On a

$$f(x) = \prod_{n=1}^{\infty} (1+x^n/n). \tag{2}$$

Posons alors

$$g(x) = (1-x) f(x), \tag{3}$$

ce qui s'écrit aussi

$$g(x) = f(x) \prod_{n=1}^{\infty} \exp(-x^n/n). \tag{4}$$

On montre que le développement en série de Taylor à l'origine de g converge normalement sur $[0, 1[$. Avec (2) et (4) on a

$$g(1) = \exp(-\gamma). \tag{5}$$

Les relations (1) et (3) donnent alors le résultat annoncé.

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Reçu le 5.2.1986
et dans la forme modifiée le 17.9.1986 (1588)

ACTA ARITHMETICA
L (1988)

Note on an index formula of elliptic units in a ring class field II

by

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This short note is a supplement to our previous note [1].

Let all of the notation and terminology be the same as in [1]. In [1] we have proved the following:

PROPOSITION 1 ([1], Prop. 1). *Let c_1 and c_2 be any two classes in $\mathrm{Cl}(K/\Sigma)$ and n the least positive rational integer such that $n\big(l(c_1)-1\big)\big(l(c_2)-1\big) \equiv 0 \pmod{24}$. Then*

$$\left\{\frac{\delta_K(c_1 c_2)}{\delta_K(c_1)\,\delta_K(c_2)}\right\}^n \in E_K^{24h}.$$

Our proof for this proposition (in [1]) contains somewhat incomplete parts. Indeed, in the arguments (in Steps 2 and 3 on pp. 208–209), some tedious verifications have been omitted. In this short note we shall give another simple and complete proof for the same proposition.

Proof of Proposition 1. It suffices to consider only the case where $K = K_f$. Moreover, since in the case where $(D, f) = (-3, 1), (-3, 2), (-3, 3), (-4, 1)$ or $(-4, 2)$, the assertion is trivial, we may exclude these cases throughout this proof.

Let C_1 and C_2 be any two classes in $\mathrm{Cl}(K_f/\Sigma)$ and let n be the least positive integer such that $n\big(l(C_1)-1\big)\big(l(C_2)-1\big) \equiv 0 \pmod{24}$. According to the arguments in [1] (Step 1 and the first half of Step 2, pp. 206–207), we have

$$\left\{\frac{\delta_f(C_1 C_2)}{\delta_f(C_1)\,\delta_f(C_2)}\right\}^{2n} \in E_{K_f}^{24h}, \tag{1.1}$$

and especially when none of three classes C_1^2, C_2^2 and $C_1^2 C_1^2$ is equal to the unit class C_0 in $\mathrm{Cl}(K_f/\Sigma)$,

$$\left\{\frac{\delta_f(C_1 C_2)}{\delta_f(C_1)\,\delta_f(C_2)}\right\}^{n} \in E_{K_f}^{24h}. \tag{1.2}$$

Even in the cases where at least one of C_1^2, C_2^2 and $C_1^2 C_2^2$ is equal to C_0, when there exists a class B in $\mathrm{Cl}(K_f/\Sigma)$ such that the order of B is equal to an odd prime number q ($\geqslant 5$), we can obtain the same conclusion. Indeed, for any i ($i = 1, 2, \ldots$)

$$\left\{\frac{\delta_f(C_1 C_2)}{\delta_f(C_1)\delta_f(C_2)}\right\}^{\sigma(B^i)} = \frac{\delta_f(C_1 C_2 B^{2i})}{\delta_f(C_1 B^i)\delta_f(C_2 B^i)} \frac{\delta_f(C_1 C_2 B^i)\delta_f(B^i)}{\delta_f(C_1 C_2 B^{2i})},$$

and since $q \geqslant 5$, it is always possible to choose a suitable i so that none of $(C_1 B^i)^2$, $(C_2 B^i)^2$, $(C_1 C_2 B^i)^2$ and $(C_1 C_2 B^{2i})^2$ is equal to C_0. Of course $l(B^i) \equiv l(B^{2i}) \equiv 1 \pmod{24}$. Therefore $\{\delta_f(C_1 C_2)/\delta_f(C_1)\delta_f(C_2)\}^{n\sigma(B^i)}$ and also $\{\delta_f(C_1 C_2)/\delta_f(C_1)\delta_f(C_2)\}^n$ are contained in $E_{K_f}^{24h}$.

If there no longer exists a class B in $\mathrm{Cl}(K_f/\Sigma)$ whose order is an odd prime number q ($\geqslant 5$), we may use the norm relation ([1], Lemma 3). Indeed, let q_1 be any one odd prime number such that $(q_1, 6f) = 1$, and for each i ($i = 1, 2$), let $\bar{C}_i$ and $\bar{\bar{C}}_i$ be any classes in $\mathrm{Cl}(K_{fq_1}/\Sigma)$ and $\mathrm{Cl}(K_{fq_1^2}/\Sigma)$ respectively such that

$$\mathrm{Res}_{K_f}\sigma(\bar{C}_i) = \mathrm{Res}_{K_f}\sigma(\bar{\bar{C}}_i) = \sigma(C_i) \quad \text{and} \quad \mathrm{Res}_{K_{fq_1}}\sigma(\bar{\bar{C}}_i) = \sigma(\bar{C}_i).$$

Here we note that $l(C_i) \equiv \bar{l}(\bar{C}_i) \equiv \bar{\bar{l}}(\bar{\bar{C}}_i) \pmod{24}$, where $\bar{l}$ (resp. $\bar{\bar{l}}$) means the homomorphism from $\mathrm{Cl}(K_{fq_1}/\Sigma)$ (resp. $\mathrm{Cl}(K_{fq_1^2}/\Sigma)$) into $(\boldsymbol{Z}/24\boldsymbol{Z})^\times$ defined in the same way as in Section 2 of [1]. Then by (4) of Lemma 3 ([1]), we have

$$\text{(1.3)} \quad \left\{\frac{\delta_{fq_1}(\bar{C}_1 \bar{C}_2)}{\delta_{fq_1}(\bar{C}_1)\delta_{fq_1}(\bar{C}_2)}\right\}^{n(q_1+1)}$$

$$= \left\{\frac{\delta_f(C_1 C_2)}{\delta_f(C_1)\delta_f(C_2)}\right\}^n \times N_{fq_1^2, fq_1}\left(\left\{\frac{\delta_{fq_1^2}(\bar{\bar{C}}_1 \bar{\bar{C}}_2)}{\delta_{fq_1^2}(\bar{\bar{C}}_1)\delta_{fq_1^2}(\bar{\bar{C}}_2)}\right\}^n\right).$$

Since there exists a class $\bar{\bar{B}}$ of order q_1 in $\mathrm{Cl}(K_{fq_1^2}/\Sigma)$, $N_{fq_1^2, fq_1}(\{-\}^n)$ (in formula (1.3)) is contained in $E_{K_{fq_1}}^{24h}$. On the other hand, by formula (1.1), the left-hand side of formula (1.3) is contained in $E_{K_{fq_1}}^{24h}$. Therefore $\{\delta_f(C_1 C_2)/\delta_f(C_1)\delta_f(C_2)\}^n$ is the $24h$-th power in $E_{K_{fq_1}}$. The same fact holds also for the other prime number q_2 such that $(q_2, 6f) = 1$. Hence $\{\delta_f(C_1 C_2)/\delta_f(C_1)\delta_f(C_2)\}^n$ must be the $24h$-th power in $E_{K_{fq_1}} \cap E_{K_{fq_2}} = E_{K_f}$. (Note that $w_{K_f} = w_{K_{fq_1}} = w_{K_{fq_2}} = w_{K_{fq_1q_2}}$.)

A computational example. Let $\Sigma = \boldsymbol{Q}(\sqrt{-8})$ and let K be the ring class field K_6 over Σ with conductor 6. Then $\mathrm{Cl}(K_6/\Sigma)$ is of type (2, 2). Let C_0, C_1, C_2 and C_3 be represented by the following four O_6-ideals

$$[1, 6\sqrt{-2}], \quad [2, 3\sqrt{-2}+1], \quad [3, 2\sqrt{-2}] \quad \text{and} \quad [6, \sqrt{-2}+3]$$

respectively. Now $[19, \sqrt{-2}+6]$ $(:= \mathfrak{p}_1)$, $[17, \sqrt{-2}+7]$ $(:= \mathfrak{p}_2)$ and $[11, \sqrt{-2}+3]$ $(:= \mathfrak{p}_3)$ are K_6-admissible prime ideals of degree 1. By the modular transformation of their basis quotients, we see that $\mathfrak{p}_i \cap O_6$ belongs to C_i ($i = 1, 2, 3$) and hence $l(C_1) \equiv 19$, $l(C_2) \equiv 17$ and $l(C_3) \equiv 11 \pmod{24}$. By the numerical computation, the following equalities can be confirmed:

$$\frac{\delta_6(C_1 C_2)}{\delta_6(C_1)\delta_6(C_2)} = \frac{\Delta([6, \sqrt{-2}+3])\Delta([1, 6\sqrt{-2}])}{\Delta([2, 3\sqrt{-2}+1])\Delta([3, 2\sqrt{-2}])} = (2+\sqrt{3})^{24},$$

$$\left\{\frac{\delta_6(C_1 C_3)}{\delta_6(C_1)\delta_6(C_3)}\right\}^2 = \left\{\frac{\Delta([3, 2\sqrt{-2}])\Delta([1, 6\sqrt{-2}])}{\Delta([2, 3\sqrt{-2}+1])\Delta([6, \sqrt{-2}+3])}\right\}^2 = \theta^{24},$$

where $\theta^2 + (10+7\sqrt{-2})\theta + 1 = 0$,

$$\left\{\frac{\delta_6(C_2 C_3)}{\delta_6(C_2)\delta_6(C_3)}\right\}^3 = \left\{\frac{\Delta([2, 3\sqrt{-2}+1])\Delta([1, 6\sqrt{-2}])}{\Delta([3, 2\sqrt{-2}])\Delta([6, \sqrt{-2}+3])}\right\}^3$$

$$= (49+20\sqrt{6})^{24}.$$

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Received on 13.5.1986 (1635)