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Units in parametrized p-adatropic number fields

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0. Introduction. In [2]–[4] H. Cohn studied fields generated by polynomials which assumed values of powers of 2 at several consecutive integers. It was felt that these fields might yield independent units parametrically. We make the following generalization:

DEFINITION 1. Let p be a prime. Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_i \in \boldsymbol{Z}$, $0 \leqslant i < n$. The polynomial, f, is said to be *p-adatropic* if there exist $n+1$ consecutive rational integers, c_i, such that $|f(c_i)|$ is a power of p.

From finite differencing the following lemma is known:

LEMMA 1. *Let $f(x)$ be a monic polynomial of degree n and let $x_0 \in \boldsymbol{R}$. Let $y_k = f(x_0+k)$, $k = 0, 1, \ldots, n$. Then*

$$(*) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} y_k = (-1)^n n!.$$

COROLLARY 1. *Every p-adatropic polynomial has degree greater than or equal to p.*

Proof. Since p divides each term on the left of $(*)$, it must also divide the degree, n.

THEOREM 1. *In a field generated by a p-adatropic polynomial of degree p, the prime ideal (p) must split completely.*

In what follows we will normalize against translation so that the powers of p occur with abscissas near 0. Specifically, $x_0 = -n/2$ if n is even and $x_0 = (1-n)/2$ otherwise. We will also avoid the symmetry $f_1(\theta) \leftrightarrow \pm f_2(-\theta)$ which gives rise to the same field since these polynomials have the same zeros.

It follows from Corollary 1 that there are no linear p-adatropic polynomials. Furthermore, this result dictates that the only p-adatropic quadratic polynomials are those where $p = 2$. These 2-adatropic polynomials were studied extensively by H. Cohn [3]. We summarize his results:

Let $v = (-1)^s 2^k$. The only parametrized family of 2-adatropic quadratic polynomials is the one given by $f(x) = x^2 + (1-v)x + v$. Let $f(\theta) = 0$. We factor the principal ideals:

$$(\theta+1) = 2_1^{k+1}, \quad (\theta) = 2_2^k, \quad (\theta-1) = 2_1.$$

Cohn easily finds the unit

$$\varepsilon = \frac{(\theta-1)^{k+1}}{(\theta+1)}.$$

1. p-adatropic cubics. Here, the difference equation $(*)$ takes the form:

$$f(-1)-3f(0)+3f(1)-f(2) = -6.$$

This gives rise to the cubic equation

$$(**) \qquad f(x) = x^3 + \frac{f(-1)-2f(0)+f(1)}{2}x^2 + \frac{f(1)-f(-1)-2}{2}x + f(0).$$

Let $|v| = p^k$, $k \in \boldsymbol{Z}^+$. Corollary 1 implies $p = 2$ or $p = 3$.

1.1. $p = 2$. In [4], Cohn listed all parametrized 2-adatropic cubic polynomials. They are:

Table 1. Parametrizations of 2-adatropic cubics

	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
A	v	4	2	v
B	v	2	$-v$	$-2v$
C	v	v	-2	$-2v$
D	-2	v	v	4
E	v	2	v	$4v$
F	v	$-v$	-2	$4v$
G	2	v	v	8

Note that cases C and G are merely different parametrizations of the same family of polynomials. This phenomenon occurs because here we assume powers of 2 at five consecutive integers as opposed to the required four.

Let $\theta = \theta_1, \theta_2, \theta_3$ be the zeros of $(**)$ and let Δ be discriminant of $(**)$. We will factor the ideals $(m-\theta)$ in $O_{Q(\theta)/Q}$ and find units.

We will discuss each case and outline some of the proofs:

A: $f(-1) = v$, $f(0) = 4$, $f(1) = 2$, $f(2) = v$

$$\Delta = (v^4 - 36v^3 + 324v^2 - 1728)/16 < 0$$

only for $v = 2, 4, 8, 16$ and in these cases the Dirichlet rank will be 1. Factoring ideals:

$(\theta-1) = 2_1$, $(\theta+1) = 2_1^k$, $(\theta) = 2_2^2$ if $k = 1$ and $2_2 2_3$ otherwise.

THEOREM 2. *When* $\Delta > 0$, $\varepsilon_1 = (\theta-1)^k/(\theta+1)$ *and* $\varepsilon_2 = \theta(\theta-1)/2$ *are independent units.*

Proof. *Case* I: $v < 0$. We find

$$\begin{array}{ll} -1 < \theta_1 < 0, & 1 < \theta_2 < 2, \\ 0 < |\theta_1| < 1, & 1 < |\theta_2| < 2, \\ 1 < |\theta_1 - 1| < 2, & 0 < |\theta_2 - 1| < 1, \\ 0 < |\theta_1 + 1| < 1, & 2 < |\theta_2 + 1| < 3. \end{array}$$

Thus,

$$|\varepsilon_1^{(1)}| = \frac{|\theta_1 - 1|^k}{|\theta_1 + 1|} > |\theta_1 - 1|^k > 1, \qquad |\varepsilon_2^{(1)}| = \frac{|\theta_1|\,|\theta_1 - 1|}{2} < \frac{|\theta_1 - 1|}{2} < 1,$$

$$|\varepsilon_1^{(2)}| = \frac{|\theta_2 - 1|^k}{|\theta_2 + 1|} < \frac{1}{2} < 1, \qquad |\varepsilon_2^{(2)}| = \frac{|\theta_2|\,|\theta_2 - 1|}{2} < 1.$$

Therefore, the regulator $\boldsymbol{R} = \ln|\varepsilon_1^{(1)}|\ln|\varepsilon_2^{(2)}| - \ln|\varepsilon_1^{(2)}|\ln|\varepsilon_2^{(1)}| < 0$.

Case II: $v \geqslant 128$.

$$\frac{8}{v} < \theta_1 < \frac{9}{v}, \quad 1 - \frac{5}{v} < \theta_2 < 1 - \frac{4}{v}, \quad \theta_3 < 0,$$

$$k\ln\left|\frac{v-9}{v}\right| - \ln\left|\frac{v+9}{v}\right| < \ln|\varepsilon_1^{(1)}| < k\ln\left|\frac{v-8}{v}\right| - \ln\left|\frac{v+8}{v}\right|,$$

$$k\ln|\tfrac{1}{2}| - \ln 2 < \ln|\varepsilon_1^{(1)}| < 0, \qquad -(k+1)\ln 2 < \ln|\varepsilon_1^{(1)}| < 0,$$

$$\ln\left|\frac{8(v-9)}{2v^2}\right| < \ln|\varepsilon_2^{(1)}| < \ln\left|\frac{9(v-8)}{2v^2}\right|,$$

$$\ln|\varepsilon_2^{(1)}| < \ln\left|\frac{8}{v}\right| = (3-k)\ln 2,$$

$$k\ln\left|\frac{4}{v}\right| - \ln\left|\frac{2v-4}{v}\right| < \ln|\varepsilon_1^{(2)}| < k\ln\left|\frac{5}{v}\right| - \ln\left|\frac{2v-5}{v}\right|,$$

$$\ln|\varepsilon_1^{(2)}| < k\ln\left|\frac{8}{v}\right| = k(3-k)\ln 2,$$

$$\ln\left|\frac{2(v-5)}{v^2}\right| < \ln|\varepsilon_2^{(2)}| < \ln\left|\frac{5(v-4)}{2v^2}\right|,$$

$$-k\ln 2 < \ln|\varepsilon_2^{(2)}| < 0,$$

$$R > [k(3-k)\ln 2][(3-k)\ln 2] - k(k+1)(\ln 2)^2,$$

$$R > (\ln 2)^2 k(k^2 - 7k + 8) > 0 \quad \text{since } k \geqslant 7.$$

Case III: $v = 32$ and $v = 64$ are easily verified by a hand calculation.

B: $f(x) = x^3-2x^2-(v+1)x+2$ and $\Delta = 4(v^3+4v^2+23v+9) > 0$ when $v > 0$ (recall, $|v| = 2^k$).

We have the unit

$$\varepsilon_2 = \frac{(\theta)^{k+1}}{(\theta-2)} = \frac{(\theta+1)(\theta-1)\theta^k}{2^k}.$$

However, we may observe that when $v = t^2$ $(t > 0)$

$$f(1+t) = -2t \quad \text{and} \quad f(1-t) = 2t$$

and

$$\varepsilon_1^* = \frac{(\theta-1-t)}{(\theta-1+t)}.$$

THEOREM 3. *For the field $Q(\theta)$ defined by the polynomial*

$$f(x) = x^3-2x^2-(t^2+1)x+2 \quad \textit{where } t = 2^m,$$

ε_1^* and ε_2 are independent units.

Proof. $-t < \theta_1 < 1-t$, $t+1 < \theta_2 < t+2$, $0 < \theta < 1$,

$$|\varepsilon_1^{(1)}| > 2t > 1, \quad |\varepsilon_2^{(1)}| > (t-1)^{2m}/(t+2) > 1 \quad \text{if } m > 1.$$

$$|\varepsilon_1^{(2)}| < \frac{1}{2t} < 1, \quad |\varepsilon_2^{(2)}| > \frac{(t+1)^{2m+1}}{t} > 1.$$

$\therefore\ m > 1 \Rightarrow R > 0.$

If $m = 1$ we find $R \simeq 7.066$.

D: $\Delta = (9v^4-28v^3-476v^2+64v)/16 < 0$ iff $v = -4, -2, 2, 4, 8$.

THEOREM 4. *When $\Delta > 0$,*

$$\varepsilon_1 = \frac{(\theta+1)^k}{(\theta-1)} \quad \textit{and} \quad \varepsilon_2 = \frac{(\theta+1)(\theta-2)}{2}$$

are independent units.

E: $\Delta = -(8v^3-49v^2+136v-36) < 0$ for $v = 2^k$.

Here, $f(x) = x^3+(v-2)x^2-x+2 = (x+1)(x-1)(x-2)+vx^2$. Factoring ideals:

$$(\theta+1) = 2_2\, 2_3^{k-1}, \quad (\theta) = 2_1 (\theta-1) = 2_2^{k-1}\, 2_3, \quad (\theta-2) = 2_1^{k+1}.$$

We also observe that $f(2-v) = v$ so $(\theta-2+v) = 2_1^k$. This leads to a 'bonus' unit $\varepsilon_3 = (\theta)^k/(\theta-2+v)$ in addition to

$$\varepsilon_1 = \frac{(\theta)^{k+2}}{(\theta-2)} \quad \text{and} \quad \varepsilon_2 = \frac{(\theta+1)(\theta-1)^k}{2^k}.$$

We note with regret that $|\varepsilon_1| = |\varepsilon_2| = |\varepsilon_3|$.

F: $\Delta = (225v^4-148v^3+724v^2-416v+576)/16 > 0$ for all powers of 2. We factor:

$$(\theta+1) = 2_1^k, \qquad (\theta-1) = 2_1,$$
$$(\theta) = 2_2^{k-1}\, 2_3 \qquad (\theta-2) = 2_2\, 2_3^{k+1}.$$

THEOREM 5.

$$\varepsilon_1 = \frac{(\theta-1)^k}{(\theta+1)} \quad \textit{and} \quad \varepsilon_2 = \frac{\theta^k(\theta-1)^{k^2-2}(\theta-2)^{k-2}}{2^{k^2-2}}$$

are independent units.

G: $f(-2) = -2v$, $f(-1) = 2$, $f(0) = v$, $f(1) = v$, $f(2) = 8$.

$\Delta = (9v^4-140v^3+244v^2-1120v+576)/16 < 0$ only for $v = 2, 4, 8$.

We factor:

$k \geqslant 3$	$k < 3$
$(\theta+2) = 2_2^k\, 2_3$	$(\theta+2) = 2_2^{k+1}$
$(\theta+1) = 2_1$	$(\theta+1) = 2_1$
$(\theta) = 2_2\, 2_3^{k-1}$	$(\theta) = 2_2^k$
$(\theta-1) = 2_1^k$	$(\theta-1) = 2_1^k$
$(\theta-2) = 2_2^2\, 2_3$	$(\theta-2) = 2_2^3$

Our units are

$$\varepsilon_1 = \frac{(\theta+1)^k}{(\theta-1)}$$

and (if $k \neq 1$)

$$\varepsilon_2 = \frac{(\theta+1)^{2k-3}(\theta)(\theta-2)^{k-2}}{2^{2k-3}}$$

(if $k = 1$)

$$\varepsilon_2 = \frac{(\theta-2)}{(\theta)^3}.$$

THEOREM 6. *When $\Delta > 0$, ε_1 and ε_2 are independent.*

1.2. $p = 3$. From the difference equation (*) we may assume, without loss of generality, that $|f(-1)| = 3$. We are then led to these four cases:

Table 2. Parametrizations of 3-adatropic cubic polynomials and their units

Case	$f(-1)$	$f(0)$	$f(1)$	$f(2)$	ε_1	ε_2
A	3	v	v	9	$\frac{(\theta+1)^2}{(\theta-2)}$	$\frac{(\theta+1)^k\theta(\theta-1)}{3^k}$
B	3	3	v	$3v$	$\frac{(\theta+1)^{k+1}}{(\theta-2)}$	$\frac{(\theta+1)^k\theta^k(\theta-1)}{3^k}$
C	3	v	-3	$-3v$	$\frac{(\theta+1)^{k+1}}{(\theta-2)}$	$\frac{(\theta+1)^k\theta(\theta-1)^k}{3^k}$
D	-3	v	v	3	$\frac{(\theta+1)}{(\theta-2)}$	$\frac{(\theta+1)^k\theta(\theta-1)}{3^k}$

THEOREM 7. *When the Dirichlet rank is 2 (here $\Delta > 0$), the parametric units defined in Table 2 are independent.*

Proof. Analogous to those for $p = 2$.

2. p-adatropic quartics. From the difference equation

$$f(-2)-4f(-1)+6f(0)-4f(1)+f(2) = 24,$$

we obtain

$$f(x) = x^4+ax^3+bx^2+cx+d,$$

where

$$a = \frac{f(2)-2f(1)+2f(-1)-f(-2)}{12}, \quad b = \frac{f(-1)-2f(0)+f(1)-2}{2}$$

$$c = \frac{f(-2)-8f(-1)+8f(1)-f(2)}{12}, \quad d = f(0).$$

2.1. 2-adatropic quartics. A table will be given of all parametrizations of powers of 2 satisfying the difference equation. In this table, type 1 implies that a choice of sign for $v = \pm 2^k$ is needed to ensure that the root, θ, of the polynomial will be an integer of the field. Type 3, indicates that neither choice of sign will make θ integral. (In general, θ will be an integer whenever $f(-2) \equiv f(1) \pmod 3$.) A type of 2 means that $f(\theta)$ is reducible over a quadratic extension of Q.

Table 3. Parametrizations of 2-adatropic polynomials of degree 4

Case	Type	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
1	1	8	v	2	$-v$	4
2	1	16	v	2	$-v$	-4
3	1	4	-2	2	v	$4v$
4	1	-4	-4	2	v	$4v$
5		4	v	2	-2	$4v$
6		-4	v	2	-4	$4v$
7	1	32	v	-2	$-v$	4
8	1	4	-8	-2	v	$4v$
9		4	v	-2	-8	$4v$
10	1, 2	$4v$	v	4	y	$4y$
11	2	$4y$	v	4	y	$4v$
12	1	y	v	4	$-v$	$-y$
13	1	32	v	-4	$-v$	16
14	1, 2	64	v	-4	$-v$	-16
15	1	16	-8	-4	v	$4v$
16		16	v	-4	-8	$4v$
17	1	-16	-16	-4	v	$4v$
18		-16	v	-4	-16	$4v$
19	1	32	-4	-4	v	$4v$
20		32	v	-4	-4	$4v$
21	1	64	4	-4	v	$4v$
22		64	v	-4	4	$4v$
23	1	v	-4	-4	-8	$-v$
24	1	32	2	v	v	$-2v$
25	1	32	2	v	$2v$	$2v$
26	1	32	2	$2v$	v	$-8v$
27	1	32	2	$-2v$	v	$16v$
28	1	-16	2	8	v	$4v$
29	1	-64	2	16	v	$4v$
30	1	128	2	-16	v	$4v$
31	1	v	2	8	4	$-v$
32	1	v	2	16	16	$-v$
33	1	v	2	-16	-32	$-v$
34	1	16	-2	v	v	$-2v$
35	1	16	-2	v	$2v$	$2v$
36	1	16	-2	$2v$	v	$-8v$
37	1	16	-2	$-2v$	v	$16v$
38	1	-32	-2	8	v	$4v$
39	1	64	-2	-8	v	4
40	1	v	-2	8	8	$-v$
41	1	v	-2	-8	-16	$-v$
42	1	-8	4	8	v	$4v$
43	1	8	-4	v	v	$-2v$
44	1	8	-4	v	$2v$	$2v$
45	1	8	-4	$2v$	v	$-8v$
46	1	8	-4	$-2v$	v	$16v$
47		32	$2v$	v	2	$2v$

Table 3 (cont.)

Case	Type	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
48		32	v	v	2	$-2v$
49		32	v	$2v$	2	$-8v$
50		32	v	$-2v$	2	$16v$
51		-64	v	16	2	$4v$
52		128	v	-16	2	$4v$
53		-16	v	8	2	$4v$
54		16	v	v	-2	$-2v$
55		16	$2v$	v	-2	$2v$
56		16	v	$2v$	-2	$-8v$
57	2	16	v	$-2v$	-2	$16v$
58		-32	v	8	-2	$4v$
59		64	v	-8	-2	$4v$
60		-8	v	8	4	$4v$
61		8	v	v	-4	$-2v$
62		8	$2v$	v	-4	$2v$
63		8	v	$2v$	-4	$-8v$
64		8	v	$-2v$	-4	$16v$
65	1	8	8	8	v	$4v$
66		8	v	8	8	$4v$
67	1	8	v	8	$-v$	-32
68	1	8	v	-8	$-v$	64
69	1	8	-16	-8	v	$4v$
70		8	v	-8	-16	$4v$
71	1	8	v	$2v$	$2v$	16
72	1	8	$2v$	$2v$	v	16
73	1	8	v	$-2v$	$-4v$	16
74	1	8	$-4v$	$-2v$	v	16
75	1	-8	-8	v	v	$-2v$
76	1	-8	-8	v	$2v$	$2v$
77	1	-8	-8	$2v$	v	$-8v$
78	1	-8	-8	$-2v$	v	$16v$
79	2	-8	v	v	-8	$-2v$
80		-8	$2v$	v	-8	$2v$
81		-8	v	$-2v$	-8	$16v$
82	1	-8	v	8	$-v$	-16
83	1	-8	16	16	v	$4v$
84		-8	v	16	16	$4v$
85	1	-8	v	16	$-v$	-64
86	1	-8	-32	-16	v	$4v$
87		-8	v	-16	-32	$4v$
88	1	-8	v	-16	$-v$	128
89	1	-8	v	$-2v$	$-4v$	32
90	1	-8	$-4v$	$-2v$	v	32
91	1	-8	$2v$	$2v$	v	32
92		$16v$	$2v$	4	v	$-4v$
93		$16v$	v	4	v	$-8v$
94		$-2v$	v	4	$-2v$	$-2v$
95		$-4v$	v	4	$-4v$	$-8v$
96	1	$2v$	2	v	-8	$-8v$
97	1	$2v$	-8	v	2	$-8v$

Table 3 (cont.)

Case	Type	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
98	1	$2v$	-2	v	-4	$-8v$
99	1	$2v$	-4	v	-2	$-8v$
100	1	$-2v$	2	v	-8	$-4v$
101	1	$-2v$	-8	v	2	$-4v$
102	1	$-2v$	-2	v	-4	$-4v$
103	1	$-2v$	-4	v	-2	$-4v$
104	3	$-4v$	$-4v$	4	v	$-8v$
105	3	$-4v$	$2v$	4	v	$16v$

Cases 10 and 11 were studied extensively by H. Cohn [2], [3]. Other interesting cases include those where six consecutive integers yield powers of 2 instead of just the required five. These occur in cases 42, 79, and 80. Actually, 42 does not deserve to be a separate case at all; its six consecutive ordinates $-v$ -8 4 8 v $4v$ make it a special subcase of case 11.

Case 79 is worthy of closer examination. In addition to the polynomial yielding a bonus power of 2 ($f(-3) = -2v$), the cubic resolvent has an integer root $\lambda = (v+12)/2$. Let $\tau = [(v+4)^2+2^7]/4$. Then

$$f(x) = x^4+2x^3-\frac{v+10}{2}x^2-\frac{v+12}{2}x+v$$

$$= \left[x^2+x+\frac{-\lambda+\sqrt{\tau}}{2}\right]\left[x^2+x+\frac{-\lambda-\sqrt{\tau}}{2}\right].$$

In fact, $f(x) = f(-1-x)$.

We factor, for $k > 2$:

$$(\theta+3) = 2_{12}^{k}\,2_{22}, \quad (\theta+1) = 2_{12}\,2_{22}^{k-1}, \quad (\theta-1) = 2_{12}^{2}\,2_{22},$$

$$(\theta+2) = 2_{11}^{2}\,2_{21}, \quad (\theta) = 2_{11}\,2_{21}^{k-1}, \quad (\theta-2) = 2_{11}^{k}\,2_{21}.$$

We find the quadratic unit

$$\varepsilon_3 = \frac{(\theta+2)^{k-1}(\theta-1)^{k-1}}{2^{k-2}(\theta+3)(\theta-2)}$$

and also the unit

$$\varepsilon_1 = \frac{(\theta+3)^{2k-3}(\theta+1)^{k-2}}{(\theta-1)^{k^2-k-1}}.$$

For a concrete example we will take $v = 16$ and so

$$\theta = \frac{-1+\sqrt{29+4\sqrt{33}}}{2}.$$

Then $\varepsilon_3 = 23+4\sqrt{33}$ and

$$\varepsilon_1 = 957776+169782\sqrt{33}-130489\sqrt{29+4\sqrt{33}}-23960\sqrt{33}\sqrt{29+4\sqrt{33}}.$$

2.2. 3-adatropic quartics.

Table 4. Parametrizations of 3-adatropic polynomials of degree 4

	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
A	$-3v$	-3	v	-3	$-3v$
B	3	v	3	$-v$	3
C	-3	v	3	$-v$	9
D	-3	v	-9	$-v$	81
E	-3	v	9	$-v$	-27
F	$-3v$	-9	v	3	$-3v$
G	$3v$	-3	v	-3	$-9v$
H	$3v$	-9	v	3	$-9v$
I	$3v$	3	v	-9	$-9v$

There are nine parametrized families which satisfy the difference equation. Each of these families leads to units (for example, in Case E we have $\varepsilon_1 = (\theta+2)^k/(\theta-1)$ and $\varepsilon_2 = (\theta+2)^{4k-6}(\theta+1)^2(\theta)^{2k-3}(\theta-2)^{2k-4}/3^{4k-6}$) but it is only in Case B that we are able to establish independence.

THEOREM 8. *Let L be one of the parametrized family of fields generated by the polynomial*

$$f(x) = x^4+\frac{v}{3}x^3-4x^2-\frac{4v}{3}x+3.$$

Let $f(\theta) = 0$ and define units as follows:

$$\varepsilon_1 = \frac{(\theta+2)^k}{(\theta-1)}, \quad \varepsilon_2 = \frac{(\theta-2)^k}{(\theta+1)}, \quad \varepsilon_3 = \frac{(\theta-2)(\theta+2)(\theta)^2}{3}.$$

Then when the Dirichlet rank is 3, the units $\varepsilon_1, \varepsilon_2, \varepsilon_3$, form an independent system.

Proof. We will outline the proof in the positive case for v sufficiently large. The negative case may be handled in an analogous manner and the exceptionally small values of $|v|$ are easily checked by a computer.

Let $v \geqslant 8$. The roots of $f(x)$ are bounded as follows:

$$\frac{2}{v} < \theta_1 < \frac{3}{v}, \quad 2-\frac{2}{v} < \theta_2 < 2-\frac{1}{v}, \quad -2-\frac{2}{v} < \theta_3 < -2-\frac{1}{v},$$

$$-\frac{v}{3} < \theta_4 < 1-\frac{v}{3}.$$

Denoting the jth conjugate of ε_i by $\varepsilon_i^{(j)}$,

$$1 < |\varepsilon_1^{(1)}|, \quad 1 < |\varepsilon_1^{(2)}| < 2^{2k+1}, \quad |\varepsilon_1^{(3)}| < \left(\frac{2}{v}\right)^k < 3^{k(1-k)} < 1,$$

$$1 < |\varepsilon_2^{(1)}| < 2^k, \quad |\varepsilon_2^{(2)}| < \left(\frac{2}{v}\right)^k < 3^{k(1-k)} < 1, \quad 1 < 4^k < |\varepsilon_2^{(3)}| < 5^k,$$

$$3^{1-2k} < |\varepsilon_3^{(1)}| < 3^{3-2k} < 1, \quad 3^{-k} < |\varepsilon_3^{(2)}| < 3^{3-k} < 1, \quad 3^{1-k} < |\varepsilon_3^{(3)}| < 1.$$

Let $Q_{ij} = \ln|\varepsilon_i^{(j)}|$, then the regulator,

$$R = \det\begin{vmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{vmatrix} = Q_{11}Q_1 - Q_{12}Q_2 + Q_{13}Q_3$$

where Q_m is the appropriate minor.

It is obvious that Q_{11}, $Q_1 > 0$. Also, it can be shown that

$$Q_{13}Q_3 - Q_{12}Q_2 > k^2(k-1)^2(2k-3)(\ln 3)^2 - k(2k+1)(2k-1)\ln 2\ln 3\ln 5$$
$$> k(\ln 3)^3(2k^4-7k^3+4k^2-3k+1) > 0.$$

The last inequality holds because $k \geqslant 3$.

$\therefore R \neq 0$.

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