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On the parity of $p(n)$

by

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1. Introduction. About twenty years ago, the second author in [4] made the conjecture that for any given integer $m \geqslant 1$ and every r, $0 \leqslant r \leqslant m-1$, the partition function $p(mn+r)$ takes even values, as well as odd values, each for infinitely many n. In the case $m = 1$, the result is due, independently, to O. Kolberg [2] and Morris Newman [3]. The case $m = 2$ is settled in [4]. We have a proof of the conjecture for $m = 4$, but are suppressing it because we prove here that the conjecture holds for $m = 16$. Note that if the conjecture holds for a positive integer m, it also holds for all divisors of m.

2. The main result. We now prove the

Theorem 2.1. *For each r, $0 \leqslant r \leqslant 15$, $p(16n+r)$ is infinitely often even, infinitely often odd.*

Proof. We have, modulo 2,

$$\sum p(n)x^n = \frac{1}{\phi(x)} = \prod_{n\geqslant 1}\frac{1}{(1-x^n)} \tag{2.1}$$

$$\equiv \prod_{n\geqslant 1}\frac{1-x^{2n}}{1-x^{2n-1}}\cdot\frac{1}{1-x^{4n}}$$

$$= \sum_{n\geqslant 0} x^{\Delta(n)}/\phi(x^4), \qquad \Delta(n) = n(n+1)/2.$$

$$\equiv \sum_{n\geqslant 0} x^{\Delta(n)} \sum_{n\geqslant 0} x^{4\Delta(n)}/\phi(x^{16}).$$

(We note in passing that continuing the iteration in (2.1) leads to a direct proof of the more important part of Theorem 1 of [1].)

It follows from (2.1) that

$$\phi(x^{16})\sum_{n\geqslant 0} p(n)x^n \equiv \sum_{n_1,n_2\geqslant 0} x^{\Delta(n_1)+4\Delta(n_2)} = \sum_{n\geqslant 0} c(n)x^n, \text{ say.} \tag{2.2}$$

Now, $\Delta(n) \equiv 0$, 1, 3 or 6 (mod 9), $4\Delta(n) \equiv 0$, 3, 4 or 6 (mod 9), so that $c(n) = 0$ if $n \equiv 2$ or 8 (mod 9).

Write $p_r(n) = p(16n+r)$. Define k_r to be the smallest k for which $p_r(k)$ is odd. A table of k_r is given below:

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
k_r	0	0	1	0	0	0	0	0	1	2	5	2	0	0	0	3

Next let l_r be given by the following table:

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
l_r	4	5	4	7	8	3	1	2	1	3	4	2	7	8	3	4

Suppose $p_r(n)$ is odd (alternatively even) for $n \geqslant n_0(r)$. We can suppose $n_0 \equiv l_r \pmod 9$ and that $2n_0+1 > k_r$.

Now let $N = N_r = (3n_0^2+n_0)/2+k_r$. Note that

$$16N+r \equiv 16((3l_r^2+l_r)/2+k_r)+r \equiv 2 \pmod 9, \quad \text{so} \quad c(16N+r) = 0.$$

It follows from (2.2) that, modulo 2,

$$(2.3)\qquad p_r(N)+p_r(N-1)+p_r(N-2)+p_r(N-5)+p_r(N-7)+\ldots \\ \ldots+p_r(n_0+k_r)+p_r(k_r) \equiv 0.$$

(The condition $2n_0+1 > k_r$ guarantees that $p_r(k_r)$ is indeed the last non-zero term on the left of (2.3).)

But the left hand side of (2.3) is odd (there is an odd number $(2n_0+1)$ of terms, the last is odd, the others are all odd (alternatively even)). So we have a contradiction, and our theorem is proved.

References

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Finiteness criteria for decomposable form equations

by

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1. Introduction. Let K be a finitely generated extension field of $\mathbf{Q}$, and R a finitely generated extension ring of $\mathbf{Z}$ in K. Let $F(X_1, \ldots, X_m)$ be a form in $m \geqslant 2$ variables with coefficients in K, and suppose that F is *decomposable* (i.e. that it factorizes into linear factors over some finite extension, G say, of K). Let b be an element of K^*(1) and consider the *decomposable form equation*

$$(1)\qquad F(x_1, \ldots, x_m) = b \quad \text{in} \quad x_1, \ldots, x_m \in R.$$

The decomposable form equations are of basic importance in the theory of diophantine equations and have many applications in algebraic number theory. Important classes of decomposable form equations are Thue equations (when $m = 2$), norm form equations, discriminant form equations and index form equations. The Thue equations are named after A. Thue [31] who proved in the case $K = \mathbf{Q}$, $R = \mathbf{Z}$, $m = 2$, that if F is a binary form having at least three pairwise linearly independent linear factors in its factorization over the field of algebraic numbers, then (1) has only finitely many solutions. After several generalizations, Lang [13] finally extended Thue's result to the general case considered above (when K is an arbitrary finitely generated extension of $\mathbf{Q}$ and R is an arbitrary finitely generated subring of K over $\mathbf{Z}$).

In the case that $K = \mathbf{Q}$, $R = \mathbf{Z}$, and F is a norm form, Schmidt [24] gave a necessary and sufficient condition for F such that (1) has only finitely many solutions for every $b \in \mathbf{Q}^*$. Later he generalized [25] this result by showing that all solutions of an arbitrary norm form equation over $\mathbf{Z}$ belong to finitely many families (cf. [25]) of solutions. These results of Schmidt were later extended by Schlickewei [20] to the case of arbitrary finitely generated subrings R of $\mathbf{Q}$ and by Laurent [14] to the above general case (when R is

(1) K^* denotes the set of non-zero elements of K. In general, for any integral domain R, R^* will denote the unit group (i.e. the multiplicative group of invertible elements) of R.