Conspectus materiae tomi L, fasciculi 4

La revue est consacrée à la Théorie des Nombres
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ACTA ARITHMETICA
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Рукописи статей редакция просит предлагать в двух экземплярах

ISBN 83-01-08174-0 ISSN 0065-1036

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Discrepancy estimates for the value-distribution of the Riemann zeta-function III

by

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1. Introduction. In the previous papers of the author ([7], [8], see also [9]), we discussed the value-distribution of the Riemann zeta-function $\zeta(s)$ in the half-plane $\operatorname{Re} s = \sigma > 1$, and obtained some refinements of Bohr–Jessen's classical results which were proved in [2]. In this paper we will consider the value-distribution of $\zeta(s)$ in a more significant region: the strip $\frac{1}{2} < \sigma \leqslant 1$.

Since the Riemann hypothesis is not yet proved, we cannot exclude the possibility of the existence of zeros of $\zeta(s)$ in this strip. Hence, to secure that $\log\zeta(s)$ is single-valued, we restrict our consideration to the set

$$G = \{\tfrac{1}{2} < \sigma\} - \bigcup_{s_j = \sigma_j + it_j} \{s = \sigma + it_j |\ \tfrac{1}{2} < \sigma \leqslant \sigma_j\},$$

where s_j's ($j = 1, 2, \ldots$) run through all zeros of $\zeta(s)$ in the region $\frac{1}{2} < \sigma \leqslant 1$. For any $s_0 = \sigma_0 + it_0 \in G$, we define $\log\zeta(s_0)$ by the analytic continuation along the path $\{s = \sigma + it_0 |\ \sigma_0 \leqslant \sigma\}$.

First we fix a $\sigma_0 \in (\frac{1}{2}, 1]$, and discuss the value-distribution of $\log\zeta(s)$ on the line $\sigma = \sigma_0$. Let R be any closed rectangle in the complex z-plane with the edges parallel to the axes, and $L(T, R)$ the (Jordan) measure of the set $\{t \in [1, T] |\ \sigma_0 + it \in G,\ \log\zeta(\sigma_0 + it) \in R\}$. Then, Bohr–Jessen [3] proved that there exists the limit

$$W(R) = \lim_{T \to \infty} L(T, R)/T, \tag{1.1}$$

which depends only on σ_0 and R. In this paper we will prove the following sharpening of (1.1):

THEOREM 1. *For any* $\sigma_0 \in (\frac{1}{2}, 1]$ *and* $\varepsilon > 0$, *we have*

$$L(T, R) = W(R)T + O\big((m(R) + \varepsilon)T(\log\log T)^{-(2\sigma_0 - 1)/15 + \varepsilon}\big), \tag{1.2}$$

where $m(R)$ *denotes the measure of* R, *and the* O-*constant depends only on* σ_0 *and* ε.

In [7], the author proved a similar result in the half-plane $\sigma > 1$. We have shown

$$L(T,R)=W(R)\,T+O\big((m(R)+\varepsilon)\,T(\log\log T)^{-(\sigma_0-1)/7+\varepsilon}\big) \tag{1.3}$$

for any $\sigma_0>1$. If $\sigma>1$, then $\zeta(s)$ has the Euler product expansion

$$\zeta(s)=\prod_{n=1}^{\infty}(1-p_n^{-s})^{-1},$$

where p_n is the nth prime number. Hence, if we put

$$f_N(s)=-\sum_{n=1}^{N}\log(1-p_n^{-s}),$$

then it is obvious that

$$\lim_{N\to\infty} f_N(s)=\log\zeta(s), \tag{1.4}$$

and the proof of (1.3) depends essentially on this fact. The basic structure of the proof of (1.2) is an analogue of that of (1.3), but in case $\frac{1}{2}<\sigma_0\leqslant 1$, the simple relation (1.4) holds no longer. So we must develop additional arguments concerning Carlson's mean-value theorem.

Next, let $\frac{1}{2}<\sigma_1<\sigma_2$, and a an arbitrary complex number. We denote by $N_a(T)$ the number of the elements of the set $\{s=\sigma+it\in G\mid \sigma_1<\sigma<\sigma_2,\ 1<t<T,\ \log\zeta(s)=a\}$. We remark that, in the definition of $N_a(T)$, and also throughout this paper, a-points are counted with multiplicity; an a-point of a function $f(s)$, that is, a zero point of $f(s)-a$, of order m is counted m times. It was proved by Bohr–Jessen [3] that there exists the limit

$$G(a)=\lim_{T\to\infty} N_a(T)/T, \tag{1.5}$$

which depends only on σ_1, σ_2 and a. The second result of this paper is the following sharpening of (1.5):

THEOREM 2. *For any $\frac{1}{2}<\sigma_1<\sigma_2$, we have*

$$N_a(T)=G(a)\,T+\begin{cases} O\big(T(\log\log T)^{-A}\big) & \text{if } \sigma_1>1,\\ O\big(T(\log\log T)^{-B/\log\log\log\log T}\big) & \text{if } \sigma_1\leqslant 1,\end{cases}$$

where A and B are positive constants which depend only on σ_1, σ_2 and a, and O-constants also depend only on σ_1, σ_2 and a.

In [8], we have shown a similar result only in the half-plane $\sigma>E$, where the number E has the properties that if $E<\sigma_1<\sigma_2$, then $|\zeta'/\zeta(s)|\geqslant C=C(\sigma_1,\sigma_2)>0$ in the strip $\sigma_1<\sigma<\sigma_2$, and that $2<E<3$ numerically. If $\sigma_1>E$, a lower-bound estimate of $|\log\zeta(s)-a|$ can be easily obtained in $\sigma_1<\sigma<\sigma_2$. (See § 6 of [8].) On the other hand, in case $\sigma_1\leqslant E$, we will deduce such a lower-bound estimate from Hilfssätze 3 and 4 of Bohr–Jessen [3], which are based on Jensen's theorem in complex function theory. And if $\sigma_1\leqslant 1$, in addition to the application of Jensen's theorem, we need rather delicate arguments concerning several mean-value lemmas.

At first we show auxiliary mean-value results in Sections 2, 3. Next we shall prove Theorem 1 in Section 4, and Sections 5–7 are devoted to the proof of Theorem 2.

In the following sections, ε denotes a small positive number, C a positive constant, and are not necessarily the same in each occurrence. The letters C_1, $C_2,\ldots$ also denote positive constants. By the symbol $\#S$ we mean the cardinality of the set S. For any subset X of the complex plane, we denote the Jordan measure of X by $m(X)$, and the boundary of X by ∂X. And $\operatorname{dist}(X,Y)=\inf\{|x-y|\mid x\in X,\ y\in Y\}$ for any two subsets X and Y.

The author expresses his gratitude to Professor Akio Fujii for constant encouragement and valuable advices; he first suggested to the author that Jensen's theorem is useful to our present problem. The author is also indebted to Professor D. R. Heath-Brown for pointing out an error in the original argument, and to Professor Leo Murata for useful discussions, both are concerning Carlson's mean-value theorem.

2. Mean-value lemmas. Let $\frac{1}{2}<\alpha_0<1$, $1\leqslant d<2$, δ a small positive number, N a positive integer, $H(d,t_0)=\{s=\sigma+it\mid \sigma>\alpha_0,\ t_0-\frac{1}{2}d<t<t_0+\frac{1}{2}d\}$ for any real t_0. In this and the next section, except for the statement and the proof of Lemma 7, the letter C and O-constants depend only on α_0, d and ε, and the letters C_1, $C_2,\ldots$ denote positive absolute constants. We put

$$R_N(s)=\log\zeta(s)-f_N(s)$$

for $\sigma\in G$, and define

$$\varphi_N^{\delta}(t_0)=\begin{cases}0 & \text{if } H(d,t_0)\subset G \text{ and } |R_N(s)|<\delta \text{ for any } s\in H(d,t_0),\\ 1 & \text{otherwise.}\end{cases}$$

We first prove the following

LEMMA 1. *We have*

$$T^{-1}\int_1^T \varphi_N^{\delta}(t_0)\,dt_0\ll \delta^{-2}\big(A_1+(N\log N)^{-4+\varepsilon}\log(\delta^{-1})\big)+T^{-1},$$

$$(=X(T,N,\delta),\ \text{say})$$

where $A_1=N^{1-2\alpha_0+\varepsilon}+T^{1-2\alpha_0+\varepsilon}\exp(CN^{1/2})$.

This lemma is a refinement of Bohr–Jessen's Satz A in [3]. This Satz is a direct consequence of Hilfssatz 5 of Bohr [1], and the proof of Hilfssatz 5 developed in [1] is based essentially on a mean-value theorem of Bohr–Landau [5].

In [1], Bohr considered the function

$$\zeta_N(s)=\zeta(s)\prod_{n=1}^{N}(1-p_n^{-s}).$$

In $\sigma>1$, $\zeta_N(s)-1$ can be written as the Dirichlet series

$$\zeta_N(s)-1=\sum_{n=1}^{\infty}a_n n^{-s}, \tag{2.1}$$

where $|a_n|\leqslant 1$ for any positive n, and $a_n=0$ for any $n<p_{N+1}$. For the proof of Hilfssatz 5 of [1], Bohr required a mean-value result for the function $\zeta_N(s)-1$. But the Dirichlet series (2.1) is not convergent if $\sigma\leqslant 1$, so in this case we cannot apply directly Bohr–Landau's theorem, which holds only for convergent Dirichlet series. Hence, Bohr applied Bohr–Landau's theorem for the function $(\zeta_N(s)-1)(1-2^{1-s})$, which has the Dirichlet series expansion, convergent in $\sigma>0$. (See Hilfssatz 2 of [1].)

Now, there is a more general result of F. Carlson [6] (see also Titchmarsh [10], § 9.51). Using Carlson's mean-value theorem, we can avoid this detour. Furthermore, the proof of Carlson's theorem is more convenient to refine than that of Bohr–Landau. In Section 3, we shall prove the following refinement of Carlson's theorem for $\zeta_N(s)-1$:

LEMMA 2. *Let α_1 be a real number which satisfies* $\max(\frac{1}{2},\alpha_0-\varepsilon)<\alpha_1<\alpha_0$. *Then,*

$$T^{-1}\int_1^T|\zeta_N(\sigma+it)-1|^2\,dt\ll A_1$$

holds uniformly in $\alpha_1\leqslant\sigma\leqslant 3$.

Besides, for $\sigma\geqslant 3$, we can show the following

LEMMA 3. *Let $\beta_1>3$. Then*

$$T^{-1}\int_1^T|\zeta_N(\sigma+it)-1|^2\,dt\ll(N\log N)^{2-2\sigma+\varepsilon}/(2\sigma-2-\varepsilon)$$

holds uniformly in $3\leqslant\sigma\leqslant\beta_1$.

Proof. By using the expression (2.1), we have

$$\begin{aligned}T^{-1}\int_1^T|\zeta_N(\sigma+it)-1|^2\,dt &=T^{-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m a_n(mn)^{-\sigma}\int_1^T(m/n)^{it}\,dt\\ &=(1-T^{-1})\sum_{n=1}^{\infty}a_n^2n^{-2\sigma}+O\Big(T^{-1}\sum_{m>n}\sum a_m a_n(mn)^{-\sigma}(\log(m/n))^{-1}\Big).\end{aligned}\tag{2.2}$$

We can evaluate the double sum in the error term of the above by the method similar to the proof of (7.2.1) of Titchmarsh [11]. The result is the inequality

$$\sum_{m>n}\sum a_m a_n(mn)^{-\sigma}(\log(m/n))^{-1}\ll\Big(\sum_{n=1}^{\infty}a_n n^{-\sigma}\Big)^2+\sum_{n=1}^{\infty}a_n n^{1-2\sigma}\log n. \tag{2.3}$$

Since $a_n=0$ for any $n<p_{N+1}$, it follows that

$$\sum_{n=1}^{\infty}a_n^2n^{-2\sigma}\ll\int_{p_N}^{\infty}x^{-2\sigma}\,dx\ll(N\log N)^{1-2\sigma}/(2\sigma-1),$$

and a similar estimate holds for the right-hand side of (2.3). These estimates with (2.2) imply the result of Lemma 3.

Now we deduce Lemma 1 from Lemmas 2 and 3. We put

$$\Phi_N(\tau)=\iint_{\substack{\alpha_1\leqslant\sigma\leqslant\beta_1\\ \tau-d\leqslant t\leqslant\tau+d}}|\zeta_N(s)-1|^2\,d\sigma\,dt\quad\text{for any }\tau\geqslant 3.$$

Then, using Lemmas 2 and 3, we have

$$\begin{aligned}\int_3^{T-2}\Phi_N(\tau)\,d\tau &\ll\iint_{\substack{\alpha_1\leqslant\sigma\leqslant\beta_1\\ 1\leqslant t\leqslant T}}|\zeta_N(s)-1|^2\,d\sigma\,dt\\ &\ll A_1(3-\alpha_1)+\int_3^{\beta_1}(N\log N)^{2-2\sigma+\varepsilon}/(2\sigma-2-\varepsilon)\,d\sigma\\ &\ll A_1+(N\log N)^{-4+\varepsilon}\log\beta_1.\end{aligned}$$

For any small positive ξ, we put $b=m\{\tau\in[3,T-2]\mid\Phi_N(\tau)\geqslant\xi\}$. Then it follows that

$$\xi b\leqslant\int_3^{T-2}\Phi_N(\tau)\,d\tau,$$

and therefore,

$$b\ll\xi^{-1}(A_1+(N\log N)^{-4+\varepsilon}\log\beta_1). \tag{2.4}$$

This is a refinement of Hilfssatz 3 of Bohr [1].

Next, let

$$2\leqslant\beta_0<\beta_1,\qquad Q(d,t_0)=H(d,t_0)\cap\{\sigma<\beta_0\},$$

and

$$P(d,t_0)=\{s=\sigma+it\mid\alpha_1\leqslant\sigma\leqslant\beta_1,\ t_0-d\leqslant t\leqslant t_0+d\}.$$

It is easily shown that, for $\sigma\geqslant 2$, the inequality $|R_N(s)|<C_1(\sigma-1)^{-1}$ holds (for any N) for some constant C_1. Hence, if we choose $\beta_0=(1+C_1\delta^{-1})\geqslant 2$

and $\beta_1 = 2\beta_0$, then $|R_N(s)| < \delta$ in the region $\sigma \geqslant \beta_0$. Under these choices of the values of β_0 and β_1, Bohr has shown, in the proof of Hilfssatz 5, that $\varphi_N^{\delta}(t_0) = 0$ if $|\zeta_N(s) - 1| < \frac{1}{2}\delta$ holds in $Q(d, t_0)$. Now we quote the following

LEMMA 4 (Bohr [1], Hilfssatz 4). *Let Γ, Γ' be two closed curves in the complex s-plane, and D, D' the open regions surrounded by Γ, Γ', respectively. We assume $\Gamma \cup D \subset D'$. If $f(s)$ is holomorphic in D' and*

$$\iint_{D'} |f(s)|^2 \, d\sigma \, dt < \pi\big(\tfrac{1}{2}\operatorname{dist}(\Gamma, \Gamma')\big)^2 (\tfrac{1}{2}\delta)^2,$$

then $|f(s)| < \frac{1}{2}\delta$ for any $s \in \Gamma \cup D$.

We apply this lemma to $\Gamma = \partial Q(d, t_0)$ and $\Gamma' = \partial P(d, t_0)$, under the above choices of the values of β_0 and β_1. Then we have that $\varphi_N^{\delta}(t_0) = 0$ if $\Phi_N(t_0) < \pi(\frac{1}{2}\lambda)^2(\frac{1}{2}\delta)^2$, where $\lambda = \operatorname{dist}(\partial Q(d, t_0), \partial P(d, t_0))$. Hence, applying (2.4) with $\xi = \pi(\frac{1}{2}\lambda)^2(\frac{1}{2}\delta)^2$, we have

$$T^{-1}\int_1^T \varphi_N^{\delta}(t_0)\, dt_0 \leqslant b + 4T^{-1} \ll \delta^{-2}\big(A_1 + (N \cdot \log N)^{-4+\varepsilon}\log(\delta^{-1})\big) + T^{-1}.$$

This is the estimate of Lemma 1.

Our next aim is to show refinements of Sätze B and C of Bohr–Jessen [3], which we shall use later in the proof of our Theorem 2. We first prove the following

LEMMA 5. *Let $f_m(s) = \exp(i^m f_N(s))$ $(m = 0, 1, 2, 3)$. Then we have*

$$T^{-1}\int_1^T \Big(\iint_{P(d,t_0)} |f_m(\sigma+it)|^2 \, d\sigma \, dt\Big) dt_0 \ll T^{-1}\exp(CN^{2(1-\alpha_1)+\varepsilon}) + \beta_1.$$

Proof. We denote the Dirichlet series expansion of $f_m(s)$ by $\sum C_n^m n^{-s}$ $(\sigma > 0)$. Then it is easily shown that $|C_n^m| \leqslant C_n^0$ for $m = 1, 2, 3$. Hence,

$$(2.5)\quad \Big|\sum_{n=1}^{\infty} C_n^m n^{-\sigma}\Big| \leqslant f_0(\sigma) = \prod_{n=1}^{N}(1 - p_n^{-\sigma})^{-1} \leqslant \exp\Big(C\sum_{n=1}^{N} p_n^{-\sigma}\Big) \leqslant \exp(CN^{1-\sigma})$$

holds uniformly in $\alpha_1 \leqslant \sigma \leqslant 3$. So, by the argument similar to the proof of Lemma 3, we have

$$T^{-1}\int_1^T |f_m(\sigma+it)|^2 \, dt \ll 1 + T^{-1}\exp(CN^{2(1-\sigma)+\varepsilon})$$

in $\alpha_1 \leqslant \sigma \leqslant 3$. On the other hand, in $\sigma \geqslant 3$, the estimate

$$T^{-1}\int_1^T |f_m(\sigma+it)|^2 \, dt \ll 1$$

is obvious. These inequalities lead to the assertion of Lemma 5.

Now we take a constant $C_2 > 2$ and choose $\beta_0 = C_2$ and $\beta_1 = C_2 + 1$. We note that there exists a constant $C_3 > 0$ for which $|f_N(s)| < C_3$ holds in $\sigma > C_2$ for any N. Let K be a large positive number, and define

$$\psi_K(t_0) = \begin{cases} 0 & \text{if } H(d, t_0) \subset G \text{ and } |\log\zeta(s)| < K \text{ for any } s \in H(d, t_0), \\ 1 & \text{otherwise.} \end{cases}$$

Bohr–Jessen's argument in the proof of Satz B implies that for any $K > C_3 + C_4$ with another positive constant C_4,

$$(2.6)\quad T^{-1}\int_1^T \psi_K(t_0)\, dt_0 \leqslant T^{-1}\int_1^T \varphi_N^{C_4}(t_0)\, dt_0 + e^{-K+C_4}\, T^{-1}\int_1^T \Big(\sum_{m=0}^{3} \sup\{|f_m(s)|^2 \mid s \in Q(d, t_0)\}\Big) dt_0$$

holds. We apply Lemma 4 again to obtain

$$|f_m(s)|^2 \ll \iint_{P(d,t_0)} |f_m(s)|^2 \, d\sigma \, dt$$

for any $s \in Q(d, t_0)$, so with Lemma 5, the second term of the right-hand side of (2.6) is estimated by $O\big(e^{-K}(T^{-1}\exp(CN^{2(1-\alpha_0)+\varepsilon}) + 1)\big)$ $(= e^{-K}Y(T, N)$, say). Combining with Lemma 1, we have

$$\begin{aligned} T^{-1}\int_1^T \psi_K(t_0)\, dt_0 &\ll X(T, N, C_4) + e^{-K}Y(T, N) \\ &\ll N^{1-2\alpha_0+\varepsilon} + T^{1-2\alpha_0+\varepsilon}\exp(CN^{1/2}) \\ &\quad + e^{-K}\big(T^{-1}\exp(CN^{2(1-\alpha_0)+\varepsilon}) + 1\big). \end{aligned}$$

Now we specify $N = [\log T]$. ($[x]$ denotes the integer part of x.) Then,

$$T^{1-2\alpha_0+\varepsilon}\exp(CN^{1/2}) \ll T^{-C} \quad \text{and} \quad T^{-1}\exp(CN^{2(1-\alpha_0)+\varepsilon}) \ll T^{-C},$$

so we arrive at the following

LEMMA 6.

$$T^{-1}\int_1^T \psi_K(t_0)\, dt_0 \ll (\log T)^{1-2\alpha_0+\varepsilon} + e^{-K}.$$

This is a refinement of Bohr–Jessen's Satz B.

Lastly we show a refinement of Satz C. We put

$$\mathscr{A} = \{s \in G \mid \log\zeta(s) = a\}, \qquad \mathscr{A}_N = \{s \mid \sigma > \tfrac{1}{2}, f_N(s) = a\},$$

$$n_a(d, t_0) = \#\big(H(d, t_0) \cap \mathscr{A}\big) \quad \text{and} \quad n_a^N(d, t_0) = \#\big(H(d, t_0) \cap \mathscr{A}_N\big).$$

We remark that in the statement and the proof of the following lemma, the constants C_5 and C_6 depend only on a, and O-constants depend only on α_0, d, a and ε.

LEMMA 7. *Let $\chi(t_0)$ be an arbitrary function defined for any real t_0, which only assumes the values* 0 *and* 1. *If*

$$T^{-1}\int_1^T \chi(t_0)\,dt_0 \leqslant \theta(T) < 1,$$

then

$$T^{-1}\int_1^T n_a(d, t_0)\chi(t_0)\,dt_0 \ll \theta(T)^{1/2}$$

and

$$T^{-1}\int_1^T n_a^N(d, t_0)\chi(t_0)\,dt_0 \ll \theta(T)^{1/2}\,Y(T, N)^{1/2}.$$

Proof. We first note that there are constants $C_5 > 1$ and $C_6 > 0$, for which the following properties hold:

(1) On the line $\sigma = C_5$,

$$|\log\zeta(s) - a| \geqslant C_6 \quad \text{and} \quad |f_N(s) - a| \geqslant C_6 \quad \text{for any } N,$$

(2) In the half-plane $\sigma > C_5$, $\log\zeta(s)$ and $f_N(s)$ (for any N) do not take the value a. (Hilfssatz 14 of Bohr–Jessen [3]).

Now we choose $\beta_0 = C_5$ and $\beta_1 = C_5 + 1$. In the proof of Satz C Bohr–Jessen showed that

$$n_a(d, t_0) \ll 1 + \iint_{P(d,t_0)} |\zeta(s)|\,d\sigma\,dt \tag{2.7}$$

and

$$n_a^N(d, t_0) \ll 1 + \iint_{P(d,t_0)} |\exp(f_N(s))|\,d\sigma\,dt. \tag{2.8}$$

From (2.7), Bohr–Jessen's argument deduces that

$$T^{-1}\int_1^T n_a(d, t_0)\chi(t_0)\,dt_0 \ll T^{-1}\int_1^T \chi(t_0)\,dt_0$$
$$+T^{-1}\Big(\int_1^T \chi(t_0)\,dt_0\Big)^{1/2}\Big\{\int_1^T m(P(d, t_0))\Big(\iint_{P(d,t_0)} |\zeta(s)|^2\,d\sigma\,dt\Big)dt_0\Big\}^{1/2}.$$

By using Theorem 7.2(A) of Titchmarsh [11], we have

$$T^{-1}\int_1^T\Big(\iint_{P(d,t_0)} |\zeta(s)|^2\,d\sigma\,dt\Big)dt_0 \ll \beta_1 - \alpha_1 \ll 1,$$

so it follows the first assertion of Lemma 7. Similarly, the second assertion can be shown from (2.8) and Lemma 5.

3. Proof of Lemma 2. In this section we show the proof of Lemma 2. Our proof is a refined version of the proof of Carlson's theorem, described in Titchmarsh's book [10], § 9.51. We remark that the following argument can be applied to many other Dirichlet series.

Let $X \geqslant 1$, $\alpha_1 \leqslant \sigma \leqslant 3$, $c > \max(0, 1-\sigma)$, and $f(s) = \zeta_N(s) - 1$. Our starting point is the following formula (Titchmarsh [10], § 9.43):

$$\sum_{n=1}^{\infty} b_n n^{-s} = \big(2\pi i(\sigma-\tfrac{1}{2})\big)^{-1}\int_{c-i\infty}^{c+i\infty} \Gamma\big(w/(\sigma-\tfrac{1}{2})\big) f(s+w)X^w\,dw,$$

where $b_n = a_n\exp\big(-(nX^{-1})^{\sigma-1/2}\big)$. We move the line of integration to $\mathrm{Re}\,w = (\frac{1}{2}+\varepsilon)-\sigma$. Then we get

$$\sum_{n=1}^{\infty} b_n n^{-s} - f(s) = (\sigma-\tfrac{1}{2})^{-1}R\cdot\Gamma\big((1-\sigma-it)/(\sigma-\tfrac{1}{2})\big)X^{1-\sigma-it} \tag{3.1}$$
$$+\big(2\pi i(\sigma-\tfrac{1}{2})\big)^{-1}\int_{(1/2+\varepsilon)-\sigma-i\infty}^{(1/2+\varepsilon)-\sigma+i\infty} \Gamma\big(w/(\sigma-\tfrac{1}{2})\big) f(s+w)X^w\,dw,$$

where R is the residue of $f(s)$ at $s = 1$. Since $|R| \leqslant 1$, by using Stirling's formula we have

$$(\sigma-\tfrac{1}{2})^{-1}R\cdot\Gamma\big((1-\sigma-it)/(\sigma-\tfrac{1}{2})\big)X^{1-\sigma-it} \ll X^{1-\sigma}e^{-C|t|}.$$

Also, using Stirling's formula again, we have

$$\int_{(1/2+\varepsilon)-\sigma-i\infty}^{(1/2+\varepsilon)-\sigma+i\infty} \Gamma\big(w/(\sigma-\tfrac{1}{2})\big) f(s+w)X^w\,dw$$
$$\ll X^{(1/2+\varepsilon)-\sigma}\int_{-\infty}^{\infty} e^{-C|v|}\big|f\big((\tfrac{1}{2}+\varepsilon)+i(t+v)\big)\big|\,dv$$
$$= X^{(1/2+\varepsilon)-\sigma}\Big(\int_{-\infty}^{-2T} + \int_{-2T}^{2T} + \int_{2T}^{\infty}\Big) = X^{(1/2+\varepsilon)-\sigma}(I_1+I_2+I_3), \text{ say}.$$

Since it is easily shown that $|f(s)| \ll (|t|+1)\exp(CN^{1-\sigma})$ (cf. (2.5)), we have

$$I_3 \ll \exp(CN^{1/2})\int_{2T}^{\infty} ve^{-Cv}\,dv \ll \exp\big(C(N^{1/2}-T)\big),$$

and a similar result holds for I_1. Also, by using Schwarz' inequality, we have

$$I_2 \leqslant \Big(\int_{-2T}^{2T} e^{-C|v|}\big|f\big((\tfrac{1}{2}+\varepsilon)+i(t+v)\big)\big|^2\,dv\Big)^{1/2}\Big(\int_{-2T}^{2T} e^{-C|v|}\,dv\Big)^{1/2}$$
$$\ll \Big(\int_{-2T}^{2T} e^{-C|v|}\big|f\big((\tfrac{1}{2}+\varepsilon)+i(t+v)\big)\big|^2\,dv\Big)^{1/2}.$$

Substituting these estimates in (3.1), we obtain

$$\sum_{n=1}^{\infty} b_n n^{-s} - f(s) \ll X^{1-\sigma} e^{-C|t|} + X^{(1/2+\varepsilon)-\sigma} \exp\bigl(C(N^{1/2}-T)\bigr)$$
$$+ X^{(1/2+\varepsilon)-\sigma} \Bigl(\int_{-2T}^{2T} e^{-C|v|} \bigl|f\bigl((\tfrac{1}{2}+\varepsilon)+i(t+v)\bigr)\bigr|^2 dv\Bigr)^{1/2},$$

and so,

$$(3.2)\qquad T^{-1}\int_1^T \Bigl|\sum_{n=1}^{\infty} b_n n^{-s} - f(s)\Bigr|^2 dt$$
$$\ll T^{-1} X^{2(1-\sigma)} + X^{(1+\varepsilon)-2\sigma} \exp\bigl(C(N^{1/2}-T)\bigr)$$
$$+ T^{-1} X^{(1+\varepsilon)-2\sigma} \int_{-2T}^{2T} e^{-C|v|} \Bigl(\int_1^T \bigl|f\bigl((\tfrac{1}{2}+\varepsilon)+i(t+v)\bigr)\bigr|^2 dt\Bigr) dv.$$

Now we note that

$$(3.3)\qquad \int_1^T \bigl|f\bigl((\tfrac{1}{2}+\varepsilon)+i(t+v)\bigr)\bigr|^2 dt \ll T\exp(CN^{1/2})$$

holds. This follows immediately from the fact that

$$|f(s)| \ll |\zeta(s)| \exp(CN^{1-\sigma+\varepsilon}) + 1$$

and Theorem 7.2 (A) of Titchmarsh [11]. From (3.2), (3.3) and Minkowski's inequality, we have

$$(3.4)\qquad \Bigl(T^{-1}\int_1^T |f(s)|^2 dt\Bigr)^{1/2} - \Bigl(T^{-1}\int_1^T \Bigl|\sum_{n=1}^{\infty} b_n n^{-s}\Bigr|^2 dt\Bigr)^{1/2}$$
$$\ll T^{-1/2} X^{1-\sigma} + X^{(1/2+\varepsilon)-\sigma} \exp(CN^{1/2}).$$

Next we estimate the second term in the left-hand side of (3.4), by a method similar to the proof of Lemma 3. In this case we apply the argument in the proof of (7.2.2), instead of (7.2.1), of Titchmarsh [11]. Then we have

$$T^{-1}\int_1^T \Bigl|\sum_{n=1}^{\infty} b_n n^{-s}\Bigr|^2 dt \ll (N\cdot\log N)^{1-2\alpha_1} + T^{-1} X^{2(1-\alpha_1)+\varepsilon}.$$

Combining this estimate with (3.4), we have

$$T^{-1}\int_1^T |f(s)|^2 dt \ll (N\cdot\log N)^{1-2\alpha_1} + T^{-1} X^{2(1-\alpha_1)+\varepsilon} + X^{1-2\alpha_1+\varepsilon} \exp(CN^{1/2}).$$

If we choose $X = T\exp(CN^{1/2})$, then we obtain the assertion of Lemma 2.

4. Proof of Theorem 1. In this section, the letters C_7, C_8, C_9 denote positive absolute constants, and the letter C and O-constants depend only on σ_0 and ε. Let R be the given rectangle, and a_p+ib_q ($1 \leqslant p,\ q \leqslant 2$, $a_1 < a_2$, $b_1 < b_2$) the four vertices of R:

$$R = \{z \mid a_1 \leqslant \operatorname{Re} z \leqslant a_2,\ b_1 \leqslant \operatorname{Im} z \leqslant b_2\}.$$

We define two rectangles R_i and R_y by

$$R_i = \{z \mid a_1+\delta \leqslant \operatorname{Re} z \leqslant a_2-\delta,\ b_1+\delta \leqslant \operatorname{Im} z \leqslant b_2-\delta\}$$

and

$$R_y = \{z \mid a_1-\delta \leqslant \operatorname{Re} z \leqslant a_2+\delta,\ b_1-\delta \leqslant \operatorname{Im} z \leqslant b_2+\delta\},$$

respectively.

Let $L_N(T, R) = m\{t\in[1, T] \mid f_N(\sigma_0+it)\in R\}$ for any rectangle R. Then, the existence of the limit

$$W_N(R) = \lim_{T\to\infty} L_N(T, R)/T$$

is a direct consequence of the Kronecker–Weyl theorem on the uniform distribution of sequences. In [7], we have shown that for any large positive integers m and r, the estimate

$$(4.1)\qquad L_N(T, R)/T - W_N(R) \ll N^2(3r)^N(m^{-1}+D'_T) + r^{-N/(N+1)} N^{(3/2)+2\sigma_0} + T^{-1}$$
$$(= A_2+A_3+T^{-1}, \text{ say})$$

holds, where

$$D'_T = T^{-1}(3+2\cdot\log m)^N \exp\bigl(C_7(mN\cdot\log N)^3(\log(mN))^2\bigr).$$

(Proposition 1, § 2 of [7]. Here we note that, though we assume $\sigma_0 > 1$ in [7], the same results hold for any $\sigma_0 > \frac{1}{2}$, except for the arguments based on Lemma 6 in [7], § 4.) Since the right-hand side of (4.1) is independent of R, we can apply this inequality to R_i and R_y, and get

$$L_N(T, R_i)/T - W_N(R_i) \ll A_2+A_3+T^{-1},$$
$$L_N(T, R_y)/T - W_N(R_y) \ll A_2+A_3+T^{-1}.$$

Furthermore, in § 9 of [7] we have shown that

$$W_N(R_i) - W_N(R) \ll \delta^{1/2} \quad\text{and}\quad W_N(R_y) - W_N(R) \ll \delta^{1/2}.$$

Hence we have

$$(4.2)\qquad L_N(T, R_i)/T - W_N(R) \ll A_2+A_3+T^{-1}+\delta^{1/2},$$

$$(4.3)\qquad L_N(T, R_y)/T - W_N(R) \ll A_2+A_3+T^{-1}+\delta^{1/2}.$$

Next, we put

$$k_N^\delta(T) = m\{t\in[1, T] \mid \sigma_0+it\in G,\ |\log\zeta(\sigma_0+it) - f_N(\sigma_0+it)| \geqslant \delta\}.$$

If $\sigma_0+it\in G$ and $|\log\zeta(\sigma_0+it)-f_N(\sigma_0+it)|<\delta$, then, by the definitions of R_i and R_y, we see that, if $\log\zeta(\sigma_0+it)\in R$ then $f_N(\sigma_0+it)\in R_y$, and, if $f_N(\sigma_0+it)\in R_i$ then $\log\zeta(\sigma_0+it)\in R$. Hence we have

$$L_N(T, R_i)-k_N^{\delta}(T)\leqslant L(T, R)\leqslant L_N(T, R_y)+k_N^{\delta}(T). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain

$$L(T, R)/T-W_N(R)\ll A_2+A_3+T^{-1}+\delta^{1/2}+k_N^{\delta}(T)/T. \tag{4.5}$$

An upper-bound estimate of the term $k_N^{\delta}(T)/T$ can be easily obtained from Lemma 1. We set $\frac{1}{2}<\alpha_1<\alpha_0<\sigma_0\leqslant 1$, $\sigma_0-\alpha_1<\varepsilon$ and $d=1$. It is obvious that

$$k_N^{\delta}(T)\leqslant\int_1^T\varphi_N^{\delta}(t_0)\,dt_0,$$

so from Lemma 1 we have

$$k_N^{\delta}(T)/T\ll\delta^{-2}A_1'+\delta^{-2}(N\cdot\log N)^{-4+\varepsilon}\log(\delta^{-1})+T^{-1}, \tag{4.6}$$

where $A_1'=N^{1-2\sigma_0+\varepsilon}+T^{1-2\sigma_0+\varepsilon}\exp(CN^{1/2})$.

Next we evaluate $|W_N(R)-W(R)|$. We first quote some results of Bohr–Jessen [4]:

(1) For any sufficiently large N ($\geqslant N_0$), there is a function $F_N(z)$ continuous in the whole plane, for which

$$W_N(R)=\iint_R F_N(z)\,dxdy\qquad(z=x+iy)$$

holds for any rectangle R.

(2) If $\sigma_0>\frac{1}{2}$, then $F_N(z)$ converges uniformly to a continuous function $F(z)$ as N tends to infinity, and

$$W(R)=\iint_R F(z)\,dxdy. \tag{4.7}$$

By virtue of these results, it is enough to evaluate $|F_N(z)-F(z)|$. Let $\varrho=\varrho_N$ be a small positive number, and $\Gamma_N=\{z\mid |z|\leqslant\varrho_N\}$. We put

$$S_{N,k}(\theta_{N+1},\ldots,\theta_{N+k})=-\sum_{n=N+1}^{N+k}\log(1-p_n^{-\sigma_0}\exp(2\pi i\theta_n))$$

for any $(\theta_{N+1},\ldots,\theta_{N+k})\in[0,1)^k$, and define

$$\Omega_{N,k}(\Gamma_N)=\{x=(\theta_{N+1},\ldots,\theta_{N+k})\in[0,1)^k\mid S_{N,k}(x)=\Gamma_N\}.$$

Then, Bohr–Jessen proved that

$$\sup_{k,z}|F_N(z)-F_{N+k}(z)|\leqslant\sup_{\mathrm{dist}(z,w)\leqslant\varrho_N}|F_{N_0}(z)-F_{N_0}(w)|+K\bigl(1-m(\Omega_{N,k}(\Gamma_N))\bigr) \tag{4.8}$$

holds for any $N\geqslant N_0$, where $K=\sup_z F_{N_0}(z)$. (Bohr–Jessen [4], § 46.) Besides, if $\sigma_0>\frac{1}{2}$, then we have

$$\varrho_N^2\bigl(1-m(\Omega_{N,k}(\Gamma_N))\bigr)<(\pi^2/6)\sum_{n=N+1}^{\infty}p_n^{-2\sigma_0}. \tag{4.9}$$

(Bohr–Jessen [4], § 50.) Since the right-hand side of (4.9) is surpassed by $O(N^{1-2\sigma_0}(\log N)^{-2\sigma_0})$, with (4.8) we have

$$\sup_z|F_N(z)-F(z)|\ll\sup_{\mathrm{dist}(z,w)\leqslant\varrho_N}|F_{N_0}(z)-F_{N_0}(w)|+\varrho_N^{-2}N^{1-2\sigma_0}(\log N)^{-2\sigma_0}.$$

We know that the first term of the right-hand side of the above can be estimated by $O(\varrho_N^{1/7}\log(\varrho_N^{-1}))$. ([7], (5.6).) Therefore, if we choose $\varrho_N=N^{7(1-2\sigma_0)/15}$, then we have

$$\begin{aligned} W_N(R)-W(R) &\ll m(R)\sup_z|F_N(z)-F(z)| \\ &\ll m(R)\,N^{(1-2\sigma_0)/15}\log N.\qquad(=m(R)A_4,\ \text{say.}) \end{aligned} \tag{4.10}$$

Now we combine (4.5), (4.6) and (4.10) to obtain

$$\begin{aligned} &L(T, R)/T-W(R) \\ &\ll\delta^{-2}A_1+A_2+A_3+m(R)A_4+\delta^{1/2}+\delta^{-2}(N\cdot\log N)^{-4+\varepsilon}\log(\delta^{-1})+T^{-1}. \end{aligned} \tag{4.11}$$

Suitable choices of the parameters m, r, δ and N in the right-hand side of (4.11) lead to the assertion of Theorem 1. At first, the method of finding the best choice of the value of m is already described in § 5 of [8]. In view of (5.4) of [8], we can assume

$$A_2\ll N^3\log(N)\cdot(3r)^N(\log T)^{-1/3}(\log\log T)^{2/3}\qquad(=A_2'(r),\ \text{say})$$

under the following conditions:

(A) $N=N(T)$ tends to infinity as T tends to infinity,

(B) $\log T\geqslant N^4$.

Next we decide the value of r by requiring $A_2'=A_3$. We assume the stronger condition

(B′) $\log T\geqslant 30^{N\cdot\log\log N}$

instead of (B). We consider the equation

$$A_2'(\varrho)=\varrho^{-N/(N+1)}N^{(3/2)+2\sigma_0} \tag{4.12}$$

under the conditions (A) and (B′). This equation can be rewritten as follows:

$$\varrho=\bigl(N^{-(3/2)+2\sigma_0}3^{-N}(\log N)^{-1}(\log T)^{1/3}(\log\log T)^{-2/3}\bigr)^{(N+1)/N(N+2)}.$$

That is, (4.12) has a unique solution ϱ, and from the condition (B′), ϱ tends to infinity as T tends to infinity. If we put $r=[\varrho]$, then we have

$$A_2'(r)\ll N^{(3/2)+2\sigma_0}(\log T)^{-1/3(N+2)}(\log\log T)^{2/3(N+2)}\qquad(=A_5,\ \text{say}),$$

and the same estimate holds for A_3. Hence we arrive at the following estimate:

(4.13) $$L(T, R)/T - W(R) \ll \delta^{-2} A_1 + m(R) A_4 + A_5 + \delta^{-2} (N \cdot \log N)^{-4+\varepsilon} \log(\delta^{-1}) + \delta^{1/2} + T^{-1}.$$

Now we decide the value of N by requiring $A_4 \geqslant A_5$. If we choose

(4.14) $$N = [C_8 \log\log T/\log\log\log T],$$

then we have

$$A_4 \ll (\log\log T)^{-(2\sigma_0 - 1)/15 + \varepsilon}$$

and

$$A_5 \ll (\log\log T)^{(3/2) + 2\sigma_0 - (1/3C_8) + \varepsilon}.$$

Hence, taking a sufficiently small value of C_8, we can assume

$$A_5 \ll (\log\log T)^{-C_9 + \varepsilon}$$

for an arbitrary large C_9. We remark that the choice (4.14) of N satisfies the technical conditions (A) and (B').

For the remaining terms in the right-hand side of (4.13), we first decide the value of δ by requiring $\delta^{-2} A_1 = \delta^{1/2}$; so that, we set $\delta = A_1^{2/5}$. On the other hand, under the choice (4.14), we have

$$A_1 \ll (\log\log T/\log\log\log T)^{1 - 2\alpha_1 + \varepsilon} + T^{1 - 2\alpha_1 + \varepsilon} \exp\left(C (\log\log T)^{1/2}\right) \ll (\log\log T)^{1 - 2\sigma_0 + \varepsilon}.$$

Hence,

$$\delta^{-2} A_1,\ \delta^{1/2} \ll (\log\log T)^{-(2\sigma_0 - 1)/5 + \varepsilon},$$

and furthermore, we see

$$\delta^{-2} (N \cdot \log N)^{-4+\varepsilon} \log(\delta^{-1}) \ll (\log\log T)^{-4 + 4(2\sigma_0 - 1)/5 + \varepsilon} \ll (\log\log T)^{-3+\varepsilon}.$$

Substituting these estimates in (4.13), we obtain

(4.15) $$L(T, R)/T - W(R) \ll m(R) (\log\log T)^{-(2\sigma_0 - 1)/15 + \varepsilon} + (\log\log T)^{-(2\sigma_0 - 1)/5 + \varepsilon}.$$

Thus our proof of Theorem 1 completes.

5. Application of Jensen's theorem. Now we start to prove Theorem 2. In this section we discuss some consequences of Hilfssätze 3 and 4 of Bohr–Jessen [3], which are based on Jensen's theorem and Carathéodory's inequality, and in particular, complete the proof of Theorem 2 in case $\sigma_1 > 1$. We note that in this section, the letters $C_{10}, C_{11}, \ldots$ and O-constants depend only on σ_1, σ_2 and a.

For given σ_1 and σ_2, we first fix a positive

$$\eta_0 = \eta_0(\sigma_1, \sigma_2) < \min\left(\tfrac{1}{2}(\sigma_2 - \sigma_1),\ \sigma_1 - \tfrac{1}{2}\right).$$

Let us remember the definition

$$Q(d, t_0) = \{s = \sigma + it \mid \alpha_0 < \sigma < \beta_0,\ t_0 - \tfrac{1}{2} d < t < t_0 + \tfrac{1}{2} d\}.$$

In the proof of Lemma 7, we remark the existence of the constant $C_5 > 1$ for which the inequalities

$$|\log\zeta(s) - a| \geqslant C_6 > 0 \quad \text{and} \quad |f_N(s) - a| \geqslant C_6 > 0$$

hold on the line $\sigma = C_5$. We take a $C_{10} > \max(C_5, \sigma_2 + \eta_0)$, and fix the values of α_0, β_0 and d for which the conditions $\frac{1}{2} < \alpha_0 = \alpha_0(\sigma_1, \sigma_2) < \sigma_1 - \eta_0$, $\beta_0 = \beta_0(\sigma_1, \sigma_2, a) \geqslant C_{10}$ and $1 + 2\eta_0 < d < 2$ hold. (In particular, if $\sigma_1 > 1$, then we require $\alpha_0 > 1$.) Since $|\log\zeta(C_5 + it_0) - a| \geqslant C_6$ and $|f_N(C_5 + it_0) - a| \geqslant C_6$ for any real t_0, we can apply Hilfssatz 3 of [3] to the function $f(s) = \log\zeta(s + it_0) - a$ and $f(s) = f_N(s + it_0) - a$ with $R = Q(d, 0)$, $s_0 = C_5$ and $k = C_6$. If we denote the set

$$\{s = \sigma + it \mid \sigma_j - \eta_0 \leqslant \sigma \leqslant \sigma_k + \eta_0,\ t_0 - \tfrac{1}{2} - \eta_0 \leqslant t \leqslant t_0 + \tfrac{1}{2} + \eta_0\}$$

by

$$A_{jk}(t_0) \quad (1 \leqslant j \leqslant k \leqslant 2),$$

then we have the following

LEMMA 8. *If* $|\log\zeta(s)| < K$ *for some large* K *in* $Q(d, t_0)$, *then*

$$\#\left(A_{jk}(t_0) \cap \mathscr{A}\right) \ll \log K.$$

Also, if $|f_N(s)| < K$ *in* $Q(d, t_0)$, *then*

$$\#\left(A_{jk}(t_0) \cap \mathscr{A}_N\right) \ll \log K \quad (1 \leqslant j \leqslant k \leqslant 2).$$

Next we define, for small positive r,

$$M_{jk}(r, t_0) = \{s \in A_{jk}(t_0) \mid |s - s_a| \geqslant r \text{ for any } s_a \in \mathscr{A}\},$$
$$M_{jk}^N(r, t_0) = \{s \in A_{jk}(t_0) \mid |s - s_a^N| \geqslant r \text{ for any } s_a^N \in \mathscr{A}_N\},$$

and consider lower-bound estimates of $|\log\zeta(s) - a|$, $|f_N(s) - a|$ in these regions. Hilfssatz 4 of [3] states such a result, and, according to the proof of Hilfssatz 2 of [3], we can write down explicitly the dependence on r in the conclusion of Hilfssatz 4. Applying this result to our case, we have the following

LEMMA 9. *There exist positive constants* C_{11} *and* C_{12} *for which the following properties hold*: *If* $|\log\zeta(s)| < K$ *in* $Q(d, t_0)$, *then*

$$|\log\zeta(s) - a| \geqslant r^{C_{11}\log K} K^{-C_{12}}$$

holds in $M_{jk}(r, t_0)$, *and also, if* $|f_N(s)| < K$ *in* $Q(d, t_0)$, *then*

$$|f_N(s)-a| \geqslant r^{C_{11}\log K} K^{-C_{12}}$$

holds in $M_{jk}^N(r, t_0)$ $(1 \leqslant j \leqslant k \leqslant 2)$.

In particular, in case $\sigma_1 > 1$, we can take $K = O(1)$ for any t_0, so we have $|\log\zeta(s)-a| \geqslant r^{C_{13}}$ for any $s \in \{\sigma_1-\eta_0 \leqslant \sigma \leqslant \sigma_2+\eta_0,\ |s-s_a| \geqslant r$ for any $s_a \in \mathscr{A}\}$. Hence, combining with the Proposition in § 5 of [8], we obtain the result of Theorem 2 for $\sigma_1 > 1$. (In the notation of [8], we choose $\delta = (\log\log T)^{-(\sigma_1-1)/C_{13}+\varepsilon}$.)

Now the only task remaining to us is to prove Theorem 2 in case $\sigma_1 \leqslant 1$. In the next section, we discuss the construction and the properties of the auxiliary function $n_a^*(t_0)$. The structure of the method, which is a refinement of Bohr–Jessen's discussion in [3], is similar to the argument developed in [8], but the details are more complicated.

6. The function $n_a^*(t_0)$. We first remark that in this and the next section, the letters $C_{14}, C_{15}, \ldots$ depend only on σ_1, σ_2 and a, and the letter C and O-constants depend only on σ_1, σ_2, a and ε. Let $R(t_0) = \{s = \sigma+it \mid \sigma_1 < \sigma < \sigma_2,\ t_0-\frac{1}{2} < t < t_0+\frac{1}{2}\}$ and $n_a(t_0) = \#(R(t_0) \cap \mathscr{A})$. It is easily shown that

$$N_a(T-\tfrac{1}{2})+O(1) < \int_1^T n_a(t_0)\,dt_0 < N_a(T+\tfrac{1}{2})+O(1).$$

Besides, applying to the function $\zeta(s)-e^a$, the same argument as in the proof of Theorem 9.2 of Titchmarsh [11], we can show

$$N_a(T+\tfrac{1}{2})-N_a(T) = O(\log T) \quad \text{and} \quad N_a(T)-N_a(T-\tfrac{1}{2}) = O(\log T).$$

So it follows that

$$N_a(T) = \int_1^T n_a(t_0)\,dt_0+O(\log T). \tag{6.1}$$

We shall construct a piecewise constant function $n_a^*(t_0)$ which "approximates" $n_a(t_0)$. Besides we require that there exists the limit

$$G^*(a) = \lim_{T\to\infty} T^{-1}\int_1^T n_a^*(t_0)\,dt_0. \tag{6.2}$$

Let $\eta < \eta_0$, and we put

$$R_i(t_0) = \{s \mid \sigma_1+\eta \leqslant \sigma \leqslant \sigma_2-\eta,\ t_0-\tfrac{1}{2}+\eta \leqslant t \leqslant t_0+\tfrac{1}{2}-\eta\},$$
$$R_y(t_0) = \{s \mid \sigma_1-\eta \leqslant \sigma \leqslant \sigma_2+\eta,\ t_0-\tfrac{1}{2}-\eta \leqslant t \leqslant t_0+\tfrac{1}{2}+\eta\},$$
$$n_a^i(t_0) = \#(R_i(t_0) \cap \mathscr{A}) \quad \text{and} \quad n_a^y(t_0) = \#(R_y(t_0) \cap \mathscr{A}).$$

Then it is obvious that

$$n_a^i(t_0) \leqslant n_a(t_0) \leqslant n_a^y(t_0). \tag{6.3}$$

If we define

$$\chi(t_0) = \begin{cases} 0 & \text{if } n_a^i(t_0) \leqslant n_a^*(t_0) \leqslant n_a^y(t_0), \\ 1 & \text{otherwise,} \end{cases} \tag{6.4}$$

then we see that $T^{-1}\int_1^T \chi(t_0)\,dt_0$ is "small", or we can show the following

LEMMA 10. *For any large positive number K and positive integer N, we shall define* $n_a^*(t_0) = n_a^*(t_0; \eta, K, N)$ *which is piecewise constant and satisfies* (6.2). *Then there exists a small positive* $\delta = \delta(\eta, K)$ *for which the following estimate holds*:

$$T^{-1}\int_1^T \chi(t_0)\,dt_0 \ll X(T, N, \delta)+(\log T)^{1-2\alpha_0+\varepsilon}+e^{-K}.$$

That is, $n_a^*(t_0)$ is an "approximate" function to $n_a(t_0)$. Furthermore, we shall prove

LEMMA 11 (Bohr–Jessen [3]). *For any real* t_0, $n_a^*(t_0) \leqslant n_a^N(d, t_0)$ *holds.*

Now we start to construct $n_a^*(t_0)$ and to prove the above lemmas. Since $R_y(t_0) \subset A_{12}(t_0)$, it follows from Lemma 8 that, if $\psi_K(t_0) = 0$, then $n_a^y(t_0) - n_a^i(t_0) \leqslant C_{14}\log K$. Hence, if we take a positive $r < \eta/(2C_{14}\log(K)+3)$, then for any t_0 with $\psi_K(t_0) = 0$, there are two positive $\tau_i = \tau_i(t_0) < \eta$ and $\tau_y = \tau_y(t_0) < \eta$, for which the following conditions hold:

$$\partial R_i(t_0, \tau_i) \subset M_{12}(r, t_0) \quad \text{and} \quad \partial R_y(t_0, \tau_y) \subset M_{12}(r, t_0),$$

where

$$R_i(t_0, \tau) = \{s \mid \sigma_1+\tau < \sigma < \sigma_2-\tau,\ t_0-\tfrac{1}{2}+\tau < t < t_0+\tfrac{1}{2}-\tau\}$$

and

$$R_y(t_0, \tau) = \{s \mid \sigma_1-\tau < \sigma < \sigma_2+\tau,\ t_0-\tfrac{1}{2}-\tau < t < t_0+\tfrac{1}{2}+\tau\}.$$

So, if we choose $r = (\eta/2)(2C_{14}\log(K)+3)^{-1}$, with Lemma 9 we have that for any t_0 which satisfies $\psi_K(t_0) = 0$,

$$|\log\zeta(s)-a| \geqslant C_{15}\,\eta^{C_{11}\log K} K^{-C_{16}\log\log K} \quad (= m_0(\eta, K),\ \text{say}) \tag{6.5}$$

holds on $\partial R_i(t_0, \tau_i) \cup \partial R_y(t_0, \tau_y)$.

Next we prove a similar result for $f_N(s)$. We first note that $|f_N(s)| \leqslant C_{17}N^{1-\alpha_0}$ in the half-plane $\sigma > \alpha_0$ (cf. (2.5)). So we can apply Lemma 8 with $K = C_{17}N^{1-\alpha_0}$, and the result is that

$$\#\big((R_y(t_0)-R_i(t_0)) \cap \mathscr{A}_N\big) \leqslant C_{18}\log N.$$

If we put $r' = (\eta/2)(2C_{18}\log(N)+3)^{-1}$, then there is a positive $\tau'_y = \tau'_y(t_0) < \eta$, for which $\partial R_y(t_0, \tau'_y) \subset M^N_{12}(r', t_0)$ holds. Hence, with Lemma 9, we have that for any real t_0,

$$(6.6) \qquad |f_N(s)-a| \geqslant C_{19}\,\eta^{C_{20}\log N} N^{-C_{21}\log\log N} \qquad (m'_0(\eta, N), \text{ say})$$

holds on $\partial R_y(t_0, \tau'_y)$.

Now we choose $\delta = \frac{1}{2}m_0(\eta, K)$, and define

$$\chi^*(t_0) = \chi^*(t_0; \eta, K, N) = \begin{cases} 0 & \text{if } \psi_K(t_0) = \varphi^\delta_N(t_0) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then it is obvious that $\chi^*(t_0) \leqslant \psi_K(t_0) + \varphi^\delta_N(t_0)$, so from Lemmas 1 and 6 we have

$$(6.7) \qquad T^{-1}\int_1^T \chi^*(t_0)\,dt_0 \ll X(T, N, \delta) + (\log T)^{1-2\alpha_0+\varepsilon} + e^{-K}.$$

For any t_0 which satisfies $\chi^*(t_0) = 0$, we see

$$|\log\zeta(s)-a| \geqslant m_0 \quad \text{and} \quad |R_N(s)| < \delta \quad \text{on} \quad \partial R_i(t_0, \tau_i) \cup \partial R_y(t_0, \tau_y),$$

from (6.5) and the definition of $\varphi^\delta_N(t_0)$. Hence, if we define a function $f^*(s) = f^*(s; t_0)$ which satisfies

$$|f^*(s)-f_N(s)| < m_0-\delta = \delta \quad \text{on} \quad \partial R_i(t_0, \tau_i) \cup \partial R_y(t_0, \tau_y),$$

and put

$$n^*_a(t_0) = \#\{s \in R(t_0) \mid f^*(s; t_0) = a\},$$

then Rouché's theorem asserts that

$$n^i_a(t_0) \leqslant n^*_a(t_0) \leqslant n^y_a(t_0) \quad \text{if } \chi^*(t_0) = 0.$$

So we see $\chi(t_0) \leqslant \chi^*(t_0)$ for any real t_0. Therefore, (6.7) implies Lemma 10. Furthermore, if we require the property that $|f^*(s)-f_N(s)| < m'_0(\eta, N)$ on $\partial R_y(t_0, \tau'_y)$, then from (6.6) and Rouché's theorem we obtain the result of Lemma 11. Hence it is sufficient to construct $f^*(s; t_0)$, which satisfies $|f^*(s) - f_N(s)| < \mu$ for some positive $\mu < \min(\delta, m'_0)$, in the half-plane $\sigma > \alpha_0$. (Besides we require that $n^*_a(t_0)$ is piecewise constant and that the limit $G^*(a)$ exists.) The method of the construction is the same as that described in § 3 of [8], so we omit the details. The difference between $T^{-1}\int_1^T n^*_a(t_0)\,dt_0$ and $G^*(a)$ is estimated by using the theory of discrepancies in § 4 of [8], and the result is that

$$T^{-1}\int_1^T n^*_a(t_0)\,dt_0 - G^*(a) \ll N^2 3^N (m^{-1}+D'_T) P^N G^*(a)$$

for any positive integer m, where $P = P(\mu)$ is a sufficiently large integer which satisfies the following condition: We put

$$f^*(s; \varphi_1, \ldots, \varphi_N) = -\sum_{n=1}^N \log(1-p_n^{-s}\exp(2\pi i\varphi_n))$$

for $\varphi_1, \ldots, \varphi_N \in [0, 1)$. If $\varphi_n < P^{-1}$ $(1 \leqslant n \leqslant N)$, then the inequality

$$\Big|f^*(s; \varphi_1, \ldots, \varphi_N) - \Big(-\sum_{n=1}^N \log(1-p_n^{-s})\Big)\Big| < \mu$$

holds uniformly in $\sigma_1 - \eta < \sigma < \sigma_2 + \eta$.

It is easily shown that we can specify $P = [C_{22}\mu^{-1}N^{1-\alpha_0}]$ (cf. § 5 of [8]). Also, by the same choice of the value of m as in § 5 of [8], we obtain

$$m^{-1} + D'_T \ll N \cdot \log N \cdot (\log T)^{-1/3} (\log\log T)^{2/3}$$

under the conditions (A) and (B) (cf. § 4). Hence we have

$$(6.8) \qquad T^{-1}\int_1^T n^*_a(t_0)\,dt_0 - G^*(a)$$
$$\ll N^3 \log N \cdot (\log T)^{-1/3} (\log\log T)^{2/3} (3C_{22}\mu^{-1}N^{1-\alpha_0})^N G^*(a)$$
$$(= Z(T, N, \mu)\, G^*(a), \text{ say})$$

under the conditions (A) and (B).

7. Completion of the proof of Theorem 2. Our starting point is the inequalities

$$n^*_a(t_0) - \big(n^y_a(t_0) - n^i_a(t_0)\big) - n^*_a(t_0)\chi(t_0) \leqslant n_a(t_0)$$
$$\leqslant n^*_a(t_0) + \big(n^y_a(t_0) - n^i_a(t_0)\big) + n_a(t_0)\chi(t_0)$$

which have appeared in the last stage of Bohr–Jessen's proof of their Satz V in [3]. These inequalities are easily obtained from (6.3) and (6.4). We integrate each term of the above inequalities to get

$$(7.1) \qquad T^{-1}\int_1^T n^*_a(t_0)\,dt_0 - T^{-1}\int_1^T \big(n^y_a(t_0) - n^i_a(t_0)\big)\,dt_0 - T^{-1}\int_1^T n^*_a(t_0)\chi(t_0)\,dt_0$$
$$\leqslant T^{-1}\int_1^T n_a(t_0)\,dt_0 \leqslant T^{-1}\int_1^T n^*_a(t_0)\,dt_0 + T^{-1}\int_1^T \big(n^y_a(t_0) - n^i_a(t_0)\big)\,dt_0$$
$$+ T^{-1}\int_1^T n_a(t_0)\chi(t_0)\,dt_0.$$

Obviously $n_a(t_0) \leqslant n_a(d, t_0)$, and Lemma 11 asserts $n^*_a(t_0) \leqslant n^N_a(d, t_0)$. Hence, from Lemmas 7 and 10, we have

$$T^{-1}\int_1^T n_a(t_0)\chi(t_0)\,dt_0 \ll X(T, N, \delta)^{1/2} + (\log T)^{1/2-\alpha_0+\varepsilon} + e^{-K/2}$$

and

$$T^{-1}\int_1^T n_a^*(t_0)\chi(t_0)\,dt_0 \ll \left(X(T,N,\delta)^{1/2}+(\log T)^{1/2-\alpha_0+\varepsilon}+e^{-K/2}\right)\cdot Y(T,N)^{1/2}.$$

Substituting these estimates in (7.1), we obtain

$$(7.2)\quad T^{-1}\int_1^T n_a(t_0)\,dt_0 - T^{-1}\int_1^T n_a^*(t_0)\,dt_0 \ll T^{-1}\int_1^T \left(n_a^y(t_0)-n_a^i(t_0)\right)dt_0 + \left(X(T,N,\delta)^{1/2}+(\log T)^{1/2-\alpha_0+\varepsilon}+e^{-K/2}\right)\left(1+Y(T,N)^{1/2}\right).$$

Next we estimate the first term of the right-hand side of (7.2). Let

$$n_a^j(t_0) = \#\left(\{s|\ \sigma_j-\eta<\sigma<\sigma_j+\eta,\ t_0-\tfrac{1}{2}-\eta<t<t_0+\tfrac{1}{2}+\eta\}\cap\mathscr{A}\right) \qquad (j=1,2).$$

Then we have (see the proof of Hilfssatz 7 of Bohr–Jessen [3])

$$(7.3)\quad T^{-1}\int_1^T \left(n_a^y(t_0)-n_a^i(t_0)\right)dt_0 \ll T^{-1}\int_1^T n_a^1(t_0)\,dt_0 + T^{-1}\int_1^T n_a^2(t_0)\,dt_0 + 4\eta\left(T^{-1}N_a(T+1)+O(T^{-1})\right).$$

From (1.5) it is obvious that $T^{-1}N_a(T+1)\ll 1$. For the integrals in the right-hand side of the above, we show the following

Lemma 12.

$$T^{-1}\int_1^T n_a^j(t_0)\,dt_0 \ll (\log T)^{1/2-\alpha_0+\varepsilon}+e^{-K/2}+\eta K^2\log K + \begin{cases}\eta^{-1}\log(K)\cdot(\log\log T)^{-(2\sigma_j-1)/5+\varepsilon} & \text{if } \sigma_j\le 1,\\ \eta^{-1}\log(K)\cdot(\log\log T)^{-(\sigma_j-1)/2} & \text{if } \sigma_j>1.\end{cases}$$

Proof. In view of Lemma 6, we can take $\theta(T)=C\left((\log T)^{1-2\alpha_0+\varepsilon}+e^{-K}\right)$ for $\chi(t_0)=\psi_K(t_0)$. Since $n_a^j(t_0)\le n_a(d,t_0)$, from Lemma 7 we have

$$(7.4)\quad T^{-1}\int_1^T n_a^j(t_0)\psi_K(t_0)\,dt_0 \ll (\log T)^{1/2-\alpha_0+\varepsilon}+e^{-K/2}.$$

Next, let t_0 be any real number for which $\psi_K(t_0)=0$ holds. Then, for any $s_0\in A_{jj}(t_0)$, the inequality $|\log\zeta(s)|<K$ holds for any $s\in C(s_0)=\{s|\ |s-s_0|=\frac{1}{2}\operatorname{dist}(A_{jj}(t_0),Q(d,t_0))\}$. Hence, at $s=s_0$,

$$|(d/ds)\log\zeta(s)| \le (2\pi)^{-1}\int_{C(s_0)}|\log\zeta(s)|/|s-s_0|^2\,|ds| \le C_{23}K.$$

Let R be the square with the edges parallel to the axes, with center a and the length of the edges $2\sqrt{2}C_{23}\eta K$. Then, Bohr–Jessen's argument in the proof

of Hilfssatz 6 of [3], combined with Lemma 8, leads to the following estimation:

$$(7.5)\quad T^{-1}\int_1^T n_a^j(t_0)\,dt_0 \ll T^{-1}\int_1^T n_a^j(t_0)\psi_K(t_0)\,dt_0+\eta^{-1}\log(K)\cdot T^{-1}\left(L(T+1,R)+1\right).$$

We have already known the asymptotic formulas of $L(T,R)$; (2.10) of [7] and (4.15) of the present paper. Since $W(R)\ll m(R)\ll\eta^2K^2$ (see (4.7)), from those asymptotic formulas we have

$$T^{-1}L(T,R)\ll\begin{cases}\eta^2K^2+(\log\log T)^{-(2\sigma_j-1)/5+\varepsilon} & \text{if } \sigma_j\le 1,\\ \eta^2K^2+(\log\log T)^{-(\sigma_j-1)/2} & \text{if } \sigma_j>1.\end{cases}$$

The result of Lemma 12 follows from (7.4), (7.5) and the above.

From (7.2), (7.3) and Lemma 12, we have

$$(7.6)\quad T^{-1}\int_1^T n_a(t_0)\,dt_0 - T^{-1}\int_1^T n_a^*(t_0)\,dt_0 \ll \left(X(T,N,\delta)^{1/2}+(\log T)^{1/2-\alpha_0+\varepsilon}+e^{-K/2}\right)\left(1+Y(T,N)^{1/2}\right) + \eta K^2\log K + \eta^{-1}\log(K)\cdot(\log\log T)^{-(2\sigma_1-1)/5+\varepsilon} + \varepsilon_2\eta^{-1}\log(K)\cdot(\log\log T)^{-(\sigma_2-1)/2}+\eta,$$

where $\varepsilon_2=0$ (if $\sigma_2\le 1$) or 1 (if $\sigma_2>1$). By (6.1) we see

$$\lim_{T\to\infty} T^{-1}\int_1^T n_a(t_0)\,dt_0 = G(a).$$

Hence, if we fix the values of η, K and N in (7.6) and increase the value of T to infinity, then with (6.2) we have

$$(7.7)\quad G(a)-G^*(a) \ll \delta^{-2}\left(N^{1-2\alpha_0+\varepsilon}+(N\cdot\log N)^{-4+\varepsilon}\log(\delta^{-1})\right) + e^{-K/2}+\eta K^2\log(K)+\eta.$$

Here we specify $\eta=e^{-K/2}$, so $\delta=\frac{1}{2}m_0(\eta,K)\gg K^{-C_{24}K}$. Hence we have

$$(7.8)\quad G(a)-G^*(a)\ll K^{2C_{25}K+\varepsilon}N^{1-2\alpha_0+\varepsilon}+e^{-K/2}\cdot K^{2+\varepsilon}.$$

Now we assume the conditions (A) and (B). Then we see $Y(T,N)\ll 1$, so, with the above choice of the value of η, we have

$$(7.9)\qquad N_a(T)/T - T^{-1}\int_1^T n_a^*(t_0)\,dt_0 \ll K^{C_{25}K+\varepsilon} N^{1/2-\alpha_0+\varepsilon} + K^{C_{24}K} T^{1/2-\alpha_0+\varepsilon}\exp(CN^{1/2}) + e^{-K/2}\cdot K^{2+\varepsilon} + e^{K/2}\cdot K^{\varepsilon}\{(\log\log T)^{-(2\sigma_1-1)/5+\varepsilon} + \varepsilon_2(\log\log T)^{-(\sigma_2-1)/2}\}$$

from (7.6) and (6.1). The above result suggests that the following condition holds:

$$(C)\qquad N \gg K^{C_{26}K}.$$

Under this condition we see $\delta \gg m_0'$, so we can choose

$$\mu = C_{27}\,m_0' = C_{19}\,C_{27}\,e^{-(1/2)C_{20}K\cdot\log N} N^{-C_{21}\log\log N}.$$

Hence we have

$$(7.10)\qquad Z(T, N, \mu) \ll (\log T)^{-1/3}(\log\log T)^{2/3} N^{N(C_{28}K + C_{29}\log\log N)}.$$

In view of this estimate, we must require $N^{C_{29}N\cdot\log\log N} \ll (\log T)^{1/3}$. Now we assume

$$N = [(\log\log T)/(\log\log\log T)^v]$$

with a positive parameter v. Under this assumption, by requiring $K^{2C_{25}K+\varepsilon} N^{1-2\alpha_0+\varepsilon} = e^{-K/2}\cdot K^{2+\varepsilon}$ in the right-hand side of (7.8), we find the following choice of the value of K;

$$K = [((2\alpha_0 - 1)\log\log\log T)/((4C_{25})\log\log\log\log T)].$$

Then, from (7.8) we have

$$(7.11)\qquad G(a) - G^*(a) \ll \exp((-C_{30}\log\log\log T)/(\log\log\log\log T) + \varepsilon),$$

and in particular, $G^*(a) = G(a) + O(1) = O(1)$. Hence, from (6.8), we have

$$(7.12)\qquad G^*(a) - T^{-1}\int_1^T n_a^*(t_0)\,dt_0 \ll Z(T, N, \mu).$$

Also, under the above choices of N and K, the right-hand side of (7.9) is estimated by $\exp((-C_{31}\log\log\log T)/(\log\log\log\log T) + \varepsilon)$. Therefore, with (7.11) and (7.12), we now arrive at the following estimation:

$$N_a(T)/T - G(a) \ll Z(T, N, \mu) + \exp((-C_{32}\log\log\log T)/(\log\log\log\log T) + \varepsilon).$$

We note that the above choices of the values of N and K satisfy the conditions (A), (B) and (C). Finally, it can be easily checked that if we set $v > 2$, then $Z(T, N, \mu) \ll (\log T)^{-1/3+\varepsilon}$. (See (7.10).) The proof of Theorem 2 is now completed.

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Received on 7.1.1986 (1583)