On the parity of \( p(n) \)

by

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1. Introduction. About twenty years ago, the second author in [4] made the conjecture that for any given integer \( m \geq 1 \) and every \( r, 0 \leq r \leq m-1 \), the partition function \( p(mn+r) \) takes even values, as well as odd values, each for infinitely many \( n \). In the case \( m = 1 \), the result is due, independently, to O. Kolberg [2] and Morris Newman [3]. The case \( m = 2 \) is settled in [4]. We have a proof of the conjecture for \( m = 4 \), but are suppressing it because we prove here that the conjecture holds for \( m = 16 \). Note that if the conjecture holds for a positive integer \( m \), it also holds for all divisors of \( m \).

2. The main result. We now prove the

Theorem 2.1. For each \( r, 0 \leq r \leq 15 \), \( p(16n+r) \) is infinitely often even, infinitely often odd.

Proof. We have, modulo 2,

\[
\sum p(n)x^n = \frac{1}{\phi(x)} = \prod_{n \geq 1} \frac{1}{(1-x^n)}
\]

\[
= \prod_{n \geq 1} \frac{1-x^{2n}}{1-x^{2n-1}} \cdot \frac{1}{1-x^{4n}}
\]

\[
= \sum_{\sigma > 0} x^{\sigma(n)}/\phi(x^4), \quad \sigma(n) = n(n+1)/2.
\]

\[
= \sum_{\sigma > 0} x^{\sigma(n)} \sum_{\sigma > 0} x^{4\sigma(n)}/\phi(x^{16n}).
\]

(We note in passing that continuing the iteration in (2.1) leads to a direct proof of the more important part of Theorem 1 of [1].)

It follows from (2.1) that

\[
\phi(x^{16}) \sum_{\sigma > 0} p(n)x^n = \sum_{\sigma_1, \sigma_2 > 0} x^{\sigma_1(n) + 4\sigma_2} = \sum_{\sigma > 0} \sigma(n)x^n, \text{ say.}
\]

Now, \( \sigma (n) \equiv 0, 1, 3 \text{ or } 6 \pmod{9} \), \( 4\sigma (n) \equiv 0, 3, 4 \text{ or } 6 \pmod{9} \), so that \( \sigma(n) = 0 \) if \( n \equiv 2 \text{ or } 8 \pmod{9} \).
Write $p_r(n) = p(16n+r)$. Define $k_r$ to be the smallest $k$ for which $p_r(k)$ is odd. A table of $k_r$ is given below:

<table>
<thead>
<tr>
<th>$r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_r$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Next let $l_r$ be given by the following table:

<table>
<thead>
<tr>
<th>$r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_r$</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>7</td>
<td>8</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Suppose $p_r(n)$ is odd (alternatively even) for $n > n_0(r)$. We can suppose $n_0 = l_r$ (mod 9) and that $2n_0+1 > k_r$.

Now let $N = N_r = (3l_r^2 + n_0)/2 + k_r$. Note that

$16N + r = 16((3l_r^2 + n_0)/2 + k_r) + r \equiv 2$ (mod 9),

so $c(16N + r) = 0$.

It follows from (2.2) that, modulo 2,

$$p_r(N) + p_r(N-1) + p_r(N-2) + p_r(N-5) + p_r(N-7) + \ldots$$

$$\ldots + p_r(n_0 + k_r) + p_r(k_r) \equiv 0.$$

(The condition $2n_0 + 1 > k_r$ guarantees that $p_r(k_r)$ is indeed the last non-zero term on the left of (2.3).)

But the left hand side of (2.3) is odd (there is an odd number ($2n_0 + 1$) of terms, the last is odd, the others are all odd (alternatively even)). So we have a contradiction, and our theorem is proved.

References


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