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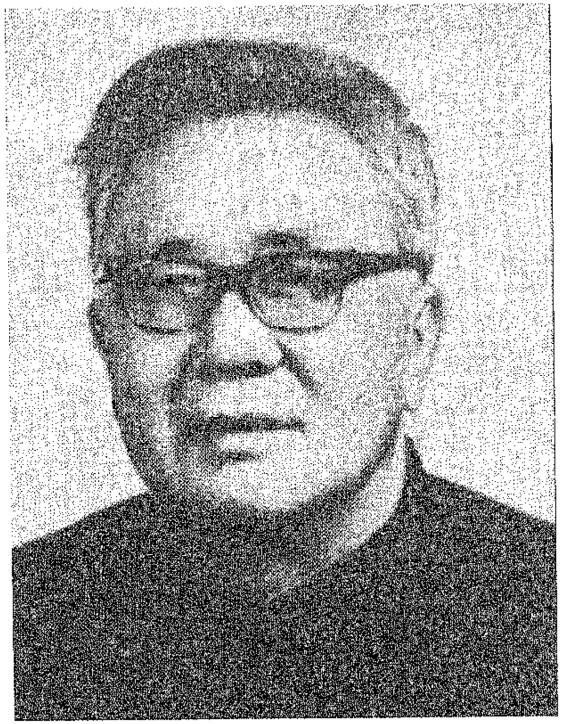

华罗庚

Loo-keng Hua: Obituary ([1])

by

H. Halberstam (Urbana, Ill.)

Hua Loo-keng was one of the leading mathematicians of his time, and with S. S. Chern the most eminent Chinese mathematician of his generation. He spent most of his working life in China. If many Chinese mathematicians are now making distinguished contributions at the frontiers of science, and if mathematics in China enjoys quite unusual popularity in public esteem, that is due in overwhelming measure to the leadership Hua gave his country, as scholar and teacher, for fifty years.

Loo-keng Hua was born([2]) on November 12, 1910, in Jinton County in the southern Jiangsu Province of China. His father managed a small general store and the family was too poor to permit Hua to finish high school. He assisted his father for a time, and then enrolled for two years at a free accountancy school, determined to continue his education. Probably he began to work seriously in mathematics during this period. Although he then returned home yet again to help in the family store, it was not long before a paper on the theory of equations drew the attention of a perceptive professor at Tsinghua University who brought Hua to Beijing. To begin with Hua worked as clerk in the library of Tsinghua University, but in 1931 he moved to an assistantship in the Mathematics Department there. His first paper had already appeared in 1929, half a dozen other publications were to follow rapidly, and by September 1932 he was an instructor in the same department. In September 1934 he was promoted to the rank of lecturer. He had arrived despite the limitations of his background, and he had done so essentially without loss of time. From 1934 onwards Hua's papers bear the unmistakable stamp of a professional mathematican.

Norbert Wiener visited China in 1935. He was impressed by Hua (note

([1]) The accompanying bibliography of Hua's papers and books derives from the *Selected papers of L.-K. Hua*, Springer 1982. Note that Books and Monographs are listed separately and are referred to in the text as [B-]. Portions of the detailed discussion of Hua's mathematical work are also reproduced from that volume with the kind permission of the Springer-Verlag.

([2]) In the personal history that he supplied when he joined the faculty of the University of Illinois, Hua gave his birthdate as October 11, 1909.

paper [27] *On Fourier transforms in L^p in the complex domain*) and on the strength of his recommendation to G. H. Hardy, Hua arrived in Cambridge, England, in 1936 to spend two crucial and formative years there. By now he was a research fellow of the China Foundation. He had already been publishing regularly on Waring's problem and was ready to profit from the stimulating environment of the Hardy–Littlewood school. At least fifteen papers date from his Cambridge period, bearing witness both to his flowering talent and his prodigious industry. Hua was a gregarious individual and he soon acquired an adequate command of English to sustain relationships with several young mathematicians in England, among them Davenport, Estermann, Heilbronn, Offord, Rankin and Titchmarsh. He liked to recall that Davenport helped him to prepare an early publication in the Journal of the London Mathematical Society.

It may be that Hua had planned to stay in England longer, but by 1938 the world was fast sinking into turmoil. In particular, Japan had invaded China proper in 1937, and Hua returned home, officially as a full professor in his old department. In fact, the invasion caused several universities to move and amalgamate at the Southwest Associated University, Kunming, in the Province of Yunan, and Hua spent the years 1938–45 there. Times must have been difficult, but the flow of Hua's publications continued unabated; moreover, towards the end of this period Hua's interests were obviously widening to embrace those parts of algebra and analysis where he was soon to make a major impact.

After World War II Hua received an invitation to visit Russia and he spent several fruitful months during 1945–46 with I. M. Vinogradov. His important formulation of Vinogradov's method for trigonometric sums as a mean value theorem [95] dates from this period, and also his influential monograph on additive prime number theory [B1], which first appeared in Russian. Hua paid one more short visit to Russia in the early fifties. He told me that he was nominated for a Stalin Prize but that owing to Stalin's death the Prize was never awarded; in view of later political developments he felt he had cause to be well-satisfied by this omission!

From 1947 to 1948 he was a visiting member of the Institute for Advanced Study, Princeton, and spent the years 1948–50 as professor at the University of Illinois at Urbana-Champaign. Though his Illinois period was all too brief, he was responsible for bringing several important faculty members into the department (P. T. Bateman, Irma and Irving Reiner) and he supervised the research of several students who have since become professional mathematicians (R. Ayoub, Josephine Mitchell, L. Schoenfeld). Although he continued to work in number theory at this time, and indeed throughout his life, he was by now fully involved also in other spheres of mathematics: matrix geometry, equations over finite fields, and automorphisms of symplectic groups.

In 1950 Hua returned finally to China, to assist with the reconstruction of the Mathematical Institute of the Academia Sinica, of which he became Director in 1952. It must have been a wrench for him to leave the flourishing American mathematical community at the very peak of his powers — much later, in a letter to me, he referred to the forties as "the golden years of my life" — but when his country's call came he did not hesitate. And indeed, return to China in no way affected the flow of his productivity nor the continued broadening of his interests. If anything, his output increased, for he now threw himself also into the instruction and training of graduate students (among them Chen Jing-run, Pan Chen-dong and Wang Yuan in number theory, Wan Zhi Xian in algebra, and Kung Sheng, Lu Qi Keng in analysis). He embarked, for their benefit and also to popularize mathematics in China, on the writing of a veritable stream of books. Especially noteworthy were his *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* [B3] (of which a 1963 AMS translation is still kept in print) — Hua received in 1957, a Prize from the Chinese Government for his now classical text — and in the same year his massive *Introduction to Number Theory* [B2], which was recently translated into English and is about to be re-issued by Springer after the rapid sale of the first printing. In 1959 appeared his *Enzyklopädie der Math. Wissenschaften* memoir on 'The estimation of trigonometric sums with applications to number theory' (in German) [B4] and in 1963, with Wan-Chie-hsien, a treatise, in Chinese, on *The Classical Groups* [B6]. As recently as 1978 he and Wang published their *Applications of Number Theory to Numerical Analysis* (English translation Springer 1981). I believe that recently Professor Wang has compiled Hua's writings in scientific popularization in a Chinese edition. At present, in English, we catch only a glimpse of this activity from his Plenary Address to the 1980 Berkeley Congress on Mathematical Instruction (Birkhäuser 1983) and from [133 A]. He involved himself also with the development of the Chinese University of Science and Technology in Hefei, in the Province of Anhwei, of which he became Vice-Rector in due course, and increasingly the Chinese Government came to use him as consultant in its drive towards industrialization. Hua travelled with a support team of scientists the length and breadth of China to show workers how to apply their reasoning faculty to a multitude of every-day problems. Whether in *ad hoc* problem-solving sessions on the shop-floor or in huge open-air meetings outside factories, he touched the multitudes with the spirit of mathematics as no mathematician had ever done before, anywhere in the world. He had a commanding presence, a genial personality and a marvelous gift for putting things simply, and he became a national hero, a teacher of his people. When Hua organized a Conference on functions of several complex variables in Hangzhou in 1984, western colleagues were astonished by the scale of the publicity accorded to it by the Chinese media. Even abroad, wherever Hua visited in the seventies, Chinese communities of all political persuasions flocked to meet him, to do him honor and to render him whatever service they could.

Hua spent the years of the "Cultural Revolution" under virtual house-arrest. He attributed his survival to the personal protection of Tsu En-lai. Even so he was exposed to harassing interrogations, many of his manuscripts were confiscated and are irretrievably lost, and attempts were made to extract from Hua's associates damaging allegations against him.

Soon after the "thaw", Hua was permitted in 1980 to accept an invitation from Professor Livingston to visit Birmingham, England, for a year as a senior visiting research fellow of the then Science Research Council of the U. K. Thus began the last period of Hua's life. Honored at home, a Vice-President of Academia Sinica and a member of the People's Congress, still a science advisor to his Government and now also a cultural ambassador charged with reestablishing links with western academics, he visited also France, Germany, Japan and the United States. During 1983–84 he was Sherman Fairchild Distinguished Scholar at the California Institute of Technology. He was by now tired and in poor health, but a characteristic zest for life never deserted him. He continued to work at mathematics and its applications to the end; his last lecture in Urbana, in the spring of 1984, to a packed lecture room, was on mathematical economics. In his last letter to me, dated May 21, 1985, he reported that most of his time now was devoted to 'non-mathematical' activities 'which are necessary for my country and my people'. He died of a heart attack on June 12, 1985 while giving a lecture in Tokyo.

Hua received honorary doctorates from the University of Nancy (1980), the Chinese University of Hong-Kong (1983), and the University of Illinois (1984); he was elected a foreign associate of the U.S. National Academy of Science (1982), a member of the Deutsche Akademie der Naturforscher Leopoldina, also of the Academy of the Third World (1983), and of the Bavarian Academy of Science (1985).

Mathematical work

The following survey of Hua's work on number theory is an expanded version of Professor Y. Wang's article in the Selecta; for a survey of the work on other parts of mathematics the reader is referred to the articles of Professor Z. X. Wan (algebra and geometry) and Professors S. Kung and K. H. Look (function theory) in the same volume.

Number Theory

1. Waring's problem and generalizations; exponential sums. In 1770 Waring conjectured (not exactly in these words) that for each integer $k \geqslant 2$ there exists an integer $s = s(k)$ depending only on k such that every positive integer N can be expressed in the form

$$N = x_1^k + \ldots + x_s^k \tag{1}$$

where the $x_i (1 \leqslant i \leqslant s)$ are non-negative integers. Lagrange had settled the case $k = 2$ in the same year by showing that $s(2) = 4$ (a best possible result), but the conjecture in full generality was not proved until 1909, by Hilbert. As is usual in number theory, the resolution of one problem leads at once to the formulation of others. For the purpose of the present discussion we identify here two major problems:

I. Determine an asymptotic formula for the number $R_s^{(k)}(N)$ of representations of N in the form (1), with an admissible s that is as small as possible.

II. Determine $G(k)$, the least value of s for which (1) is true for all sufficiently large N.

Clearly a solution of problem I sheds some light on problem II: if there exists an asymptotic formula for $R_s^{(k)}(N)$ when $s \geqslant \bar{G}(k)$, then $G(k) \leqslant \bar{G}(k)$.

In 1918 Hardy and Ramanujan devised a method for I in the case $k = 2$. As an outgrowth of this work, Hardy and Littlewood began, in 1920, their famous 'Partitio Numerorum' series of memoirs in which they developed the famous 'circle' method to attack problem I for $k \geqslant 3$. More than half a century later, neither I nor II is fully resolved.

The circle method is easy to explain in principle, but its application involves great technical difficulties, largely having to do with the estimation of exponential sums. These difficulties have posed problems of independent interest and have been a seminal influence in contemporary number theory. Let

$$T(\alpha) = \sum_{x=1}^{P} e(\alpha x^k), \quad e(\theta) = e^{2\pi i\theta} \quad \text{and} \quad P = [N^{1/k}].$$

Then

$$R_s^{(k)}(N) = \int_0^1 T(\alpha)^s e(-N\alpha)\, d\alpha.$$

(The use of the finite generating function $T(\alpha)$ derives from Vinogradov's 1928 formulation of the circle method.) The interval $[0, 1]$ is now separated into two parts: $\mathscr{M}$, consisting of short intervals centered at rational points a/q, $(a, q) = 1$, with small denominations q (often called the 'major' arcs), and m, the rest of the unit interval. On each major arc, $T(\alpha)$ can be well approximated by a known function and as a consequence, if s is sufficiently large,

$$\int_{\mathscr{M}} T(\alpha)^s e(-N\alpha)\, d\alpha = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} N^{s/k-1} G_s(N) + o(N^{s/k-1}) \tag{2}$$

where the singular series $G_s(N)$ is given by

$$G_s(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{x=1}^{q} e(ax^k/q)\Big)^s. \tag{3}$$

This provides the leading term in the asymptotic formula for $R_s^{(k)}(N)$ (once one has proved that

$$G_s(N) \gg 1). \tag{4}$$

Also

$$\Big|\int_m T(\alpha)^s e(-N\alpha)\,d\alpha\Big| \leqslant \int_m |T(\alpha)|^s\,d\alpha = o(N^{s/k-1}) \tag{5}$$

provided s is large enough. It is clear that both parts of this classical argument require the estimation of exponential sums. By a happy chance, Weyl, in his famous memoir on the uniform distribution of sequences of 1916, had provided this key ingredient: let $f(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \ldots + \alpha_k$ where $\alpha_0, \ldots, \alpha_k$ are real and $|\alpha_0 - a/q| \leqslant q^{-2}$ with $(a, q) = 1$. Then Weyl proved that

$$\sum_{x=1}^{Q} e\big(f(x)\big) \ll Q^{1+\varepsilon}(q^{-1}+Q^{-1}+qQ^{-k})^{2^{1-k}}. \tag{6}$$

With this inequality alone, Hilbert's theorem can be proved provided only that s exceeds some sufficiently large $s_0(k)$.

The most critical component of the classical argument is the proof of (5). By suitable choice of $\mathscr{M}$ it turns out that if α is **not** in $\mathscr{M}$, that is, if $\alpha \in m$, then α lies near a rational a/q with a large denominator, specifically $P < q \leqslant P^{k-1}$. Hence, by (6)

$$\sup_{\alpha \in m} |T(\alpha)| \ll P^{1-2^{1-k}+\varepsilon} \tag{7}$$

so that (cf. (5))

$$\begin{aligned}\int_m T(\alpha)^s e(-N\alpha)\,d\alpha &\ll P^{(s-4)(1-2^{1-k}+\varepsilon)} \int_0^1 |T(\alpha)|^4\,d\alpha \\ &= P^{2+\varepsilon+(s-4)(1-2^{1-k}+\varepsilon)} = o(N^{s/k-1}),\end{aligned}$$

provided that $s \geqslant (k-2)\,2^{k-1}+5$. This, essentially, is the argument of Hardy and Littlewood on m. Hua's first distinctive contribution to the circle method was, in 1938 (see [36]), to show that, for every j, $1 \leqslant j \leqslant k$,

$$\int_0^1 |T(\alpha)|^{2^j}\,d\alpha \ll P^{2^j-j+\varepsilon}. \tag{8}$$

For $j = 1$ it is obvious even that $\int_0^1 |T(\alpha)|^2\,d\alpha = P$, and we used the almost trivial case $j = 2$ above (in the argument following (7)). However, using (8) with $j = k$ in this argument we find that (5) holds for $s \geqslant 2^k+1$. For all such s, (2) and (4) follow without new difficulties so that Hua settled I for $s \geqslant 2^k+1$. Automatically, it follows for II that

$$G(k) \leqslant 2^k+1. \tag{9}$$

For small values of k these both are very good results. For example, by (9) $G(4) \leqslant 17$; in fact, Davenport proved that $G(4) = 16$ but the corresponding problem I remains open. By (9) also, $G(3) \leqslant 9$. Linnik and G. L. Watson have proved, by quite different methods, that $G(3) \leqslant 7$, but in the context of I there has been an improvement only quite recently, when Vaughan showed $k = 3$, $s = 8$ to be admissible.

For larger k ($k > 10$) one can do much better, as will be explained below.

I have already indicated that the chief difficulty in implementing the circle method resides on m. In 1957 (see [119], [120]) Hua showed that, even with the major arcs made as wide and as numerous as possible, (2) remains valid provided only that $s \geqslant k+1$. The key ingredient in Hua's argument is the estimation of the complete exponential sum

$$\sum_{x=1}^{q} e\big((ax^k+bx)/q\big) \ll q^{1/2+\varepsilon}(q, b), \tag{10}$$

and this depends on a deep theorem of Weil. This exponential sum is but a special case of the more general sum

$$S(q, f) = \sum_{x=1}^{q} e\big(f(x)/q\big),$$

where $f(x) = a_1 x^k + \ldots + a_k x$ and $a_1, \ldots, a_k$ are integers such that $(a_1 \ldots a_k, q) = 1$. This general sum has relevance to Waring's problem (see section 7.2 of Vaughan([3]) and was studied by Hua as early as 1938 (see [38]). In 1940 Hua [49A] proved that

$$S(q, f) \ll q^{1-1/k+\varepsilon} \tag{11}$$

where the O-constant depends at most on k and ε. Apart from the ε, this result is essentially best possible; when $f(x) = x^2$, Gauss showed that $|S(q, f)| = q^{1/2}$ and it is known that when $f(x) = x^k$, $q = p^k$ (p a prime, $p \nmid k$), $|S(p^k, f)| = p^{k(1-1/k)}$. Before Hua, only these and other special results had been known. Since Hua, Chen Jing-run proved in 1977 that

$$S(q, f) \leqslant e^{7k} q^{1-1/k};$$

([3]) R. C. Vaughan, *The Hardy–Littlewood Method*, Cambridge 1981.

for another refinement, using more information about f, see Loxton and Smith. [4]

With the major arcs presenting no problem provided only that $s \geqslant k+1$, the chief difficulty with improving Hua's requirement $s \geqslant 2^k+1$ rests with (5). Here Vinogradov made crucial progress by improving (6) (and, in particular (7)) for large k. Vinogradov's method is decidedly complicated.

Hua improved and simplified Vinogradov's method for the estimation of Weyl's sums by pointing out that the essence of Vinogradov's method is the following mean value theorem [95]:

Let $f(x) = \alpha_k x^k + \ldots + \alpha_1 x$ and

$$C_k = C_k(P) = \sum_{x=a+1}^{a+P} e(f(x)).$$

Let $t_1 = t_1(k) \geqslant 1/4k(k+1) + lk$. Then

$$\int_0^1 \ldots \int_0^1 |C_k|^{2t_1} d\alpha_1 \ldots d\alpha_k \leqslant (7t_1)^{4t_1 l} P^{2t_1 - (1/2)k(k+1)+\delta} (\log P)^{2l},$$

where $\delta = \delta(k) = 1/2k(k+1)(1-1/k)^l$.

From this one derives immediately the following theorem:

Suppose that $k \geqslant 12$, $2 \leqslant r \leqslant k$ and

$$\left|\alpha_r - \frac{h}{q}\right| \leqslant \frac{1}{q^2}, \quad (h, q) = 1, \ 1 \leqslant q \leqslant P^r.$$

Then for $P \leqslant q \leqslant P^{r-1}$, we have

$$S = \sum_{x=1}^{P} e(f(x)) \ll P^{1-1/\sigma_k + \varepsilon}$$

where $\sigma_k = 2k^2(2\log k + \log\log k + 3)$.

In all current monographs on analytic number theory, Vinogradov's method is stated according to Hua's formulation (see for example R. C. Vaughan, *loc. cit.*). It should be said that the most recent account, by Vaughan, incorporates later improvements of Karacuba and Bombieri. Actually Vaughan shows that problem I is solved, provided $s \geqslant s_0(k)$ where $s_0(k) \sim 4k^2 \log k$ as $k \to \infty$ and $s_0(k) < 2^k+1$ for $k > 10$.

Vinogradov developed also a method for attacking problem II. The most elaborate form of this method uses an asymptotic version of the Vinogradov–Hua mean value theorem and also Hua's (11) to prove that (Vaughan, *loc. cit.*, Ch.7)

(12) $$G(k) \leqslant 2k(\log k)(1+o(1)) \quad \text{as } k \to \infty.$$

[4] J. H. Loxton and R. A. Smith, *Estimates for multiple exponential sums*, J. Austral. Math. Soc. Ser A, 33 (1982), 125–134.

The mean-value theorem of Vinogradov–Hua has a very important application to the estimation of the order of magnitude of the Riemann zeta function $\zeta(s)$ in the critical strip. For an account of this application see, for example, Walfisz [5], Ch. V.

Hua investigated several variants of (1), each presenting some novel technical difficulty. Thus he dealt with the following two generalizations of Hilbert's theorem, each preserving the additive character of the original Waring's problem. Suppose that $f(x)$ denotes an integral polynomial of degree k, i.e.,

$$f(x) = a_k \binom{x}{k} + a_{k-1} \binom{x}{k-1} + \ldots + a_1 \binom{x}{1},$$

where $\binom{x}{r} = x(x-1)\ldots(x-r+1)/r!$ and a_i $(1 \leqslant i \leqslant k)$ are integers which we assume always to be positive. Hua proved between 1937 and 1940 that the number of solutions of the equation

(13) $$N = f(x_1) + \ldots + f(x_s)$$

has an asymptotic formula for $s \geqslant 2^k+1$ $(1 \leqslant k \leqslant 10)$ [36] and (implicitly from Lemma 12.3 of [104]) for $s \geqslant 2k^2(2\log k + \log\log k + 2.5)$ $(k > 10)$. The result is true also if (13) is replaced by

$$N = f_1(x_1) + \ldots + f_s(x_s),$$

where the $f_i(x)$ $(1 \leqslant i \leqslant s)$ are any given s integral polynomials of degree k. Let $G(f)$ denote the least integer s such that (13) is solvable for all sufficiently large N. Let $\partial^0 f$ denote the degree of f. Hua [30], [45] established that

$$G(f \mid \partial^0 f = k) \leqslant (k-1)\, 2^{k+1}$$

and [47]

$$G(f \mid \partial^0 f = 3) \leqslant 8.$$

He also proved [45] that

$$\operatorname*{Max}_{f(x)} G(f) \geqslant 2^k - 1,$$

where $f(x)$ runs over all integral polynomials of degree k, in which $k \geqslant 5$.

Hua went on to study systematically the so-called Waring–Goldbach problem, of the solvability of (1) and its generalizations in which the variables x_u $(1 \leqslant u \leqslant s)$ are restricted to prime numbers. The fruits of his

[5] A. Walfisz, *Weylsche Exponentialsummen in der neuren Zahlentheorie*, Berlin 1963.

researches are given in the well-known monograph *Additive prime number theory*. For example, he proves there (cf. [39]) that the number of solutions of the equation

$$N = p_1^k + \ldots + p_s^k \tag{14}$$

has an asymptotic formula for $s \geqslant 2^k+1$ $(k \leqslant 10)$ and $s \geqslant 2k^2(2\log k + \log\log k + 2.5)$ $(k > 10)$; and he obtains analogues for prime arguments of many of the results sketched above.

Finally, there is Hua's work on the so-called Tarry's problem.

Let $N(k)$ denote the least integer t such that the system of equations

$$x_1^h + \ldots + x_t^h = y_1^h + \ldots + y_t^h, \quad 1 \leqslant h \leqslant k \tag{15}$$

has a non-trivial solution, i.e., $x_1, \ldots, x_t, y_1, \ldots, y_t$ are positive integers but the set of $x_1, \ldots, x_t$ is not a permutation of $y_1, \ldots, y_t$. Let $M(k)$ be the least integer t such that (15) is solvable and

$$x_1^{k+1} + \ldots + x_t^{k+1} \neq y_1^{k+1} + \ldots + y_t^{k+1}.$$

Evidently, we have

$$k+1 \leqslant N(k) \leqslant M(k).$$

Hua [37] proved that, for $k \geqslant 12$,

$$M(k) \leqslant (k+1)\left[\left[\frac{\log \frac{1}{2}(k+2)}{\log(1+(1/k))}\right]+1\right] \sim k^2 \log k$$

which improves significantly on the early result

$$M(k) < 7k^2(k-11)(k+3)/216$$

due to Wright. Hua's argument is elementary and straightforward.

It was pointed out by Hua that Vinogradov's method may be used to treat Tarry's problem. In this way Hua [104] obtained the following results: Let t_0 be given by the table:

k	2	3	4	5	6	7	8	9	10	$k \geqslant 11$
t_0	3	8	23	55	120	207	336	540	885	$[k^2(3\log k + \log\log k + 9)]$

Let $r_1(p)$ denote the number of solutions of (15) satisfying

$$1 \leqslant x_i, y_i \leqslant P, \quad 1 \leqslant i \leqslant t.$$

Then

$$\lim_{P\to\infty} P^{1/2k(k+1)-2t} r_t(p) = c(t)$$

holds for $t > t_0$, where $c(t)$ is a positive constant depending on t only.

To sum up, Hua made contributions to the development of the circle method that, together with Davenport's, rank only second to those of Hardy, Littlewood and I. M. Vinogradov, and will assuredly stand the test of time. His two integral mean value theorems are major technical advances that have made a lasting impact even outside the domain of Hilbert's theorem; and his studies of variations of Waring's problem, and notably of the Waring–Goldbach problem, were crucial pioneering investigations to prove the power and versatility of the circle method. Several generations of number-theoreticians derived their knowledge of the method from Hua's still influential 1947 monograph [B1]. Vaughan's more recent tract gives an excellent account of the major achievements of the circle method to-date; even a casual perusal of it leaves one in no doubt of the prominent place of Hua's own contributions in its evolution.

2. Other contributions to number theory. Hua's papers in the theory of numbers are virtually an index to the major activities in that subject that held the stage in the thirties and forties of our own century. Given his background, it is quite extraordinary how broad Hua's interests were, how deep his command of existing techniques and how sure his instinct for what was important. By 1945 Hua was clearly one of the leading number-theoreticians of his day.

In [60], soon after the appearance of Rademacher's famous work on the partition function, Hua obtained the following exact formula for $q(n)$, the number of partitions of n into unequal parts (or into odd parts):

$$q(n) = \frac{1}{2^{1/2}} \sum_{\substack{k=1\\ k\,\mathrm{odd}}}^{\infty} \sum_{\substack{(h,k)=1\\ 0<h\leqslant k}} \omega_{h,k}\, e\left(-\frac{hn}{k}\right) \frac{d}{dn} J_0\left[\frac{i\pi}{k}\left\{\frac{2}{3}\left[n+\frac{1}{24}\right]\right\}^{1/2}\right],$$

where $J_0(x)$ is the Bessel function of order 0. (Because of the second World War, publication of this paper was delayed.)

In [63] Hua tackled the circle problem and proved that if $A(x)$ denotes the number of lattice points (u, v) in the disc $u^2+v^2 \leqslant x$, then

$$A(x) = \pi(x) + O(x^{\theta+\varepsilon})$$

with $\theta = 13/40$. This represented an improvement of Titchmarsh's $\theta = 15/46$, and inspired several Chinese mathematicians subsequently to make further improvements.

In the two papers [64] Hua (also Hua and Min) made a contribution to the question which real quadratic fields $R(\sqrt{d})$, with d square free, possess a euclidean algorithm (EA). In the first of these papers Hua showed that there is no EA if $d > e^{250}$, and stated in a footnote that he could reduce the exponent 250 to 160. He derived his result from the independently interesting study of least quadratic non residues modulo a prime.

In [103] Hua improved a result of Buchstab on the rate of convergence of the solution of a certain differential difference equation with retarded argument and prescribed initial behavior. The elegant account contains a very useful lemma (Lemma 5): Suppose $F(u)$ is a positive function satisfying the functional inequality

$$F(u) \leqslant \frac{1}{u} \int_{u-1}^{u} F(t)\, dt, \qquad u \geqslant u_0.$$

Then

$$F(u) \leqslant \exp\left\{-u\left(\log u + \log\log u - 1 + \frac{\log\log u}{\log u} + O\left(\frac{1}{\log u}\right)\right)\right\}.$$

Differential difference equations of a similar kind have featured in many arithmetical investigations of late, and Hua's lemma has proved very helpful. (See also Hua's survey [B4], pp. 70–71.)

Hua and Wang wrote several papers ([131], [138], [147], [148]) on replacing statistical Monte Carlo methods in numerical analysis by effective deterministic procedures based on number theoretic ideas; their work, and other contributions chiefly from the Chinese & Russian schools, are described in their book [B9]. Their methods rest on the construction of special integral bases of algebraic number fields, the construction of simultaneous rational approximations to these by means of a given set of independent units or by linear recurrences, and on discrepancy measures. Numerical information for dimensions 2 to 18 is given in an appendix.

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Publications of Loo-keng Hua

I. Articles

1. *Study on Sturm Theorem* (in Chinese), Science 14 (1929), 545–548.
2. *On incorrectness of Shu Jia Ju's paper* (in Chinese), Science 15 (1930), 307–309.
3. *Study on function $T^{-1}\{H(x)\}$* (in Chinese), Science 15 (1931), 871–888, 1055–1062.
4. *A Theorem of integral calculus*, Science 15 (1931), 1716–1719.
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