$ and k be a real number. Then the following two conditions are equivalent:*

(A) $f_1(-1/Nz) = (\sqrt{N}z/i)^k f_2(z)$.

(B) *$\xi_1(s)$ and $\xi_2(s)$ can be continued through the whole s-plane as meromorphic functions satisfying $\xi_1(s) = \xi_2(-s)$, and*

$$\xi_1(s) + \frac{k}{s^2} + \frac{B}{s} + \frac{2\pi\delta_1}{\sqrt{N}}\frac{1}{1+s} + \frac{2\pi\delta_2}{\sqrt{N}}\frac{1}{1-s}$$

is entire and bounded in every vertical strip.

Proof. It follows easily from (17) and Mellin's inversion formula that

$$(18) \qquad G_1(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} \xi_1(s)y^{-s}ds$$

and

$$(19) \qquad G_2(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} \xi_2(s)y^{-s}ds,$$

where σ is chosen large enough to be in the intersection of two domains of absolute convergence of $\varphi_1(s)$ and $\varphi_2(s)$. Assume now (B). Then shifting the line of integration in (18) to $\mathrm{Re}(s) = -\sigma$, we have

$$G_1(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-\sigma} \xi_1(s)y^{-s}ds + \{\text{sum of residues of integrand at } s = 1, 0, -1\}.$$

By (19) and the functional equation $\xi_1(s) = \xi_2(-s)$, we see that

$$G_2(1/y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-\sigma} \xi_1(s)y^{-s}ds.$$

The residues in the sum are as follows:

$$\operatorname*{Res}_{s=1} \xi_1(s)y^{-s} = \frac{2\pi\delta_2}{\sqrt{N}y},$$

$$\operatorname*{Res}_{s=0} \xi_1(s)y^{-s} = k\log y - B,$$

$$\operatorname*{Res}_{s=-1} \xi_1(s)y^{-s} = -\frac{2\pi\delta_1 y}{\sqrt{N}}.$$

Thus we have

$$G_1(y) = G_2(1/y) + \frac{2\pi\delta_2}{\sqrt{N}y} + k\log y - B - \frac{2\pi\delta_1 y}{\sqrt{N}},$$

which yields

$$\log f_1(iy/\sqrt{N}) = -k\log y + \log f_2(i/\sqrt{N}y).$$

Therefore

$$f_1(iy/\sqrt{N}) = y^{-k} f_2(i/\sqrt{N}y).$$

Substituting $1/\sqrt{N}y$ for y in the above equality, we obtain

$$f_1(i/Ny) = (\sqrt{N}y)^k f_2(iy),$$

which is (A).

Hereafter, $\mathrm{Re}(s)$ is taken so large to ensure the validity of the later argument. Noticing that

$$\Gamma(s) = \int_0^\infty e^{-t} t^s d^\times t \quad \text{where} \quad d^\times t = \frac{dt}{t},$$

we can deduce from (17) that

$$\xi_1(s) = \int_0^\infty G_1(y) y^s d^\times y.$$

Then it is easy to see that

$$\xi_1(s) = \int_1^\infty G_1(y) y^s d^\times y + \int_1^\infty G_1(1/y) y^{-s} d^\times y. \tag{20}$$

Assuming now (A),

$$\log f_1(i/\sqrt{N}y) = k\log y + \log f_2(iy/\sqrt{N}),$$

so that

$$G_1(1/y) = G_2(y) + \frac{2\pi\delta_2 y}{\sqrt{N}} - \frac{2\pi\delta_1}{\sqrt{N}y} - k\log y - B.$$

Hence, from (20) we have

$$\xi_1(s) + \frac{k}{s^2} + \frac{B}{s} + \frac{2\pi\delta_1}{\sqrt{N}}\frac{1}{1+s} + \frac{2\pi\delta_2}{\sqrt{N}}\frac{1}{1-s} = \int_1^\infty G_1(y) y^s d^\times y + \int_1^\infty G_2(y) y^{-s} d^\times y. \tag{21}$$

Lemma 1 shows that the above integrals converge absolutely and uniformly in every vertical strip in the s-plane and therefore define entire functions of s, so that (21) gives us an analytic continuation of $\xi_1(s)$ into the whole s-plane as a meromorphic function. By the same way as above, we know that

$$\xi_2(-s) + \frac{k}{s^2} + \frac{B}{s} + \frac{2\pi\delta_1}{\sqrt{N}}\frac{1}{1+s} + \frac{2\pi\delta_2}{\sqrt{N}}\frac{1}{1-s} = \int_1^\infty G_1(y) y^s d^\times y + \int_1^\infty G_2(y) y^{-s} d^\times y.$$

holds in the whole s-plane. From this and (21), we have the functional equation $\xi_1(s) = \xi_2(-s)$.

This completes the proof of the lemma.

LEMMA 3. *Let $N > 0$ and k be a real number. Then* (3) *holds, if, and only if, $\varphi_1(s)$ and $\varphi_2(s)$ satisfy the following four conditions:*

(a) *$\varphi_1(s)$ and $\varphi_2(s)$ can be continued through the whole s-plane as meromorphic functions.*

(b) *$s(s-1)\varphi_1(s)\zeta(s+1)$ and $s(s-1)\varphi_2(s)\zeta(s+1)$ are entire of finite order.*

(c) *$M^s\varphi_1(s)\zeta(-s) = \varphi_2(s)\zeta(s)$.*

(d) *$\varphi_1(0) = -k$, $\varphi_1(-1) = -2\lambda_1\delta_1$, $B = -\varphi_1'(0) - k\log(2\pi/\lambda_1\sqrt{N})$ and*

$$\operatorname*{Res}_{s=1}\varphi_1(s) = 24\delta_2/\lambda_1 N.$$

Proof. The method of the proof is almost the same as that of Theorem 2 in [3]. By Lemma 2, we have only to prove that (B) is equivalent to the above four conditions. By noticing that

$$2\Gamma(s)\zeta(s)\cos(\pi s/2) = (2\pi)^s\zeta(1-s),$$

we find that $\xi_1(s) = \xi_2(-s)$ is equivalent to (c). In the following, σ is taken large enough to be in the intersection of two domains of absolute convergence of $\varphi_1(s)$ and $\varphi_2(s)$, and, for brevity, we put

$$E(s) = \xi_1(s) + \frac{k}{s^2} + \frac{B}{s} + \frac{2\pi\delta_1}{\sqrt{N}}\frac{1}{1+s} + \frac{2\pi\delta_2}{\sqrt{N}}\frac{1}{1-s}.$$

Assume now (B). Then (a) and (d) are clear. Further it is easy to see that $s(s-1)\varphi_1(s)\zeta(s+1)$ is entire in the whole s-plane. Since $\varphi_1(s)\zeta(s+1)$ and $\varphi_2(s)\zeta(s+1)$ are bounded in $\mathrm{Re}(s) \geqslant \sigma$, by Stirling's formula and $\xi_1(s) = \xi_2(-s)$, we know that $\xi_1(s)$ is entire of order 1 there and also in $\mathrm{Re}(s) \leqslant -\sigma$, as is $\varphi_1(s)\zeta(s+1)$, because $\varphi_1(s)\zeta(s+1) = (2\pi/\lambda_1\sqrt{N})^s\xi_1(s)/\Gamma(s)$ and $1/\Gamma(s)$ is an entire function of order 1. Similarly, we can deduce that $s(s-1)\varphi_1(s)\zeta(s+1)$ has order 1 in the strip $-\sigma \leqslant \mathrm{Re}(s) \leqslant \sigma$, since $E(s)$ is entire and bounded in every vertical strip. Thus $s(s-1)\varphi_1(s)\zeta(s+1)$ is entire of order 1. Since $\xi_1(s) = \xi_2(-s)$, we see that $s(s-1)\varphi_2(s)\zeta(s+1)$ is entire of order 1 by using the same way as above.

Conversely, let us suppose that the conditions (a), (b), (c) and (d) are fulfilled. Then we find that $E(s)$ can be continued through the whole s-plane as an entire function of finite order. Taking $v \geqslant \sigma$, then $E(s) = O(1)$ on the line $\mathrm{Re}(s) = v$ and also on the line $\mathrm{Re}(s) = -v$ by Stirling's formula and $\xi_1(s) = \xi_2(-s)$. Hence, $E(s) = O(1)$ in the strip $-v \leqslant \mathrm{Re}(s) \leqslant v$ by the Phragmen–Lindelöf theorem, which proves (B).

3. On certain Dirichlet series.

LEMMA 4. *Let $D(s)$ be a convergent Dirichlet series defined by*

$$D(s) = \sum_{n=1}^{\infty} c(n)n^{-s}.$$

Assume that $D(s)$ can be continued through the whole s-plane as a meromorphic function of finite order with a finite number of poles and that there exists a positive number K such that $D(-s) = O(|K^s|)$ for Re(s) *sufficiently large. Then, if $K < 1$, $D(s) = 0$ and if $K \geqslant 1$,*

$$D(s) = \sum_{n=1}^{[K]} c(n)n^{-s}.$$

where $[K]$ denotes the integral part of K.

Proof. See Lemma 5 in [3].

LEMMA 5. *Let $D_1(s)$ and $D_2(s)$ be two convergent Dirichlet series. Assume that they can be continued through the whole s-plane as non-zero meromorphic functions of finite order, which have a finite number of poles and which satisfy $d^s D_1(s) = D_2(-s)$ for some positive number d. Then d is a positive integer,*

$$D_1(s) = \sum_{n|d} c(n)n^{-s} \quad \text{and} \quad D_2(s) = \sum_{n|d} c(d/n)n^{-s},$$

where $c(n)$, defined for n dividing d, are complex numbers.

Proof. Put

$$D_1(s) = \sum_{n=1}^{\infty} c(n)n^{-s} \quad \text{and} \quad D_2(s) = \sum_{n=1}^{\infty} b(n)n^{-s}.$$

By our assumptions, we see that

$$D_2(-s) = O(|d^s|)$$

for Re(s) sufficiently large. Then it follows from Lemma 4 that $d \geqslant 1$ and

$$D_2(s) = \sum_{n=1}^{[d]} b(n)n^{-s}.$$

Since $d^s D_1(s) = D_2(-s)$, we have

$$\sum_{n=1}^{\infty} c(n)(d/n)^s = \sum_{n=1}^{[d]} b(n)n^s. \tag{22}$$

Let p denote the first value of the index n such that $c(n) \neq 0$ and let q denote the last value of the index n such that $b(n) \neq 0$. Then from (22) we find that $c(p)(d/p)^s = b(q)q^s$, which implies $c(p) = b(q)$ and $pq = d$. Thus d is an integer. Repeating the same argument as above, we can prove the lemma.

4. Proof of the theorem.

We shall prove the first assertion. By our assumptions, $\varphi_1(s)$ and $\varphi_2(s)$ satisfy the four conditions of Lemma 3. Hence, if we put $D_1(s) = \varphi_1(s)/\zeta(s)$ and $D_2(s) = \varphi_2(s)/\zeta(s)$, then $D_1(s)$ and $D_2(s)$ can be continued through the whole s-plane as non-zero meromorphic functions of finite order, and $M^s D_1(s) = D_2(-s)$. Further we know that $D_1(s)$ and $D_2(s)$ have a finite number of poles in the whole s-plane by (b), (c) and the assumptions of $\varphi_1(s)$ and $\varphi_2(s)$. Then, from Lemma 4 we see that M is a positive integer and

$$D_1(s) = \sum_{m|M} c(m)m^{-s} \quad \text{and} \quad D_2(s) = \sum_{m|M} c(M/m)m^{-s},$$

where $c(m)$, defined for m dividing M, are complex numbers. Hence we have

$$\varphi_1(s) = \Big(\sum_{m|M} c(m)m^{-s}\Big)\zeta(s) \tag{23}$$

and

$$\varphi_2(s) = \Big(\sum_{m|M} c(M/m)m^{-s}\Big)\zeta(s). \tag{24}$$

By noting that $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$ and $\zeta'(0) = -(\log 2\pi)/2$, from (d) and (23) we can easily obtain (6), (7), (8) and (9). Moreover, from (1), (2), (23) and (24), we find that

$$f_1(z) = \exp(2\pi i\delta_1 z) \prod_{m|M} \prod_{n=1}^{\infty} (1-q(\lambda_1)^{mn})^{c(m)}$$

and

$$f_2(z) = \exp(2\pi i\delta_2 z + B) \prod_{m|M} \prod_{n=1}^{\infty} (1-q(\lambda_2)^{mn})^{c(M/m)},$$

which yield (4) and (5).

The remaining part of the theorem follows at once from Lemma 3, since $\varphi_1(s)$ and $\varphi_2(s)$ are given by (23) and (24), respectively.

5. Proofs of corollaries. Corollary 1 is a direct consequence of the theorem. Now we shall prove Corollary 2. From the theorem and its proof, we know that M is a positive integer, $\varphi_1(s)$ and $\varphi_2(s)$ are given by (23) and (24), respectively. If $a_1(n) = a_2(n)$ for all n, then $\varphi_1(s) = \varphi_2(s)$, so that $c(m) = c(M/m)$ for any divisor m of M. Since $M = \lambda^2 N$, from (9) we have

$$\begin{aligned} 2B &= \tfrac{1}{2}(\log M)\sum_{m|M} c(m) - \sum_{m|M} c(m)\log m \\ &= \tfrac{1}{2}(\log M)\sum_{m|M} c(m) - \sum_{m|M} c(M/m)\log m \\ &= \tfrac{1}{2}(\log M)\sum_{m|M} c(m) - \sum_{m|M} c(m)\log(M/m) = -2B, \end{aligned}$$

which implies $B = 0$. Thus Corollary 2 follows easily from the theorem. Corollary 3 is an immediate consequence of Corollary 2.

6. Proofs of remarks. By (23), (24) and Möbius' inversion formula, we can easily show Remark 1. Now we shall prove Remark 2. As is easily verified, $\vartheta(z)$ may be written as

$$\vartheta(z) = \prod_{n=1}^{\infty} \frac{(1-q(2)^{2n})^5}{(1-q(2)^n)^2(1-q(2)^{4n})^2}.$$

Hence we have

$$\vartheta(z) = \prod_{n=1}^{\infty} (1-q(2)^n)^{a(n)},$$

where

$$a(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4, \\ 3 & \text{if } n \equiv 2 \pmod 4, \\ -2 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = (-2+5\cdot 2^{-s}-2\cdot 4^{-s})\zeta(s).$$

Since, by (23),

$$\varphi(s) = (c(1)+c(2)2^{-s}+c(4)4^{-s})\zeta(s),$$

we find that $c(1) = c(4) = -2$ and $c(2) = 5$. Then, by the theorem, the remark is proven.

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