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INSTITUT FÜR MATHEMATIK
KARL-FRANZENS-UNIVERSITÄT
Halbärthgasse 1/1
A-8010 Graz

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A generalization of Bombieri's prime number theorem to algebraic number fields

by

JÜRGEN G. HINZ (Marburg)

1. Introduction. For many important problems in the theory of numbers we need some information about the average distribution of primes in arithmetic progressions. It is convenient to introduce the classical device of "weighting" the primes with von Mangoldt's function $\Lambda(m)$. Let

$$\psi(y;q,l)=\sum_{\substack{m\leqslant y\\ m\equiv l \bmod q}}\Lambda(m). \tag{1.1}$$

We ask for inequalities of type ($A>0$ arbitrary but fixed)

$$\sum_{q\leqslant Q=Q(x)}\max_{y\leqslant x}\max_{(l,q)=1}\left|\psi(y;q,l)-\frac{y}{\varphi(q)}\right|\ll x(\log x)^{-A}. \tag{1.2}$$

The first attempt to obtain a "non-trivial" estimate of this kind was made by Renyi. He showed that (1.2) is true with $Q=x^a$ for some small positive a. Subsequent refinements of Bombieri [1] enable us to take $Q=x^{1/2}(\log x)^{-B}$ for some $B=B(A)>0$. A slightly weaker result has been derived independently by A. I. Vinogradov [18], using a different method. Gallagher [4] later introduced major simplifications in Bombieri's arguments. More recently Vaughan [17] developed an ingenious new method which differs significantly from all approaches used previously and which gives a still simpler proof by essentially elementary means.

The main advantage of Bombieri's theorem becomes clear, if we note that the classical prime number theorem of Page, Siegel and Walfisz only leads to the limit $Q=(\log x)^C$ for the moduli q in (1.2). Moreover, Bombieri's bound Q is as good, apart from the logarithmic factor, as one can obtain on the assumption of the generalized Riemann hypothesis.

Many important applications of (1.2) are to be found in the literature. The major results are too well known to need elaboration. In the sequel, let K be an algebraic number field of finite degree $n=r_1+2r_2$ (in the usual notation) over the rationals with discriminant d. Z_K will denote the ring of integers in K.

There are various ways in which (1.2) can be extended to K. The first generalization is due to Wilson [19] and Huxley [11]. Instead of rational prime numbers they consider prime ideals of K. Following Landau, the $\varphi(q)$ residue classes l mod q coprime to the modulus are then replaced by $h(\mathfrak{q})$ reduced narrow ideal classes H modulo an ideal $\mathfrak{q}$ of Z_K. In such terminology the counting function (1.1) is of the form

$$\Psi(y;\mathfrak{q},H) = {\sum_{\substack{N\mathfrak{a}\leqslant y\\ \mathfrak{a}\in H}}}^{*} \Lambda(\mathfrak{a}).$$

In the present state of knowledge it seems to be impossible to derive a Bombieri-type result for this function because of our ignorance about the number of reduced residue classes mod $\mathfrak{q}$ containing totally positive units.

Wilson and Huxley evade this difficulty by introducing weights of the form $h(\mathfrak{q})\Phi^{-1}(\mathfrak{q})$. Consequently, they only consider the "weighted" inequality

$$\sum_{N\mathfrak{q}\leqslant Q}\frac{h(\mathfrak{q})}{\Phi(\mathfrak{q})}\max_{y\leqslant x}\max_{H}\left|\Psi(y;\mathfrak{q},H)-\frac{y}{h(\mathfrak{q})}\right| \ll x(\log x)^{-A}. \tag{1.3}$$

In his paper Wilson proves (1.3) with $Q = x^{1/(n+1)}(\log x)^{-B(A)}$. Using another method Huxley shows that the above estimate is true even if $Q = x^{1/2}(\log x)^{-B(A)}$.

Other types of generalizations, not discussed here, have been considered by Fogels [3] in the special case of a quadratic number field and by Johnson [12] for imaginary quadratic fields.

In view of applications the following formulation of (1.2) in the language of algebraic number fields is of some interest.

Let $y_1, \ldots, y_{r+1}$, $r = r_1+r_2-1$, be positive real numbers and write $y = y_1\ldots y_{r+1}$. Consider the set $\mathfrak{R} = \mathfrak{R}(y_1, \ldots, y_{r+1})$ of integers $\alpha\in Z_K$ subject to the conditions

$$\begin{aligned} &0<\alpha^{(k)}\leqslant y_k, \quad k=1,\ldots,r_1,\\ &0<|\alpha^{(k)}|^2\leqslant y_k, \quad k=r_1+1,\ldots,r+1. \end{aligned} \tag{1.4}$$

It is convenient to assume for the computations that

$$c_1 y^{1/(r+1)} \leqslant y_k \leqslant c_2 y^{1/(r+1)}, \quad k=1,\ldots,r+1, \tag{1.5}$$

where the positive constants c_1, c_2 depend on the field K only. If the y_k do not obey the inequalities (1.5) one can restore (1.5) by multiplying the y_k by a suitably chosen totally positive unit of K (cf. [9], (13)).

An element $\omega\in Z_K$ is said to be a *prime number in K* or simply *prime*, if the principal ideal (ω) is a prime ideal. Let us denote by $\Pi(\mathfrak{R};\mathfrak{q},\gamma)$ the number of primes $\omega\in\mathfrak{R}$ satisfying $\omega\equiv\gamma \bmod \mathfrak{q}$, where $\mathfrak{q}$ is an integral ideal of K and $\gamma\in Z_K$. For brevity we put

$$I = I(y_1,\ldots,y_{r+1}) = \frac{w}{2^{r_1}hR}\int_2^{y_k}\!\!\ldots\!\int \frac{du_1\ldots du_{r+1}}{\log(u_1\ldots u_{r+1})}.$$

Here, w denotes the number of roots of unity, h the class number and R the regulator of the field K.

THEOREM. *Let $x_1,\ldots,x_{r+1}$ be positive real numbers, and write x for the product $x_1\ldots x_{r+1}$. Then, for any constant $A>0$,*

$$\sum_{N\mathfrak{q}\leqslant Q}\max_{2\leqslant y_k\leqslant x_k}\max_{(\gamma,\mathfrak{q})=1}\left|\Pi(\mathfrak{R};\mathfrak{q},\gamma)-\frac{I}{\Phi(\mathfrak{q})}\right| \ll x(\log x)^{-A}, \tag{1.6}$$

provided that

$$Q = \begin{cases} x^{1/2}(\log x)^{-B} & \textit{for some } B=B(A,n)>0, \textit{ if } r_2=0,\\ x^{(r+5/2)^{-1}-\varepsilon} & \textit{for any } \varepsilon>0, \textit{ if } r_2>0. \end{cases} \tag{1.7}$$

As is customary in analytic number theory we introduce the sum

$$\Psi(\mathfrak{R};\mathfrak{q},\gamma) = \sum_{\substack{\alpha\in\mathfrak{R}\\ \alpha\equiv\gamma \bmod \mathfrak{q}}}\Lambda(\alpha) = \sum_{\substack{\alpha\in\mathfrak{R},(\alpha)=\mathfrak{p}^m\\ \alpha\equiv\gamma \bmod \mathfrak{q}}}\log N\mathfrak{p},$$

since there are certain technical advantages in doing so. The transition to the function $\Pi(\mathfrak{R};\mathfrak{q},\gamma)$ results from Grotz's formula [5] of the multidimensional partial summation in K.

We continue by making a few remarks about the proof of (1.6). First the method is influenced by Vaughan's work [17] on the corresponding result in the rational case. The principle underlying the treatment of Bombieri's theorem is that of the large sieve. Here, the basic inequality of the large sieve method in K is required in a new form which has been derived by the author in [10]. Moreover, the Pólya–Vinogradov character sum estimate is an essential ingredient in Vaughan's proof of (1.2). By appealing to the Siegel–Grotz summation formula the author [9] succeeded in extending this inequality to K. Finally, we introduce the following version of Vaughan's identity

$$\zeta_K'/\zeta_K = (\zeta_K'/\zeta_K+F)(1-\zeta_K G)+(\zeta_K'+\zeta_K F)G-F, \tag{1.8}$$

where F, G are partial sums of the Dirichlet series for $-\zeta_K'/\zeta_K$, $1/\zeta_K$ respectively. ζ_K denotes Dedekind's zeta-function in K. It turns out that some additional difficulties arise in connection with the application of (1.8) to sums over algebraic integers instead of ideals. This problem does not occur in the rational case.

Inequality (1.6) plays an important part in many of the most significant applications of sieve theory in K, and we mention just two. Our motivation for studying estimates of the form (1.6) comes from this observation.

First, we ask for the order of magnitude of the sum
$$\sum_{\omega\in\mathfrak{R},\,\omega\ \mathrm{prime}} D(\omega-1), \quad \text{where } D(\alpha)=\sum_{\mathfrak{q}\mid(\alpha)} 1.$$
This problem used to be known as the Titchmarsh divisor problem. Generally, we obtain
$$y \ll \sum_{\omega\in\mathfrak{R},\,\omega\ \mathrm{prime}} D(\omega-1) \ll y.$$
Specially, in a totally real field K, we can derive the asymptotic formula
$$\sum_{\omega\in\mathfrak{R},\,\omega\ \mathrm{prime}} D(\omega-1)=\frac{1}{|\sqrt{d}|}\prod_{\mathfrak{p}}\left(1+\frac{1}{N\mathfrak{p}(N\mathfrak{p}-1)}\right)\cdot y+O\left(\frac{y\cdot\log\log y}{\log y}\right).$$

The second application is connected with the binary Goldbach problem. In a totally real number field inequality (1.6) enables us to show that every totally positive even integer with sufficiently large norm is the sum of a totally positive prime and a totally positive integer having at most three prime ideal factors. Let us describe the result as (1, 3). Even in the general case of an arbitrary number field it is possible to obtain for the first time a result of type $(1, b)$, where b is a natural number depending only on r. We recall that an element $\alpha\in Z_K$ is said to be *even*, if $\alpha\neq 0$ and all prime ideals $\mathfrak{p}$ with $N\mathfrak{p}=2$ divide (α).

The numbers $c_3, \ldots, c_8$ coming up in the sequel are positive constants which depend at most on the field K. Throughout the paper, small German letters stand for integral ideals of K; particularly, $\mathfrak{p}$ always denotes a prime ideal. The letter ω will be used for primes in K only. Finally, ε is an arbitrarily small positive real constant that is not necessarily the same at each occurrence.

2. Reduction of the problem. We start from the relation
$$\Pi(\mathfrak{R};\,\mathfrak{q},\gamma)=\frac{1}{\Phi(\mathfrak{q})}\sum_{\chi \bmod \mathfrak{q}}\bar{\chi}(\gamma)\sum_{\omega\in\mathfrak{R}}\chi(\omega), \qquad (\gamma,\mathfrak{q})=1,$$
where the first sum on the right is over all characters χ of the multiplicative group of non-zero residue classes mod $\mathfrak{q}$. The contribution of the principal character χ_0 provides the main term. It is obvious that
$$\sum_{\omega\in\mathfrak{R}}\chi_0(\omega)=\sum_{\omega\in\mathfrak{R}}1+O\Big(\sum_{\substack{\omega\in\mathfrak{R}\\ \omega\mid\mathfrak{q}}}1\Big).$$
As for the error term, let $\omega\in\mathfrak{R}$ be kept fixed. Then, by (1.4), we have to estimate the number $L(\omega)$ of units η satisfying
$$0\leqslant\eta^{(k)}\omega^{(k)}\leqslant y_k, \quad k=1,\ldots,r_1,$$
$$0<|\eta^{(k)}\omega^{(k)}|^2\leqslant y_k, \quad k=r_1+1,\ldots,r+1.$$

It is easy to confirm (see e.g. [16], p. 269) that

(2.1) $$L(\omega)\ll\log^r y$$

and therefore
$$\sum_{\omega\in\mathfrak{R}}\chi_0(\omega)=\sum_{\omega\in\mathfrak{R}}1+O(\log^r y\cdot\log N\mathfrak{q}).$$
Now we can apply Mitsui's extension of the prime number theorem to K, namely a very special case of the Main Theorem in [14],
$$\sum_{\omega\in\mathfrak{R}}1=I+O\{y\cdot\exp(-c_3(\log y)^{1/2})\}.$$
Hence
$$\Pi(\mathfrak{R};\,\mathfrak{q},\gamma)-\frac{I}{\Phi(\mathfrak{q})}\ll\frac{1}{\Phi(\mathfrak{q})}\sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi\neq\chi_0}}\Big|\sum_{\omega\in\mathfrak{R}}\chi(\omega)\Big|$$
$$+\log^r y\cdot\log N\mathfrak{q}+\frac{y}{\Phi(\mathfrak{q})}\exp(-c_3(\log y)^{1/2}).$$
Since $N\mathfrak{q}\leqslant Q\leqslant x^{1/2}$, this implies that
$$\max_{2\leqslant y_k\leqslant x_k}\max_{(\gamma,\mathfrak{q})=1}\left|\Pi(\mathfrak{R};\,\mathfrak{q},\gamma)-\frac{I}{\Phi(\mathfrak{q})}\right|$$
$$\ll\frac{1}{\Phi(\mathfrak{q})}\sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi\neq\chi_0}}\max_{2\leqslant y_k\leqslant x_k}\Big|\sum_{\omega\in\mathfrak{R}}\chi(\omega)\Big|+\frac{x}{\Phi(\mathfrak{q})}\exp(-c_3(\log x)^{1/2}).$$
Summation of the latter term over $N\mathfrak{q}\leqslant Q$ gives a contribution of order less than $x(\log x)^{-A}$. It remains to consider the expression

(2.2) $$\sum_{N\mathfrak{q}\leqslant Q}\frac{1}{\Phi(\mathfrak{q})}\sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi\neq\chi_0}}\max_{2\leqslant y_k\leqslant x_k}\Big|\sum_{\omega\in\mathfrak{R}}\chi(\omega)\Big|.$$

Each character $\chi\neq\chi_0$ occurring here is induced by a primitive character χ^* to a modulus $\mathfrak{q}^*$ satisfying $\mathfrak{q}^*\neq 1$, $\mathfrak{q}^*\mid\mathfrak{q}$. Since $\chi(\omega)=\chi^*(\omega)$ if $(\omega,\mathfrak{q})=1$, we have
$$\sum_{\omega\in\mathfrak{R}}\chi(\omega)-\sum_{\omega\in\mathfrak{R}}\chi^*(\omega)\ll\log^r y\cdot\log N\mathfrak{q},$$
arguing as in (2.1). Thus the difference contributes at most $\ll Q(\log x)^{r+1}$, which is negligible. Hence (2.2) can be replaced by
$$\sum_{1<N\mathfrak{q}\leqslant Q}\ \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}}\max_{2\leqslant y_k\leqslant x_k}\Big|\sum_{\omega\in\mathfrak{R}}\chi(\omega)\Big|\cdot\sum_{N\mathfrak{q}'\leqslant Q/N\mathfrak{q}}\frac{1}{\Phi(\mathfrak{q}\mathfrak{q}')},$$
where the asterisk indicates summation over primitive characters mod $\mathfrak{q}$.

Using the elementary fact that

$$\sum_{N\mathfrak{q}'\leqslant Q}\frac{1}{\Phi(\mathfrak{q}\mathfrak{q}')}\leqslant\frac{1}{\Phi(\mathfrak{q})}\sum_{N\mathfrak{q}'\leqslant Q}\frac{1}{\Phi(\mathfrak{q}')}\ll\frac{\log Q}{\Phi(\mathfrak{q})},$$

the above term is

$$\ll\log x\cdot\sum_{1<N\mathfrak{q}\leqslant Q}\frac{1}{\Phi(\mathfrak{q})}\sum_{\chi\bmod\mathfrak{q}}^{*}\max_{2\leqslant y_k\leqslant x_k}\Big|\sum_{\omega\in\mathfrak{R}}\chi(\omega)\Big|.$$

We now consider small and large values of $N\mathfrak{q}$ separately. If $1<N\mathfrak{q}\leqslant Q_1$, where $Q_1=(\log x)^C$ with a positive constant C that will be chosen later, Mitsui's generalized prime number theorem for arithmetic progressions [14] enables us to bound the sum under consideration. Since χ is a non-principal character mod $\mathfrak{q}$, we have

$$\begin{aligned}\sum_{\omega\in\mathfrak{R}}\chi(\omega)&=\sum_{\substack{\xi\bmod\mathfrak{q}\\(\xi,\mathfrak{q})=1}}\chi(\xi)\sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv\xi\bmod\mathfrak{q}}}1\\&=\frac{I}{\Phi(\mathfrak{q})}\sum_{\xi\bmod\mathfrak{q}}\chi(\xi)+O\big(y\cdot\exp(-c_4(\log y)^{1/2})\big)\\&\ll y\cdot\exp(-c_4(\log y)^{1/2}),\end{aligned}$$

which is acceptable as before. It remains to deal with the range $Q_1<N\mathfrak{q}\leqslant Q$. As we have indicated in the Introduction, it suffices to show that

$$(2.3)\qquad\sum_{Q_1<N\mathfrak{q}\leqslant Q}\frac{1}{\Phi(\mathfrak{q})}\sum_{\chi\bmod\mathfrak{q}}^{*}\max_{y_k\leqslant x_k}\Big|\sum_{\alpha\in\mathfrak{R}}\chi(\alpha)\Lambda(\alpha)\Big|\ll x(\log x)^{-A-r-2}.$$

The transition to an inequality for $\sum\chi(\omega)$, instead of for $\sum\chi(\alpha)\Lambda(\alpha)$, is essentially an exercise in multidimensional partial summation. First, we find by the argument that was used in (2.1)

$$\begin{aligned}(2.4)\qquad\sum_{\alpha\in\mathfrak{R}}\chi(\alpha)\Lambda(\alpha)&=\sum_{\substack{\alpha\in\mathfrak{R}\\(\alpha)=\mathfrak{p}^m}}\chi(\alpha)\log N\mathfrak{p}\\&=\sum_{\omega\in\mathfrak{R}}\chi(\omega)\log N\omega+O\Big\{\log^r y\cdot\sum_{\substack{N\mathfrak{p}^m\leqslant y\\m\geqslant2}}\log N\mathfrak{p}\Big\}.\end{aligned}$$

The last error term is $\ll y^{1/2}(\log y)^r$, which is acceptable for (2.3) provided that

$$(2.5)\qquad Q\leqslant x^{1/2}(\log x)^{-A-2r-2},$$

as we henceforth assume. Next, we make use of Grotz's version of the partial

summation in the setting of an algebraic number field. In our case we obtain

$$\begin{aligned}(2.6)\qquad\sum_{\omega\in\mathfrak{R}}\chi(\omega)&=\frac{1}{\log y}\sum_{\omega\in\mathfrak{R}}\chi(\omega)\log N\omega\\&+\sum_{j=1}^{r+1}(-1)^j\sum_{1<l_1<\ldots<l_j\leqslant r+1}\int_0^{y_{l_j}}\ldots\int_0^{y_{l_1}}\Big\{\sum_{\omega\in\mathfrak{R}}^{0}\chi(\omega)\log N\omega\Big\}\\&\times\frac{\partial^j\left(\dfrac{1}{\log\dfrac{u_{l_1}\ldots u_{l_j}y}{y_{l_1}\ldots y_{l_j}}}\right)}{\partial u_{l_1}\ldots\partial u_{l_j}}du_{l_1}\ldots du_{l_j}.\end{aligned}$$

The mark at the sign of summation indicates that the sum runs only over those $\omega\in\mathfrak{R}$ which satisfy (1.4) with $u_{l_1},\ldots,u_{l_j}$ in place of $y_{l_1},\ldots,y_{l_j}$ respectively.

Let us now study the integral in (2.6). We begin by remarking that

$$(2.7)\qquad u_{l_i}\geqslant 2y_{l_i}y^{-1}\quad\text{for } i=1,\ldots,j,$$

since otherwise the $\omega\in\mathfrak{R}$ counted in $\sum^0$ satisfy

$$N\omega=\omega^{(1)}\ldots\omega^{(r_1)}|\omega^{(r_1+1)}|^2\ldots|\omega^{(r+1)}|^2<2.$$

But this contradicts the fact that (ω) is a prime ideal in K. Next, it is easy to check that

$$\frac{\partial^j\left(\dfrac{1}{\log\dfrac{u_{l_1}\ldots u_{l_j}y}{y_{l_1}\ldots y_{l_j}}}\right)}{\partial u_{l_1}\ldots\partial u_{l_j}}=\frac{(-1)^j j!}{u_{l_1}\ldots u_{l_j}\left(\log\dfrac{u_{l_1}\ldots u_{l_j}y}{y_{l_1}\ldots y_{l_j}}\right)^{j+1}}.$$

Arguing in the same way as in (2.7) we obtain

$$\frac{u_{l_1}\ldots u_{l_j}y}{y_{l_1}\ldots y_{l_j}}\geqslant 2.$$

Now our problem has been reduced to that of establishing a suitable bound for the expression

$$\sum_{j=1}^{r+1}\sum_{1\leqslant l_1<\ldots<l_j\leqslant r+1}\sum_{Q_1<N\mathfrak{q}\leqslant Q}\frac{1}{\Phi(\mathfrak{q})}\sum_{\chi\bmod\mathfrak{q}}^{*}\max_{y_k\leqslant x_k}\int_{2y_{l_1}y^{-1}}^{y_{l_j}}\ldots\int\Big|\sum_{\omega\in\mathfrak{R}}^{0}\chi(\omega)\log N\omega\Big|\frac{du_{l_1}\ldots du_{l_j}}{u_{l_1}\ldots u_{l_j}}.$$

As for the domain of integration, we have

$$y_{l_i} \leqslant x_{l_i}, \qquad 2y_{l_i} y^{-1} = 2y_1^{-1} \dots y_{l_i-1}^{-1} y_{l_i+1}^{-1} \dots y_{r+1}^{-1} \geqslant 2x_{l_i} x^{-1}, \qquad i = 1, \dots, j.$$

In view of (2.4), an application of (2.3) leads us to the desired estimate, namely

$$\sum_{Q_1 < N\mathfrak{q} \leqslant Q} \frac{1}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k \leqslant x_k} \Big|\sum_{\omega \in \Re} \chi(\omega)\Big| \ll x (\log x)^{-A-1}.$$

3. Application of the large sieve inequality in K. We begin by quoting from [10] the author's version of the large sieve in the setting of an algebraic number field.

Let $\mathfrak{a}$ be an integral ideal of K and suppose that $Q \geqslant 1$. Then

$$(3.1) \qquad \sum_{N\mathfrak{q} \leqslant Q} \frac{N\mathfrak{q}}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \Big|\sum_{\alpha \in \Re}{}' c(\alpha)\chi(\alpha)\Big|^2 \ll \Big(Q^2 + \frac{x}{N\mathfrak{a}}\Big) \sum_{\alpha \in \Re}{}' |c(\alpha)|^2,$$

where the coefficients $c(\alpha)$ are arbitrary complex numbers. The dash indicates here and later that the sum is restricted to those $\alpha \in \Re = \Re(x_1, \dots, x_{r+1})$ which are divisible by $\mathfrak{a}$.

The object of this section is to apply (3.1) to the estimation of the following expression (3.3) that plays a fundamental role in the proof of (2.3).

Let $z_1, \dots, z_{r+1}$ be positive real numbers satisfying

$$(3.2) \qquad z^{1/(r+1)} \ll z_k \ll z^{1/(r+1)}, \qquad k = 1, \dots, r+1,$$

and suppose that $z_0 \leqslant z = z_1 \dots z_{r+1} \ll xy$. We require, in an obvious notation, a bound for

$$(3.3) \qquad \sum_{N\mathfrak{q} \leqslant Q} \frac{N\mathfrak{q}}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{z_k} \Big| \sum_{\alpha_1 \in \Re_1, \alpha_2 \in \Re_2}{}'' c_1(\alpha_1) c_2(\alpha_2) \chi(\alpha_1 \alpha_2)\Big|.$$

Here $\sum''$ is used to mean summation over such integers $\alpha_1 \in \Re_1 = \Re_1(x_1, \dots, x_{r+1})$, $\alpha_2 \in \Re_2 = \Re_2(y_1, \dots, y_{r+1})$ for which both

$$\alpha_j \equiv 0 \bmod \mathfrak{a}_j, \quad j = 1, 2,$$

and

$$(3.4) \qquad \begin{aligned} &0 < \alpha_1^{(k)} \alpha_2^{(k)} \leqslant z_k, \qquad k = 1, \dots, r_1; \\ &0 < |\alpha_1^{(k)} \alpha_2^{(k)}|^2 \leqslant z_k, \qquad k = r_1+1, \dots, r+1 \end{aligned}$$

are satisfied.

In order to attack (3.3) we first note that by (3.1) and Cauchy's inequality,

$$(3.5) \qquad \sum_{N\mathfrak{q} \leqslant Q} \frac{N\mathfrak{q}}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \Big|\sum_{\alpha_1 \in \Re_1}{}' \sum_{\alpha_2 \in \Re_2}{}' c_1(\alpha_1) c_2(\alpha_2) \chi(\alpha_1 \alpha_2)\Big|$$
$$\ll \Big(Q^2 + \frac{x}{N\mathfrak{a}_1}\Big)^{1/2} \Big(Q^2 + \frac{y}{N\mathfrak{a}_2}\Big)^{1/2} \Big\{\sum_{\alpha_1 \in \Re_1}{}' |c_1(\alpha_1)|^2\Big\}^{1/2} \Big\{\sum_{\alpha_2 \in \Re_2}{}' |c_2(\alpha_2)|^2\Big\}^{1/2}.$$

Let us now introduce the condition (3.4). To this end we require the following result (see [2], p. 165):

If $T \geqslant 1$, $b > 0$, and a is real, then

$$(3.6) \qquad \int_{-T}^{T} e^{iat} \frac{\sin bt}{t} dt = \begin{cases} \pi + O\big(T^{-1}(b - |a|)^{-1}\big) & \text{if } |a| \leqslant b, \\ O\big(T^{-1}(|a| - b)^{-1}\big) & \text{if } |a| > b. \end{cases}$$

We also make use of the trivial bound

$$(3.7) \qquad \int_{-T}^{T} e^{iat} \frac{\sin bt}{t} dt \ll \int_{0}^{T} \frac{|\sin bt|}{t} dt \ll \log 2T + |\log b|.$$

It is convenient at this point to introduce the notations ($j = 1, 2$)

$$S_j(t_1, \dots, t_{r+1}; \chi) = \sum_{\alpha_j \in \Re_j}{}' c_j(\alpha_j) \chi(\alpha_j) \prod_{k=1}^{r+1} |\alpha_j^{(k)}|^{-ie_k t_k},$$

where

$$e_k = \begin{cases} 1 & \text{for } k = 1, \dots, r_1, \\ 2 & \text{for } k = r_1+1, \dots, r+1. \end{cases}$$

In this terminology we find that

$$\int_{-T}^{T} \dots \int_{-T}^{T} S_1(t_1, \dots, t_{r+1}; \chi) S_2(t_1, \dots, t_{r+1}; \chi) \prod_{k=1}^{r+1} \frac{\sin(t_k \log z_k)}{\pi t_k} dt_1 \dots dt_{r+1}$$
$$= \sum_{\alpha_1 \in \Re_1}{}' \sum_{\alpha_2 \in \Re_2}{}' c_1(\alpha_1) c_2(\alpha_2) \chi(\alpha_1 \alpha_2) \prod_{k=1}^{r+1} \int_{-T}^{T} |\alpha_1^{(k)} \alpha_2^{(k)}|^{-ie_k t_k} \frac{\sin(t_k \log z_k)}{\pi t_k} dt_k.$$

We dissect the double sum over α_1, α_2 into four parts $\sum_1', \dots, \sum_4'$. The first sum $\sum_1'$ contains all $\alpha_j \in \Re_j$, $\alpha_j \equiv 0 \bmod \mathfrak{a}_j$, $j = 1, 2$, for which there exists $1 \leqslant l \leqslant r+1$ such that $|\alpha_1^{(l)} \alpha_2^{(l)}| < 1$. Taking account of (3.7), their contribution is

$$(3.8) \qquad \ll (\log 2T + \log\log xy)^{r+1} \sum_1{}' |c_1(\alpha_1)| |c_2(\alpha_2)|.$$

The sum $\sum_2'$ relates to the property that $\alpha_j \in \Re_j$, $\alpha_j \equiv 0 \bmod \mathfrak{a}_j$, $j = 1, 2$, satisfy the conditions

$$1 \leqslant |\alpha_1^{(k)} \alpha_2^{(k)}|^{e_k} \leqslant z_k - 1, \qquad k = 1, \dots, r+1.$$

For these values we use (3.6) which yields

$$\sum_2' c_1(\alpha_1)c_2(\alpha_2)\chi(\alpha_1\alpha_2)\prod_{k=1}^{r+1}\left\{1+O\left(T^{-1}\left(\log\frac{z_k}{|\alpha_1^{(k)}\alpha_2^{(k)}|^{e_k}}\right)^{-1}\right)\right\}.$$

We have now only to observe that, in view of (3.2),

$$\log z_k - e_k\log|\alpha_1^{(k)}\alpha_2^{(k)}| \geqslant \frac{z_k-|\alpha_1^{(k)}\alpha_2^{(k)}|^{e_k}}{z_k} \geqslant z_k^{-1} \geqslant (xy)^{-1/(r+1)}.$$

This leads us to the expression

$$\sum_2' c_1(\alpha_1)c_2(\alpha_2)\chi(\alpha_1\alpha_2)\left\{1+O\left(\frac{(xy)^{1/(r+1)}}{T}\right)\right\}, \tag{3.9}$$

provided that

$$T \geqslant (xy)^{1/(r+1)}. \tag{3.10}$$

In $\sum_3'$ we group together all α_1, α_2 under consideration which satisfy, in addition,

$$|\alpha_1^{(l)}\alpha_2^{(l)}|^{e_l} > z_l+1$$

for at least one integer l with $1 \leqslant l \leqslant r+1$. In conjunction with (3.6) and (3.7), this sum gives a contribution of order

$$\ll \frac{(xy)^{1/(r+1)}}{T}(\log 2T+\log\log xy)^r \sum_3' |c_1(\alpha_1)||c_2(\alpha_2)|. \tag{3.11}$$

The remaining sum $\sum_4'$ leads, by (3.7), to

$$\ll (\log 2T+\log\log xy)^{r+1}\sum_4' |c_1(\alpha_1)||c_2(\alpha_2)|.$$

If we combine this with (3.8), (3.9) and (3.11), subject to (3.10), we arrive readily at

$$\int_{-T}^{T}\ldots\int_{-T}^{T} S_1(t_1,\ldots,t_{r+1};\chi)S_2(t_1,\ldots,t_{r+1};\chi)\prod_{k=1}^{r+1}\frac{\sin(t_k\log z_k)}{\pi t_k}dt_1\ldots dt_{r+1}$$

$$= \sum_{\alpha_1\in\mathfrak{N}_1,\alpha_2\in\mathfrak{N}_2}'' c_1(\alpha_1)c_2(\alpha_2)\chi(\alpha_1\alpha_2)$$

$$+O\left\{\frac{(xy)^{1/(r+1)}}{T}(\log 2T+\log\log xy)^r \sum_{\alpha_1\in\mathfrak{N}_1}'\sum_{\alpha_2\in\mathfrak{N}_2}' |c_1(\alpha_1)||c_2(\alpha_2)|\right\}$$

$$+O\left\{(\log 2T+\log\log xy)^{r+1}\left(\sum_1' |c_1(\alpha_1)||c_2(\alpha_2)|+\sum_4' |c_1(\alpha_1)||c_2(\alpha_2)|\right)\right\}.$$

As for the integral, we have

$$\sin(t_k\cdot\log z_k) \ll \min(1, |t_k|\log xy),$$

so that the left-hand side above can be bounded by

$$\ll \int_{-T}^{T}\ldots\int_{-T}^{T} |S_1(t_1,\ldots,t_{r+1};\chi)S_2(t_1,\ldots,t_{r+1};\chi)|$$

$$\times\prod_{k=1}^{r+1}\min\left(\frac{1}{|t_k|}, \log xy\right)dt_1\ldots dt_{r+1}.$$

Making use of (3.5) and Cauchy's inequality the expression (3.3) we wish to estimate is at most

$$\ll \left(Q^2+\frac{x}{N\mathfrak{a}_1}\right)^{1/2}\left(Q^2+\frac{y}{N\mathfrak{a}_2}\right)^{1/2}\Big\{\sum_{\alpha_1\in\mathfrak{N}_1}' |c_1(\alpha_1)|^2\Big\}^{1/2}\Big\{\sum_{\alpha_2\in\mathfrak{N}_2}' |c_2(\alpha_2)|^2\Big\}^{1/2}$$

$$\times\prod_{k=1}^{r+1}\int_{-T}^{T}\min\left(\frac{1}{|t_k|}, \log xy\right)dt_k$$

$$+Q^2T^{-1}(xy)^{\frac{1}{2}+\frac{1}{r+1}}(\log 2T+\log\log xy)^r\Big\{\sum_{\alpha_1\in\mathfrak{N}_1}' |c_1(\alpha_1)|^2\Big\}^{1/2}\Big\{\sum_{\alpha_2\in\mathfrak{N}_2}' |c_2(\alpha_2)|^2\Big\}^{1/2}$$

$$+Q^2(\log 2T+\log\log xy)^{r+1}\left(\sum_1' |c_1(\alpha_1)||c_2(\alpha_2)|+\sum_4' |c_1(\alpha_1)||c_2(\alpha_2)|\right).$$

In accordance with (3.10) we choose

$$T = (xy)^{1/2+1/(r+1)},$$

so that finally

$$\sum_{N\mathfrak{q}\leqslant Q}\frac{N\mathfrak{q}}{\Phi(\mathfrak{q})}\sum_{\chi \bmod \mathfrak{q}}^{*}\max_{z_k}\Big|\sum_{\alpha_1\in\mathfrak{N}_1,\alpha_2\in\mathfrak{N}_2}'' c_1(\alpha_1)c_2(\alpha_2)\chi(\alpha_1\alpha_2)\Big| \tag{3.12}$$

$$\ll \left(Q^2+\frac{x}{N\mathfrak{a}_1}\right)^{1/2}\left(Q^2+\frac{y}{N\mathfrak{a}_2}\right)^{1/2}(\log xy)^{r+1}\Big\{\sum_{\alpha_1\in\mathfrak{N}_1}' |c_1(\alpha_1)|^2\Big\}^{1/2}\Big\{\sum_{\alpha_2\in\mathfrak{N}_2}' |c_2(\alpha_2)|^2\Big\}^{1/2}$$

$$+Q^2(\log xy)^{r+1}\left(\sum_1' |c_1(\alpha_1)||c_2(\alpha_2)|+\sum_4' |c_1(\alpha_1)||c_2(\alpha_2)|\right).$$

It should be noted that the second error term on the right of (3.12) does not occur in the rational case, where we may assume without loss of generality that z is of the form $z = g+\frac{1}{2}$, $g\in \mathbf{Z}$.

4. Vaughan's identity in K and proof of the Theorem. Our object in this section will be to prove (2.3). We begin by explaining how Vaughan's identity can be extended to K. Let $\zeta_K(s)$ denote Dedekind's zeta-function in K. With a parameter $U \geqslant 1$ we introduce two sums

$$F(s) = \sum_{N\mathfrak{a} \leqslant U} \Lambda(\mathfrak{a})(N\mathfrak{a})^{-s}, \qquad G(s) = \sum_{N\mathfrak{b} \leqslant U} \mu(\mathfrak{b})(N\mathfrak{b})^{-s}$$

which approximate to $-\zeta_K'/\zeta_K$ and $1/\zeta_K$ respectively. Following Vaughan, we note the identity

$$-\frac{\zeta_K'}{\zeta_K}(s) = F(s) - \zeta_K(s)F(s)G(s) - \zeta_K'(s)G(s) + \left\{-\frac{\zeta_K'}{\zeta_K}(s) - F(s)\right\}\{1-\zeta_K(s)G(s)\},$$

valid for $\operatorname{Re} s > 1$. Calculating the Dirichlet series coefficients of the four functions on the right-hand side, we find that

$$\Lambda(\mathfrak{a}) = a_1(\mathfrak{a}) + a_2(\mathfrak{a}) + a_3(\mathfrak{a}) + a_4(\mathfrak{a}), \tag{4.1}$$

where

$$a_1(\mathfrak{a}) = \begin{cases} \Lambda(\mathfrak{a}) & \text{if } N\mathfrak{a} \leqslant U, \\ 0 & \text{if } N\mathfrak{a} > U; \end{cases}$$

$$a_2(\mathfrak{a}) = -\sum_{\substack{N\mathfrak{b} \leqslant U, N\mathfrak{c} \leqslant U \\ \mathfrak{b}\mathfrak{c} \mid \mathfrak{a}}} \Lambda(\mathfrak{b})\mu(\mathfrak{c});$$

$$a_3(\mathfrak{a}) = \sum_{\substack{N\mathfrak{b} \leqslant U \\ \mathfrak{b}\mathfrak{c} = \mathfrak{a}}} \mu(\mathfrak{b}) \cdot \log N\mathfrak{c}$$

and

$$a_4(\mathfrak{a}) = -\sum_{\substack{\mathfrak{b}\mathfrak{c} = \mathfrak{a} \\ N\mathfrak{b} > U, N\mathfrak{c} > 1}} \Lambda(\mathfrak{b}) \sum_{\substack{N\mathfrak{d} \leqslant U \\ \mathfrak{d} \mid \mathfrak{c}}} \mu(\mathfrak{d}).$$

Formula (4.1) leads us to the following decomposition of the inner sum in (2.3)

$$\sum_{\alpha \in \mathfrak{R}} \chi(\alpha)\Lambda(\alpha) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_j = \sum_{\alpha \in \mathfrak{R}} \chi(\alpha) a_j((\alpha)), \quad j = 1, \ldots, 4.$$

In S_1 we merely must apply (2.1) to obtain

$$S_1 = \sum_{\substack{\alpha \in \mathfrak{R} \\ N\alpha \leqslant U}} \chi(\alpha)\Lambda(\alpha) \ll \log^r x \cdot \sum_{N\mathfrak{a} \leqslant U} \Lambda(\mathfrak{a}) \ll U \log^r x. \tag{4.2}$$

The second sum S_2 we write in the form

$$S_2 = -\sum_{N\mathfrak{a} \leqslant U^2} \sum_{\substack{N\mathfrak{b}, N\mathfrak{c} \leqslant U \\ \mathfrak{b}\mathfrak{c} = \mathfrak{a}}} \Lambda(\mathfrak{b})\mu(\mathfrak{c}) \sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{a}}} \chi(\alpha).$$

Then, by appealing to the author's version of the Pólya–Vinogradov inequality in K (see [9], (15)),

$$S_2 \ll_\varepsilon N\mathfrak{q}^{\frac{1}{2(r+2)}} x^{1-\frac{1}{r+2}+\varepsilon} \sum_{N\mathfrak{a} \leqslant U^2} \sum_{\mathfrak{b} \mid \mathfrak{a}} \Lambda(\mathfrak{b}) \ll U^2 N\mathfrak{q}^{\frac{1}{2(r+2)}} x^{1-\frac{1}{r+2}+\varepsilon} \log U, \tag{4.3}$$

for any real $\varepsilon > 0$. To deal with the sum S_3 we first note that

$$S_3 = \sum_{N\mathfrak{b} \leqslant U} \mu(\mathfrak{b}) \sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{b}}} \chi(\alpha)\{\log N\alpha - \log N\mathfrak{b}\}. \tag{4.4}$$

Proceeding now in a way that is similar to, but much easier than, that at the ending of Section 2 we obtain .

$$\sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{b}}} \chi(\alpha) \log N\alpha = \log y \cdot \sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{b}}} \chi(\alpha) - \sum_{l=1}^{r+1} \int_{y_l y^{-1}}^{y_l} \Big\{ \sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{b}}}^{0} \chi(\alpha) \Big\} \frac{du_l}{u_l}.$$

Finally, again by the Pólya–Vinogradov inequality in K, it is easily seen that the bound for S_2 also holds for S_3. Accordingly (4.2) and (4.3) leads us to the estimate

$$\sum_{Q_1 < N\mathfrak{q} \leqslant Q} \frac{1}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k \leqslant x_k} |S_1 + S_2 + S_3| \ll U^2 Q^{1+\frac{1}{2(r+2)}} x^{1-\frac{1}{r+2}+\varepsilon} \log U. \tag{4.5}$$

Choosing

$$U = x^\varepsilon, \tag{4.6}$$

we find that the error term due to (4.5) may be absorbed into that required in (2.3) if

$$Q \leqslant x^{(r+5/2)^{-1}-\varepsilon}. \tag{4.7}$$

As for the special case of a totally real algebraic number field, we choose

$$U = (\log x)^D, \quad D \geqslant 1, \tag{4.8}$$

to be specified later.

If $y < QN\mathfrak{a}$, then by a crude estimate,

$$\sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha \equiv 0 \bmod \mathfrak{a}}} \chi(\alpha) \ll \frac{y}{N\mathfrak{a}} + 1 \ll Q.$$

In the second case $QN\mathfrak{a} \leqslant y$ we apply (23) of [9] which leads us to the upper bound

$$\sum_{\substack{\alpha\in\mathfrak{R}\\ \alpha\equiv 0 \bmod \mathfrak{a}}} \chi(\alpha) \ll (QN\mathfrak{a})^{1-1/2(n+1)}\cdot y^{\varepsilon}, \qquad N\mathfrak{q}\leqslant Q.$$

Hence we obtain in the present situation

$$\sum_{Q_1<N\mathfrak{q}\leqslant Q} \frac{1}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k\leqslant x_k} |S_1+S_2+S_3| \ll QU\log^r x + Q^2U^2\log U + Q^{2-1/2(n+1)}U^{4-1/(n+1)}x^{\varepsilon}.$$

This bound is admissible provided that

(4.9) $$Q^2 \leqslant x(\log x)^{-A-n-2D-2}.$$

Now our problem has been reduced to showing that

(4.10) $$\sum_{Q_1<N\mathfrak{q}\leqslant Q} \frac{1}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k\leqslant x_k} |S_4| \ll x(\log x)^{-A-r-2}.$$

The proof of (4.10) is longer, constituting in fact one of the main difficulties to be overcome. We transform the left-hand side of (4.10) into an expression in preparation for the application of the large sieve inequality (3.12). First we note that the above sum over $N\mathfrak{q}$ can be included in a sum of $\ll \log x$ expressions, each of the form

$$\frac{2}{P} \sum_{P/2<N\mathfrak{q}\leqslant P} \frac{N\mathfrak{q}}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k\leqslant x_k} |S_4|,$$

where $Q_1 < P \leqslant Q$. Our main problem is to develop S_4 into a more convenient form. To this end we use

$$\sum_{\mathfrak{d}|\mathfrak{c},\, N\mathfrak{d}\leqslant U} \mu(\mathfrak{d}) = 0 \qquad \text{for } 1 < N\mathfrak{c} \leqslant U,$$

so that

$$S_4 = -\sum_{U<N\mathfrak{a}\leqslant y/U} \Lambda(\mathfrak{a}) \sum_{\substack{\alpha\in\mathfrak{R}\\ \alpha\equiv 0 \bmod \mathfrak{a}\\ N\alpha > UN\mathfrak{a}}} \chi(\alpha) \sum_{\substack{N\mathfrak{c}\leqslant U\\ \mathfrak{c}|(\alpha/\mathfrak{a})}} \mu(\mathfrak{c}).$$

It is necessary to decompose the sum over $N\mathfrak{a}$ into $\ll \log x$ sums of the type

$$\sum_{M/2<N\mathfrak{a}\leqslant M}, \qquad \text{where } U < M = 2^m \leqslant y/U \leqslant x/U.$$

From our point of view, this representation is of little use as the ideals $\mathfrak{a}$ need not be principal. We may, however, use the following technique to carry out the reduction from ideals to algebraic integers. Let $\mathfrak{a}$ be an integral ideal with norm $M/2 < N\mathfrak{a} \leqslant M$ belonging to a given narrow ideal class $\mathfrak{C}$ modulo (1). Then the conditions of summation $\alpha\equiv 0 \bmod \mathfrak{a}$, α totally positive imply

that $(\alpha) = \mathfrak{a}\mathfrak{b}$ with $\mathfrak{b}\in\mathfrak{C}^{-1}$. Next, we choose, once and for all, fixed prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ satisfying

(4.11) $$P^{r+1} < N\mathfrak{p}_j \leqslant 2P^{r+1}, \qquad j = 1, 2,$$

in the classes $\mathfrak{C}^{-1}$, $\mathfrak{C}$ respectively. This can clearly be done in view of the prime ideal theorem for ideal classes modulo (1) (see e.g. [13], Satz LXXXV). We note that

(4.12) $$(\mathfrak{p}_1, \mathfrak{q}) = (\mathfrak{p}_2, \mathfrak{q}) = 1 \qquad \text{for all ideals } \mathfrak{q} \text{ with norm } P/2 < N\mathfrak{q} \leqslant P.$$

The product $\mathfrak{p}_1\cdot\mathfrak{p}_2$ is a principal ideal. By the theory of units there exists a totally positive generator ϱ_0 of $\mathfrak{p}_1\cdot\mathfrak{p}_2$ such that

(4.13) $$c_5(N\mathfrak{p}_1 N\mathfrak{p}_2)^{1/(r+1)} \leqslant |\varrho_0^{(k)}|^{e_k} \leqslant c_6(N\mathfrak{p}_1 N\mathfrak{p}_2)^{1/(r+1)}, \qquad k = 1, \ldots, r+1.$$

Hence

$$(\varrho_0\alpha) = (\mathfrak{p}_1\mathfrak{a})(\mathfrak{p}_2\mathfrak{b}) = (\alpha_1)(\beta), \qquad \text{say}.$$

Again, we may assume that α_1, a representative totally positive generator of $\mathfrak{p}_1\mathfrak{a}$, satisfies

(4.14) $$c_7(N\alpha_1)^{1/(r+1)} \leqslant |\alpha_1^{(k)}|^{e_k} \leqslant c_8(N\alpha_1)^{1/(r+1)}, \qquad k = 1, \ldots, r+1.$$

Summarizing, we have shown that, if $\alpha\in\mathfrak{R}$, $\alpha\equiv 0 \bmod \mathfrak{a}$, then $\varrho_0\alpha = \alpha_1\alpha_2$, where $\alpha_j \equiv 0 \bmod \mathfrak{p}_j$, $j = 1, 2$ and

(4.15) $$0 < |\alpha_1^{(k)}\alpha_2^{(k)}|^{e_k} \leqslant |\varrho_0^{(k)}|^{e_k} y_k,$$

(4.16) $$0 < |\alpha_2^{(k)}|^{e_k} \leqslant \frac{c_2c_6}{c_7}\left(\frac{2xN\mathfrak{p}_2}{M}\right)^{1/(r+1)}, \qquad k = 1, \ldots, r+1.$$

Furthermore, by (4.12),

$$\chi(\alpha) = \chi(\alpha_1)\chi(\alpha_2)\bar{\chi}(\varrho_0).$$

Returning to S_4 we infer at once from these transformations that

$$S_4 = -\sum_m \sum_{\mathfrak{C}} \bar{\chi}(\varrho_0) \sum_{\substack{\alpha_1\in\mathfrak{R}_1^{\times},\, \alpha_2\in\mathfrak{R}_2\\ N\alpha_2 > UN\mathfrak{p}_2}}^{\prime\prime} \chi(\alpha_1\alpha_2)\Lambda\left(\frac{\alpha_1}{\mathfrak{p}_1}\right) \sum_{\substack{N\mathfrak{c}\leqslant U\\ \mathfrak{c}|(\alpha_2/\mathfrak{p}_2)}} \mu(\mathfrak{c}),$$

where $\mathfrak{R}_2$ is defined by (4.16) and where $\mathfrak{R}_1^{\times}$ denotes a set of mod (1) nonassociated numbers $\alpha_1 \in Z_K$ satisfying (4.14) and $(M/2)N\mathfrak{p}_1 < N\alpha_1 \leqslant MN\mathfrak{p}_1$, $M = 2^m$. $\sum''$ again signifies (cf. (3.3)) that both $\alpha_j \equiv 0 \bmod \mathfrak{p}_j$, $j = 1, 2$ and (4.15) are satisfied.

Let us introduce an abbreviated notation for some of the terms occurring above. First, we put for $\alpha_2 \in \mathfrak{R}_2$, $\alpha_2 \equiv 0 \bmod \mathfrak{p}_2$,

$$(4.17)\qquad c_2(\alpha_2) = \begin{cases} \sum\limits_{N\mathfrak{c} \leqslant U, \mathfrak{c} | (\alpha_2/\mathfrak{p}_2)} \mu(\mathfrak{c}), & \text{if } N\alpha_2 > UN\mathfrak{p}_2, \\ 0, & \text{otherwise.} \end{cases}$$

We next define a set $\mathfrak{R}_1$ of totally positive integers $\alpha_1 \in Z_K$ subject to the conditions

$$(4.18)\qquad 0 < |\alpha_1^{(k)}|^{e_k} \leqslant c_8 (MN\mathfrak{p}_1)^{1/(r+1)}, \quad k = 1, \ldots, r+1.$$

Finally we set for $\alpha_1 \in \mathfrak{R}_1$, $\alpha_1 \equiv 0 \bmod \mathfrak{p}_1$,

$$(4.19)\qquad c_1(\alpha_1) = \begin{cases} \Lambda\left(\dfrac{\alpha_1}{\mathfrak{p}_1}\right), & \text{if } \alpha_1 \in \mathfrak{R}_1^{\times}, \\ 0, & \text{if } \alpha_1 \in \mathfrak{R}_1 \setminus \mathfrak{R}_1^{\times}. \end{cases}$$

Then we have

$$S_4 = -\sum_m \sum_{\mathfrak{C}} \bar{\chi}(\varrho_0) \sum_{\alpha_1 \in \mathfrak{R}_1, \alpha_2 \in \mathfrak{R}_2}'' c_1(\alpha_1) c_2(\alpha_2) \chi(\alpha_1 \alpha_2).$$

If now we appeal to (3.12), (4.11), (4.15), (4.16) and (4.18), we are able to conclude that

$$(4.20)\qquad \frac{1}{P} \sum_{P/2 < N\mathfrak{q} \leqslant P} \frac{N\mathfrak{q}}{\Phi(\mathfrak{q})} \sum_{\chi \bmod \mathfrak{q}}^{*} \max_{y_k \leqslant x_k} |S_4|$$

$$\ll \frac{1}{P} (\log x)^{r+1} \sum_m \sum_{\mathfrak{C}} (P^2+M)^{1/2} \left(P^2 + \frac{x}{M}\right)^{1/2} \Big\{\sum_{\alpha_1 \in \mathfrak{R}_1}' |c_1(\alpha_1)|^2\Big\}^{1/2}$$

$$\times \Big\{\sum_{\alpha_2 \in \mathfrak{R}_2}' |c_2(\alpha_2)|^2\Big\}^{1/2}$$

$$+ P(\log x)^{r+1} \sum_m \sum_{\mathfrak{C}} \Big(\sum\nolimits_1' |c_1(\alpha_1)|\, |c_2(\alpha_2)| + \sum\nolimits_4' |c_1(\alpha_1)|\, |c_2(\alpha_2)|\Big).$$

By standard calculations it follows at once from (4.19) that

$$\sum_{\alpha_1 \in \mathfrak{R}_1}' |c_1(\alpha_1)|^2 = \sum_{\substack{\alpha_1 \in \mathfrak{R}_1^{\times} \\ \alpha_1 \equiv 0 \bmod \mathfrak{p}_1}} \Lambda^2\left(\frac{\alpha_1}{\mathfrak{p}_1}\right) \ll \sum_{M/2 < N\mathfrak{a} \leqslant M} \Lambda^2(\mathfrak{a}) \ll M \log M.$$

Similarly, by (2.1), (4.11), (4.16) and (4.17),

$$\sum_{\alpha_2 \in \mathfrak{R}_2}' |c_2(\alpha_2)|^2 \ll \sum_{\substack{\alpha_2 \in \mathfrak{R}_2 \\ \alpha_2 \equiv 0 \bmod \mathfrak{p}_2}} \Big\{\sum_{\mathfrak{c} | (\alpha_2/\mathfrak{p}_2)} 1\Big\}^2 \ll \log^r x \cdot \sum_{N\mathfrak{a} \ll x/M} \Big\{\sum_{\mathfrak{c} | \mathfrak{a}} 1\Big\}^2 \ll \frac{x}{M} (\log x)^{r+3}.$$

Thus we find that the first error term on the right-hand side of (4.20) is

$$\ll x^{1/2} (\log x)^{3r/2+4} (P + x^{1/2} U^{-1/2} + x^{1/2} P^{-1}).$$

Using our choice of U in (4.6) or in (4.8) with $D = 2A+5n+9$, this is acceptable in (4.10) if

$$(4.21)\qquad Q \leqslant x^{1/2} (\log x)^{-A-5r/2-6}, \quad Q_1 = (\log x)^{A+5r/2+6}.$$

It remains to deal with the sums $\sum_1'$ and $\sum_4'$ in (4.20). As to $\sum_1'$ we put

$$v = \frac{c_2 c_6 c_8}{c_7} (2xN\mathfrak{p}_1 N\mathfrak{p}_2)^{1/(r+1)}$$

so that, by (4.16), (4.17), (4.18) and (4.19),

$$\sum\nolimits_1' |c_1(\alpha_1)|\, |c_2(\alpha_2)|$$

$$\ll U \sum_{l=1}^{r+1} \sum_{\substack{\alpha_1 \in \mathfrak{R}_1^{\times} \alpha_2 \in \mathfrak{R}_2 \\ |\alpha_1^{(l)} \alpha_2^{(l)}| < 1}}' \sum' \Lambda(\alpha_1/\mathfrak{p}_1) \leqslant U \sum_{l=1}^{r+1} \sum_{\substack{\alpha \in \mathfrak{N}(v, \ldots, v) \\ \alpha \equiv 0 \bmod \mathfrak{p}_1 \mathfrak{p}_2 \\ |\alpha^{(l)}| < 1}} \sum_{\mathfrak{a} | (\alpha)} \Lambda(\mathfrak{a})$$

$$\ll U \log x \left\{\frac{(xN\mathfrak{p}_1 N\mathfrak{p}_2)^{1-1/(r+1)}}{N\mathfrak{p}_1 N\mathfrak{p}_2} + 1\right\} \ll U \log x \{P^{-2} x^{1-1/(r+1)} + 1\},$$

on using in the last step the condition (4.11). This leads us again to an estimate which is negligible compared with the right side of (4.10).

We now turn to $\sum_4'$, as defined in Section 3. Putting

$$z_k = |\varrho_0^{(k)}|^{e_k} y_k, \quad k = 1, \ldots, r+1,$$

we arrive at once at

$$(4.22)\qquad \sum\nolimits_4' |c_1(\alpha_1)|\, |c_2(\alpha_2)| \ll U \log x \Big\{ \sum_{\substack{\alpha \in \mathfrak{N}(z_1+1, \ldots, z_{r+1}+1) \\ \alpha \equiv 0 \bmod \mathfrak{p}_1 \mathfrak{p}_2}} 1 - \sum_{\substack{\alpha \in \mathfrak{N}(z_1-1, \ldots, z_{r+1}-1) \\ \alpha \equiv 0 \bmod \mathfrak{p}_1 \mathfrak{p}_2}} 1 \Big\}.$$

By appealing to the Siegel–Grotz summation formula (see [9], Lemma 3) it is easy to verify that

$$(4.23)\qquad \sum_{\substack{\alpha \in \mathfrak{N}(x_1, \ldots, x_{r+1}) \\ \alpha \equiv 0 \bmod \mathfrak{a}}} 1 = \frac{(2\pi)^{r_2}}{|\sqrt{d}|} \frac{x}{N\mathfrak{a}} + O(x^{1-1/(r+2)+\varepsilon}).$$

To see this we have in [9] only to choose $\mathfrak{q} = (1)$ and

$$Q_k = P_k \cdot P^{-1/(r+2)}, \quad k = 1, \ldots, r+1.$$

Let $(\alpha) = \mathfrak{a}\mathfrak{b}$. There exists ideals $\mathfrak{a}_0$, $\mathfrak{b}_0$ in the ideal classes of $\mathfrak{a}$, $\mathfrak{b}$ respectively having

$$N\mathfrak{a}_0,\ N\mathfrak{b}_0 \leqslant |\sqrt{d}|.$$

We write $\mathfrak{a}\mathfrak{b}_0 = (\beta_0)$, $\mathfrak{a}_0\mathfrak{b}_0 = (\gamma_0)$, where

$$\left.\begin{aligned}(N\mathfrak{a}N\mathfrak{b}_0)^{1/(r+1)} &\ll |\beta_0^{(k)}|^{e_k} \ll (N\mathfrak{a}N\mathfrak{b}_0)^{1/(r+1)};\\ (N\mathfrak{a}_0\mathfrak{b}_0)^{1/(r+1)} &\ll |\gamma_0^{(k)}|^{e_k} \ll (N\mathfrak{a}_0\mathfrak{b}_0)^{1/(r+1)},\end{aligned}\right\} \quad k = 1, \ldots, r+1.$$

Using the abbreviations

$$y_k = |\gamma_0^{(k)}|^{e_k} |\beta_0^{(k)}|^{-e_k} x_k, \quad k = 1, \ldots, r+1,$$

it follows from (4.23)

$$\sum_{\substack{\alpha\in\mathfrak{R}(x_1,\ldots,x_{r+1})\\ \alpha\equiv 0 \bmod \mathfrak{a}}} 1 = \sum_{\substack{\beta\in\mathfrak{R}(y_1,\ldots,y_{r+1})\\ \beta\equiv 0 \bmod \mathfrak{a}_0}} 1 = \frac{(2\pi)^{r_2}}{|\sqrt{d}|}\frac{y}{N\mathfrak{a}_0} + O(y^{1-1/(r+2)+\varepsilon}) = \frac{(2\pi)^{r_2}}{|\sqrt{d}|}\frac{x}{N\mathfrak{a}} + O\left(\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/(r+2)+\varepsilon} + 1\right).$$

Substituting in (4.22) we find that

$$\sum\nolimits_4' |c_1(\alpha_1)||c_2(\alpha_2)| \ll U\log x\left\{\frac{1}{N\mathfrak{p}_1 N\mathfrak{p}_2}\left(\prod_{k=1}^{r+1}(z_k+1) - \prod_{k=1}^{r+1}(z_k-1)\right) + O\left(\left(\frac{z}{N\mathfrak{p}_1 N\mathfrak{p}_2}\right)^{1-1/(r+2)+\varepsilon}\right)\right\} \ll Ux^{1-1/(r+2)+\varepsilon}.$$

If (4.6) and (4.7) hold then the term arising from $\sum_4'$ in (4.20) may be ignored.

In the special case of a totally real algebraic number field the estimation of $\sum_4'$ can be modified to give a sharper result. Starting from (4.22) we deduce that

$$\sum\nolimits_4' |c_1(\alpha_1)||c_2(\alpha_2)| \ll U\log x \sum_{l=1}^{n} \sum_{\substack{\alpha\in\mathfrak{R}(z_1+1,\ldots,z_n+1)\\ \alpha^{(l)}>z_l-1\\ \alpha\equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2}} 1 \ll U\log x\,(x^{1-1/n}(N\mathfrak{p}_1 N\mathfrak{p}_2)^{-1/n} + 1).$$

For the last step we have used Hilfssatz 9 of [15]. Hence, by (4.8) and (4.11),

$$P(\log x)^{r+1}\sum_{m}\sum_{\mathfrak{C}}\sum\nolimits_4' |c_1(\alpha_1)||c_2(\alpha_2)| \ll P^{-1}x^{1-1/n}(\log x)^{r+D+3} \ll P^{-1}x(\log x)^{-A-n-2}.$$

By summing this over $P = 2^j$ for an appropriate range of j, the desired result is shown.

To complete the proof of (1.6) it only remains to choose Q in accordance with (2.5), (4.9) and (4.21), namely

$$Q = x^{1/2}(\log x)^{-B}, \quad \text{where } B = \frac{5A}{2} + \frac{11n}{2} + 10.$$

5. Applications. First we are concerned with the behaviour of the sum

$$\sum_{\omega\in\mathfrak{R}(y_1,\ldots,y_{r+1})} D(\omega-1), \quad y_k \geqslant 1 \text{ for } k = 1, \ldots, r+1,$$

where $D(\alpha)$ denotes the number of integral ideal divisors of (α). Since

$$D(\alpha) = 2\sum_{\substack{\mathfrak{q}|(\alpha)\\ N\mathfrak{q}<|N\alpha|^{1/2}}} 1 + \sum_{\substack{\mathfrak{q}|(\alpha)\\ N\mathfrak{q}=|N\alpha|^{1/2}}} 1,$$

it follows that

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) = 2\sum_{\omega\in\mathfrak{R}} \sum_{\substack{\mathfrak{q}|(\omega-1)\\ N\mathfrak{q}<|N(\omega-1)|^{1/2}}} 1 + \sum_{\omega\in\mathfrak{R}} \sum_{\substack{\mathfrak{q}|(\omega-1)\\ N\mathfrak{q}=|N(\omega-1)|^{1/2}}} 1.$$

By (1.4) we have

$$|N(\omega-1)| = \prod_{k=1}^{r+1} |\omega^{(k)}-1|^{e_k} \leqslant \prod_{k=1}^{r+1} (y_k^{1/e_k}+1)^{e_k} \leqslant 2^n y =: y_0.$$

An application of the Brun–Titchmarsh theorem in K (see e.g. [15]) leads us to the estimate

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) \ll \sum_{N\mathfrak{q}\leqslant y_0^{1/2}} \sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}}} 1 \ll \frac{y}{\log 2y} \sum_{N\mathfrak{q}\leqslant y_0^{1/2}} \frac{1}{\Phi(\mathfrak{q})} \ll y.$$

On the other hand, we merely need to observe that

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) \gg \sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} \sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}}} 1 + \sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} \sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}\\ |N(\omega-1)|\leqslant N\mathfrak{q}^2}} 1.$$

Using (2.1), the second expression gives a contribution of order

$$\ll \sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} \sum_{\substack{|\alpha^{(k)}|^{e_k}\leqslant 2^{e_k}y_k\\ \alpha\equiv 0 \bmod \mathfrak{q}\\ |N\alpha|\leqslant N\mathfrak{q}^2}} 1 \ll \log^r 2y \sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} N\mathfrak{q} \ll y^{2/3}\log^r 2y.$$

The first expression can be brought into the form

$$I\sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} \frac{1}{\Phi(\mathfrak{q})} + \sum_{N\mathfrak{q}\leqslant y^{1/(n+2)}} \left\{\sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}}} 1 - \frac{I}{\Phi(\mathfrak{q})}\right\}.$$

An appeal to (1.6) now shows that

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) \gg y,$$

since, by Lemma 6 of [6],

$$I = \frac{w}{2^{r_1} hR}\frac{y}{\log 2y}+O\left(\frac{y}{\log^2 2y}\right).$$

In the special case of a totally real algebraic number field K we are able to derive an asymptotic formula for the sum under consideration. We begin as above and obtain at once

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) = 2I \sum_{N\mathfrak{q}\leqslant y^{1/2}(\log y)^{-B}} \frac{1}{\Phi(\mathfrak{q})}+O\Bigg\{\sum_{N\mathfrak{q}\leqslant y^{1/2}(\log y)^{-B}}\Bigg|\sum_{\substack{\omega\in\mathfrak{R},\\ \omega\equiv 1 \bmod \mathfrak{q}}} 1-\frac{I}{\Phi(\mathfrak{q})}\Bigg|\Bigg\}$$

$$+O\Bigg\{\sum_{N\mathfrak{q}\leqslant y^{1/2}(\log y)^{-B}}\sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}\\ |N(\omega-1)|\leqslant N\mathfrak{q}^2}} 1\Bigg\}+O\Bigg\{\sum_{y^{1/2}(\log y)^{-B}<N\mathfrak{q}\leqslant y_0^{1/2}}\sum_{\substack{\omega\in\mathfrak{R}\\ \omega\equiv 1 \bmod \mathfrak{q}}} 1\Bigg\}.$$

Arguing now in the same way as at the corresponding stage in the general case we find that

$$\sum_{\omega\in\mathfrak{R}} D(\omega-1) = \frac{1}{|\sqrt{d}|}\prod_{\mathfrak{p}}\left(1+\frac{1}{N\mathfrak{p}(N\mathfrak{p}-1)}\right)\cdot y+O\left(\frac{y\cdot\log\log y}{\log y}\right),$$

since it is well known that

$$\sum_{N\mathfrak{q}\leqslant x}\frac{1}{\Phi(\mathfrak{q})} = \frac{(2\pi)^{r_2}}{|\sqrt{d}|}\frac{2^{r_1} hR}{w}\prod_{\mathfrak{p}}\left(1+\frac{1}{N\mathfrak{p}(N\mathfrak{p}-1)}\right)\cdot\log x+O(1).$$

The approximation to Goldbach's conjecture in number fields stated in the Introduction is based on Selberg's weighted sieve method, which has been developed by the author in [7] and [8]. Particularly, the argument of [8] remains applicable with slight modifications. We therefore omit the proof.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARBURG
D-3550 Marburg/Lahn

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