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On linear forms of G-functions

by

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1. Introduction. In his fundamental paper in 1929 Siegel [9] developed a method for studying the arithmetic properties of the values of certain classes of analytic functions known as E- and G-functions. He proved the algebraic independence of the values of certain E-functions at algebraic points, and also pointed out that his method could be used to investigate G-functions, giving some examples of the results that could be obtained.

This suggestion of Siegel has been followed more recently by Nurmagomedov [8], Galochkin [5], [6], Flicker [4], Väänänen [10], [11], Matveev [7] and Xu [12], [13], for example, but the results of these papers use the additional Galochkin's condition on G-functions. This restrictive condition is usually not trivial to verify, see e.g. [1], where Gauss hypergeometric functions are considered. In an important paper of Bombieri [2] this condition is replaced by another condition, he considers G-functions which are "Fuchsian of arithmetic type" (for the definition, see [2]).

Using very interesting new ideas, Chudnovsky [3] recently succeeded in considering the arithmetic properties of the values of classical G-functions without any further restrictions. In particular, he gave a lower bound for linear forms in the values of G-functions at certain rational points. Our aim in the present paper is to obtain a generalization of this result to algebraic number fields, in both the archimedian and the p-adic case. Our proof is based on the ideas of Chudnovsky [3] regarding the use of Padé approximations of the second kind, and on local to global technique as used in the work of Bombieri [2].

2. Notation and main results. Let $\boldsymbol{K}$ be an algebraic number field of degree d over $\boldsymbol{Q}$, and let $O_{\boldsymbol{K}}$ denote the domain of integers in $\boldsymbol{K}$. For every place v of $\boldsymbol{K}$ we write $d_v = [\boldsymbol{K}_v : \boldsymbol{Q}_v]$. If the finite place v of $\boldsymbol{K}$ lies over the prime p, we write $v|p$, for infinite place v of $\boldsymbol{K}$ we write $v|\infty$. We normalize the absolute value $|\ |_v$ so that

(i) if $v|p$, then $|p|_v = p^{-d_v/d}$

(ii) if $v|\infty$, then $|x|_v = |x|^{d_v/d}$,

here $|\ |$ denotes the ordinary absolute value in $\boldsymbol{R}$ or in $\boldsymbol{C}$.

Clearly we have the product formula

$$\prod_v |x|_v = 1, \qquad x \in \boldsymbol{K},\ x \neq 0,$$

and, for all $x_1, \ldots, x_n \in \boldsymbol{K}$,

$$|x_1 + \ldots + x_n|_v \leqslant \begin{cases} \max_i |x_i|_v & \text{if} \quad v|p, \\ n^{d_v/d} \max_i |x_i|_v & \text{if} \quad v|\infty. \end{cases}$$

For any polynomial $P(z) = \sum_{i=0}^{n} p_i z^i \in \boldsymbol{K}[z]$ we denote

$$|P|_v = \max(1, \max_i |p_i|_v).$$

The absolute height $h(x)$ of $x \in \boldsymbol{K}$ is defined by the formula

$$h(x) = \prod_v \max(1, |x|_v),$$

and absolute height $h(X)$ of the vector $X = (x_1, \ldots, x_n)^t$, $x_i \in \boldsymbol{K}$, by

$$h(X) = \prod_v \max(1, \max_i |x_i|_v).$$

Analogously we define the absolute heights of a matrix $A = (a_{ij})$, $a_{ij} \in \boldsymbol{K}$, and a polynomial $P(z) \in \boldsymbol{K}[z]$ by the formulae

$$h(A) = \prod_v \max(1, \max_{i,j} |a_{ij}|_v),$$

and

$$h(P) = \prod_v |P|_v,$$

respectively.

We shall write $\log^+ a = \log\max(1, a)$ for all $a \geqslant 0$. We then denote

$$\alpha_v = \begin{cases} 1, & \text{if} \quad v|p, \\ d_v/d, & \text{if} \quad v|\infty, \end{cases} \qquad \beta_v = \begin{cases} 0, & \text{if} \quad v|p, \\ d_v/d, & \text{if} \quad v|\infty. \end{cases}$$

The power series

$$(1) \qquad y_i(z) = \sum_{m=0}^{\infty} a_{m,i} z^m, \qquad i = 1, \ldots, n,$$

are called ***KG*-functions**, if the following conditions are satisfied:

(i) $a_{m,i} \in \boldsymbol{K}$, $i = 1, \ldots, n$, $m = 0, 1, \ldots$;

(ii) there exists a constant $C \geqslant 1$ such that, for every $v|\infty$,

$$\max_i |a_{m,i}|_v \leqslant C^{\alpha_v(m+1)}, \qquad m = 0, 1, \ldots;$$

(iii) there exists a sequence of natural numbers (r_l) such that $r_l a_{m,i} \in O_{\boldsymbol{K}}$, $i = 1, \ldots, n$, $m = 0, 1, \ldots, l$, $l = 1, 2, \ldots$, and $r_l \leqslant C^l$, $l = 1, 2, \ldots$

From the above condition (iii) it follows that

$$\max_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant m \leqslant l}} |a_{m,i}|_v \leqslant |r_l|_v^{-1} \leqslant r_l \leqslant C^l$$

for every finite place of $\boldsymbol{K}$ and $l = 1, 2, \ldots$ Thus the set $y_1(z), \ldots, y_n(z)$ of $\boldsymbol{K}G$-functions is v-adically convergent in $|z|_v < C^{-\alpha_v}$.

In the following we suppose that the functions (1) satisfy a system of linear differential equations

$$(2) \qquad \frac{d}{dz} Y = AY,$$

where $Y = (y_1(z), \ldots, y_n(z))^t$, $A = (A_{ij}(z))_{n \times n}$, $A_{ij} \in \boldsymbol{K}(z)$. Let $T(z) \in \boldsymbol{K}[z]$ denote the common denominator of A_{ij}, and put

$$s = \max(\deg T, \deg TA_{ij},\ i, j = 1, \ldots, n).$$

We shall consider Padé approximations of the second kind for $\boldsymbol{K}G$-functions (1). These are defined in the following way. Let D_0, D and M be natural numbers. Let $Q(z)$ be a non-zero polynomial of degree $\leqslant D_0$. Then, for every $i = 1, \ldots, n$, there exists a unique polynomial

$$P_i(z) = [Q \cdot y_i]_D$$

of degree $\leqslant D$ such that $\operatorname{ord}_{z=0}(Q(z) y_i(z) - P_i(z)) \geqslant D+1$. If we now have

$$\operatorname{ord}_{z=0}(Q(z) y_i(z) - P_i(z)) \geqslant M + D + 1$$

for every $i = 1, \ldots, n$, then the system of polynomials $(Q(z); P_1(z), \ldots, P_n(z))$ is called a system of *Padé approximations of the second kind* for the functions (1) with weights D_0 and D and order M of approximation. We shall say briefly that the system $(Q(z); P_1(z), \ldots, P_n(z))$ of Padé approximations of the second kind has the parameters (D_0, D, M). According to Dirichlet's box principle the system $(Q(z); P_1(z), \ldots, P_n(z))$ with parameters (D_0, D, M) exists whenever $D_0 \geqslant nM$.

In the present paper we assume that the functions $1, y_1(z), \ldots, y_n(z)$ are linearly independent over $\boldsymbol{K}(z)$. We consider a linear form l in $1, y_1(z), \ldots, y_n(z)$,

$$l(z) = H_0 + \sum_{i=1}^{n} H_i y_i(z),$$

where H_i are elements of $\boldsymbol{K}$, not all zero. Let $H = (H_0, H_1, \ldots, H_n)^t$. By l_v we mean a linear form obtained by considering the form l in the corresponding completion K_v, i.e. we think of H_i and all the coefficients of $y_i(z)$ as elements of K_v. For $\theta \in K$, $|\theta|_v < C^{-\alpha_v}$, the value $l_v(\theta)$ is then defined in K_v.

Our main result is the following theorem.

THEOREM. *Let u and ε, $0 < u$, $\varepsilon < 1$, be given. There exists an effective constant λ, depending only on u, ε and the functions (1), such that if $\theta \in K$ satisfies $\theta T(\theta) \neq 0$,*

$$\log h(\theta) > \lambda,$$

$$\log|\theta|_v \leqslant \min\left(\left(\frac{u\varepsilon}{(n+1)(n+\varepsilon)} - 1\right)\log h(\theta),\ -\alpha_v \log 2C\right),$$

then

$$\log|l_v(\theta)|_v > -(n+1+\varepsilon)\log h(H) + \log^+ \max_i |H_i|_v$$

for all $h(H) > C_0$, where C_0 is a positive constant depending on u, ε, θ and the system (2).

Remark. The constants λ and C_0 are given explicitly in (12) and (13), respectively. Of course, as we show on p. 262, our Theorem implies the linear independence of the numbers $1, y_1(\theta), \ldots, y_n(\theta) \in K_v$ over K.

We also obtain the following corollaries.

COROLLARY 1. *Let $K = Q$ and $v|\infty$, and let u and ε be as in the Theorem. Let the coefficients H_i of l belong to Z. There exists a positive constant c_0, depending only on u, ε and the functions (1), such that if $\theta = a/b \in Q$ ($(a, b) = 1$, $b > 0$) satisfies $\theta T(\theta) \neq 0$ and*

$$|a/b| \leqslant 1/(2C), \qquad b^{u\varepsilon} > c_0 |a|^{(n+1)(n+\varepsilon)},$$

then

$$\log|l(a/b)| > -(n+\varepsilon)\log H$$

for all $H = \max(|H_0|, \ldots, |H_n|) > C_0$.

This corollary is analogous to Theorem I of [3], but the proof gives the constants explicitly. As a p-adic analogue to Corollary 1 we propose the following

COROLLARY 2. *Let $K = Q$ and $v|p$, and let u and ε be as in the Theorem. Let the coefficients H_i of l belong to Z. There exists a positive constant c_1, depending only on u, ε and the functions (1), such that if $\theta = a/b \in Q$ ($(a, b) = 1$) satisfies $\theta T(\theta) \neq 0$ and*

$$|a/b|_p \leqslant 1/(2C),$$

$$(\max(|a|, |b|))^{u\varepsilon} > c_1 (|a/b|_p \max(|a|, |b|))^{(n+1)(n+\varepsilon)},$$

then

$$\log|l_p(a/b)|_p > -(n+1+\varepsilon)\log H$$

for all $H = \max(|H_0|, \ldots, |H_n|) > C_0$.

Our results can be applied e.g. to the functions $(1+\alpha z)^{\nu_i}$ with pairwise distinct rational ν_i, $0 < \nu_i < 1$, $i = 1, \ldots, n$, and nonzero $\alpha \in K$, or to certain hypergeometric functions with rational parameters.

3. Lemmas.

The proof of our Theorem is based on the ideas of Chudnovsky [3], and our first lemmas are results of this work. Lemma 1 is Theorem 1.1 of [3].

LEMMA 1. *Let $(Q(z); P_1(z), \ldots, P_n(z))$ be a system of Padé approximations of the second kind with parameters (D_0, D, M) for the functions (1). Let $k \in N$ and assume that $M \geqslant k(s+1)$. We define*

$$Q^{\langle k\rangle}(z) = T^k(z)\left(\frac{d}{dz}\right)^k Q(z)/k!,$$

$$P_i^{\langle k\rangle}(z) = [Q^{\langle k\rangle}(z)\cdot y_i(z)]_{(D+ks)}, \qquad i = 1, \ldots, n.$$

Then $(Q^{\langle k\rangle}(z); P_1^{\langle k\rangle}(z), \ldots, P_n^{\langle k\rangle}(z))$ is a system of Padé approximations of the second kind with parameters $(D_0+ks, D+ks, M-k(s+1))$ for the functions (1).

The following very important result follows from Theorem 1.2 of [3], Chudnovsky's proof of this result is highly ingenious.

LEMMA 2. *Let δ, $0 < \delta < 1/(n+n^2(s+1))$, be given, and define $D_0 = D$, $M = [(n^{-1}-\delta)D]$. There exists a positive constant N, depending only on the system (2) and δ, such that, for all $D > N$ and arbitrary $z_0 \neq 0$ satisfying $T(z_0) \neq 0$, there are integers $k_0, k_1, \ldots, k_n$,*

$$0 \leqslant k_0 < k_1 < \ldots < k_n \leqslant D - nM + n(n+1)(s+1)/2,$$

for which the $n+1$ linear forms in $x_0, x_1, \ldots, x_n$

$$Q^{\langle k_j\rangle}(z_0)x_0 + \sum_{i=1}^{n} P_i^{\langle k_j\rangle}(z_0)x_i, \qquad j = 0, 1, \ldots, n,$$

are linearly independent.

In the following we need Siegel's lemma which we give in the form presented by Bombieri [2].

LEMMA 3. *Let $\gamma = 4d^{2d}|D_K|^{1/2}$, where D_K is the discriminant of K. Let $K < L$. Then there is a non-trivial solution $X \in K^n$ of*

$$\sum_{j=1}^{L} a_{ij}x_j = 0, \qquad i = 1, \ldots, K,\ a_{ij} \in K,$$

with

$$h(X) \leqslant \gamma(2L\gamma)^{K/(L-K)}\Big(\prod_{i=1}^{K}\prod_{v}\max_{j}|a_{ij}|_v\Big)^{1/(L-K)}.$$

LEMMA 4. *Let $y_1(z), \ldots, y_n(z)$ be a set of KG-functions. Then, for any δ, $0<\delta<1/n$, and an arbitrary positive integer D satisfying $D>nM$, $M=[(n^{-1}-\delta)D]$, there exists a system $(Q(z); P_1(z), \ldots, P_n(z))$ of Padé approximations of the second kind with parameters (D, D, M) for the functions $y_1(z), \ldots, y_n(z)$ such that $Q(z)$, $P_i(z)\in \boldsymbol{K}[z]$ and*

$$\log h(Q) \leqslant 2(1+n^{-1}-\delta)\big((\delta n)^{-1}-1\big)D\log C+\log\gamma/(\delta n)$$
$$+\big((\delta n)^{-1}-1\big)\log 2(D+1).$$

Proof. Let q_m, $m=0, 1, \ldots, D$, be the undetermined coefficients of Q,

$$Q(z)=\sum_{m=0}^{D} q_m z^m.$$

By the hypothesis of the lemma and the definition of Padé approximations of the second kind, the unknowns q_k must satisfy the system of linear equations

$$\sum_{k=0}^{D} q_k a_{m-k,i}=0, \quad m=D+1, \ldots, D+M,\ i=1, \ldots, n.$$

By multiplying these equations by r_{D+M} we obtain a system of linear equations in q_k with coefficients in $O_{\boldsymbol{K}}$. The number of equations is $K=nM$ and the number of unknowns is $L=D+1$. Thus $K<L$.

By Lemma 3 and the inequalities

$$|r_{D+M}a_{m-k,i}|_v \leqslant 1$$

valid for all $v|p$ and $i=1, \ldots, n$, $k=0, 1, \ldots, D$, $m=D+1, \ldots, D+M$, we can find a non-trivial solution $q_k\in \boldsymbol{K}$ satisfying

$$h(Q)\leqslant \gamma\big(2(D+1)\gamma\big)^{(1-\delta n)/\delta n}C^{2(1+n^{-1}-\delta)(1-\delta n)D/\delta n}.$$

Obviously also $P_i(z)=[Q(z)\cdot y_i(z)]_D\in \boldsymbol{K}[z]$. Thus Lemma 4 is proved.

LEMMA 5. *Let $(Q(z); P_1(z), \ldots, P_n(z))$ be the system constructed in Lemma 4. If $M=[(n^{-1}-\delta)D]\geqslant k(s+1)$ and $\theta\in\boldsymbol{K}$ satisfies $\theta T(\theta)\neq 0$, then the polynomials $Q^{\langle k\rangle}(z)$ and $P_i^{\langle k\rangle}(z)$ defined in Lemma 1 have, for each place v of $\boldsymbol{K}$, the estimates*

(3) $$|r_{D+ks}Q^{\langle k\rangle}(\theta)|_v \leqslant c(D)^{\beta_v}C^{(D+ks)\beta_v}|Q|_v|T|_v^k\max(1, |\theta|_v^{D+ks}),$$

(4) $$|r_{D+ks}P_i^{\langle k\rangle}(\theta)|_v \leqslant c(D)^{3\beta_v}C^{2(D+ks)\beta_v}|Q|_v|T|_v^k\max(1, |\theta|_v^{D+ks}),$$

$$i=1, \ldots, n,$$

where $c(D)=(s+1)^k(D+1)2^D$.

Proof. We denote $T(z)=\sum_{i=0}^{s} t_i z^i$. If $v|\infty$, then we have

$$|Q^{\langle k\rangle}(\theta)|_v=\Big|\Big(\sum_{i=0}^{s} t_i\theta^i\Big)^k\Big|_v\Big|\sum_{j=k}^{D}\binom{j}{k}q_j\theta^{j-k}\Big|_v$$
$$\leqslant\big((s+1)^k(D+1)2^D\big)^{d_v/d}|T|_v^k|Q|_v\max(1, |\theta|_v^{D+ks}).$$

This implies the estimate (3) in this case.

By denoting

$$Q^{\langle k\rangle}(z)=\sum_{j=0}^{D+ks}q_j^{\langle k\rangle}z^j,$$

we obviously have $(v|\infty)$

(5) $$|q_j^{\langle k\rangle}|_v\leqslant\big((ks+1)(s+1)^{k-1}(D+1)2^D\big)^{d_v/d}|T|_v^k|Q|_v,$$

$$j=0, 1, \ldots, D+ks.$$

Thus we obtain

$$|P_i^{\langle k\rangle}(\theta)|_v=\Big|\sum_{j=0}^{D+ks}\Big(\sum_{m=0}^{j}q_m^{\langle k\rangle}a_{j-m,i}\Big)\theta^j\Big|_v$$
$$\leqslant(D+ks+1)^{d_v/d}\max_{0\leqslant j\leqslant D+ks}\Big|\Big(\sum_{m=0}^{j}q_m^{\langle k\rangle}a_{j-m,i}\Big)\theta^j\Big|_v$$
$$\leqslant(D+ks+1)^{2d_v/d}\max_{0\leqslant j\leqslant D+ks}|q_j^{\langle k\rangle}|_v C^{(D+ks)d_v/d}\max(1, |\theta|_v^{D+ks}),$$

which, by the above estimate for $|q_j^{\langle k\rangle}|_v$, proves (4) in the case $v|\infty$.

If $v|p$, then

$$|r_{D+ks}Q^{\langle k\rangle}(\theta)|_v\leqslant|T|_v^k|Q|_v\max(1, |\theta|_v^{D+ks})$$

and

$$|r_{D+ks}P_i^{\langle k\rangle}(\theta)|_v\leqslant\max_{\substack{0\leqslant j\leqslant D+ks\\0\leqslant m\leqslant j}}\big(|q_m^{\langle k\rangle}|_v|r_{D+ks}a_{j-m,i}|_v\big)\max(1, |\theta|_v^{D+ks})$$
$$\leqslant|Q|_v|T|_v^k\max(1, |\theta|_v^{D+ks}), \quad i=1, \ldots, n,$$

thus proving Lemma 5.

For the remainder functions we use the notation

$$R_i^{\langle k\rangle}(z)=Q^{\langle k\rangle}(z)y_i(z)-P_i^{\langle k\rangle}(z), \quad i=1, \ldots, n.$$

We then have the following lemma.

LEMMA 6. *Let the hypothesis of Lemma 5 be valid. If $|\theta|_v<(2C)^{-\alpha_v}$, then we have the estimates*

$$|R_i^{\langle k\rangle}(\theta)|_v\leqslant c(D)^{2\beta_v}|Q|_v|T|_v^k(C^{\alpha_v}|\theta|_v)^{D+M+1-k}, \quad i=1, \ldots, n.$$

Proof. If $v|\infty$ and $|\theta|_v < (2C)^{-\alpha_v}$, then $|\theta| < 1/(2C)$. By (5), we have

$$\Big|\sum_{j=0}^{D+ks} q_j^{\langle k\rangle} a_{m-j,i}\Big| \leqslant (D+ks+1)(ks+1)(s+1)^{k-1}(D+1)2^D |Q|_v^{d/d_v} |T|_v^{kd/d_v} C^m,$$

$$i = 1, \ldots, n,\ m = D+M+1-k, \ldots$$

This implies, for each $i = 1, \ldots, n$,

$$|R_i^{\langle k\rangle}(\theta)| = \Big|\sum_{m=D+M+1-k}^{\infty} \Big(\sum_{j=0}^{D+ks} q_j^{\langle k\rangle} a_{m-j,i}\Big)\theta^m\Big|$$

$$\leqslant c(D)^2 |Q|_v^{d/d_v} |T|_v^{kd/d_v} (C|\theta|)^{D+M+1-k}.$$

Hence Lemma 6 is true in the case $v|\infty$.

If $v|p$ and $|\theta|_v < 1/(2C)$, then we obtain

$$|R_i^{\langle k\rangle}(\theta)|_v \leqslant \max_{m\geqslant D+M+1-k} \Big|\Big(\sum_{j=0}^{D+ks} q_j^{\langle k\rangle} a_{m-j,i}\Big)\theta^m\Big|_v \leqslant |Q|_v |T|_v^k (C|\theta|_v)^{D+M+1-k}.$$

Thus Lemma 6 is proved.

Lemma 7. *Let δ and δ_1, $0<\delta<1/(3n^2(s+1))$, $0<\delta_1<1$, be given. Assume that*

$$\delta D \geqslant 1+(n+1)(s+1)/2, \quad (1-3\delta n^2(s+1))D \geqslant n,$$

$$k \leqslant D-nM+n(n+1)(s+1)/2.$$

If $\log|\theta|_v < \min((\delta_1-1)\log h(\theta),\ -\alpha_v \log 2C)$, then we have the estimates

(6) $$\log|r_{D+ks} R_i^{\langle k\rangle}(\theta)|_v \leqslant \{2\beta_v(2\delta n\log(s+1)+\log 2+1)$$

$$+(\beta_v(1+2\delta ns)+\alpha_v(1+n^{-1}-3\delta n))\log C$$

$$+2\delta n\log|T|_v+(\delta_1-1)(1+n^{-1}-3\delta n)\log h(\theta)\}D$$

$$+\log|Q|_v, \quad i = 1, \ldots, n.$$

Proof. From the hypothesis it follows that

$$k \leqslant 2\delta nD,\ M = [(n^{-1}-\delta)D] \geqslant k(s+1).$$

Thus the hypothesis of Lemma 6 are satisfied.

If $v|\infty$, then Lemma 6 and our hypothesis on $\log|\theta|_v$ imply

$$\log|r_{D+ks} R_i^{\langle k\rangle}(\theta)|_v \leqslant \beta_v(2\log c(D)+(D+ks)\log C)+\log|Q|_v$$

$$+k\log|T|_v+(D+M+1-k)\log(C^{\alpha_v}|\theta|_v)$$

$$\leqslant \{\beta_v(4\delta n\log(s+1)+2(\log 2+1)+(1+2\delta ns)\log C)$$

$$+2\delta n\log|T|_v+\alpha_v(1+n^{-1}-3\delta n)\log C$$

$$+(\delta_1-1)(1+n^{-1}-3\delta n)\log h(\theta)\}D+\log|Q|_v.$$

Similarly, if $v|p$, then we have

$$\log|r_{D+ks} R_i^{\langle k\rangle}(\theta)|_v \leqslant \log|Q|_v+(2\delta n\log|T|_v+\alpha_v(1+n^{-1}-3\delta n)\log C$$

$$+(\delta_1-1)(1+n^{-1}-3\delta n)\log h(\theta))D.$$

This proves Lemma 7.

4. Proof of the Theorem. First we shall prove the following Theorem A which then implies the truth of our Theorem.

Let $\theta\in \boldsymbol{K}$, $\theta T(\theta)\neq 0$, be given. Suppose that δ and δ_1 satisfying $0<\delta<1/(3n^2(s+1))$ and $0<\delta_1<1$ are given. We shall use the following notations:

$$\mu_1 = n^{-1}+3\delta\delta_1 n-\delta n(2s+3)-\delta_1(1+n^{-1}),$$

$$A = 3(1+n^{-1})\log C/(\delta n)+2/(\delta n)+2\delta n\log h(T),$$

$$\Lambda = \gamma^{1/(\delta n)}(n+1),$$

$$B = \frac{(\mu_1+1+2\delta ns)\log h(\theta)}{\mu_1\log h(\theta)-A}\log\Lambda+A+(1+2\delta ns)\log h(\theta).$$

Theorem A. *Assume that*

$$\mu_1\log h(\theta)-A>0$$

and

$$\log|\theta|_v < \min((\delta_1-1)\log h(\theta),\ -\alpha_v\log 2C).$$

There then exists a positive constant C_1 (given explicitly in (11)) such that

$$\log|l_v(\theta)|_v > -\frac{(\mu_1+1+2\delta ns)\log h(\theta)}{\mu_1\log h(\theta)-A}\log h(H)-B+\log^+\max_i|H_i|_v$$

for all $h(H)>C_1$.

Proof. Let $(Q(z);\ P_1(z), \ldots, P_n(z))$ be the system of Padé approximations with parameters $(D, D, [(n^{-1}-\delta)D])$ constructed in Lemma 4. Let D be large enough to satisfy the hypothesis of Lemmas 2, 4, 5 and 7.

By Lemma 2, we can find an integer k_j,

$$0\leqslant k_j\leqslant D-n[(n^{-1}-\delta)D]+n(n+1)(s+1)/2$$

satisfying

$$I = r_{D+ks}\Big(H_0 Q^{\langle k_j\rangle}(\theta)+\sum_{i=1}^{n} H_i P_i^{\langle k_j\rangle}(\theta)\Big)\neq 0.$$

Obviously $I\in\boldsymbol{K}$, whence we obtain, by the product formula,

$$\prod_v |I|_v = 1.$$

Denoting $k_j = k$, this gives

$$(7)\qquad \log|I|_v = -\sum_{v_1 \neq v} \log|I|_{v_1}$$
$$\geqslant -\sum_{v_1 \neq v} \beta_{v_1}\log(n+1) - \sum_{v_1 \neq v} \log^+ \max_i |H_i|_{v_1}$$
$$-\sum_{v_1 \neq v} \log\max\left(|r_{D+ks}Q^{\langle k\rangle}(\theta)|_{v_1},\ \max_i |r_{D+ks}P_i^{\langle k\rangle}(\theta)|_{v_1}\right).$$

On the other hand, we have

$$(8)\qquad \log|I|_v \leqslant \log|r_{D+ks}Q^{\langle k\rangle}(\theta)\, l_v(\theta) - \sum_{i=1}^{n} H_i r_{D+ks} R_i^{\langle k\rangle}(\theta)|_v$$
$$\leqslant \beta_v \log(n+1) + \log\max\left(|r_{D+ks}Q^{\langle k\rangle}(\theta)\, l_v(\theta)|_v,\ \max_{1\leqslant i\leqslant n} |H_i r_{D+ks} R_i^{\langle k\rangle}(\theta)|_v\right).$$

First we prove that, for a sufficiently large D,

$$(9)\qquad \log(n+1) + \sum_{v_1\neq v} \log^+ \max |H_i|_{v_1}$$
$$+\sum_{v_1 \neq v} \log\max\left(|r_{D+ks}Q^{\langle k\rangle}(\theta)|_{v_1},\ \max_{1\leqslant i\leqslant n} |r_{D+ks}P_i^{\langle k\rangle}(\theta)|_{v_1}\right)$$
$$+\log \max_{1\leqslant i\leqslant n}\left(|H_i|_v |r_{D+ks}R_i^{\langle k\rangle}(\theta)|_v\right) < 0.$$

From Lemmas 4, 5 and 7 we obtain the following upper estimate for the left-hand side of (9):

$$\log(n+1) + \log h(H) + \sum_{v_1\neq v} \{(3\beta_{v_1}(2\delta n\log(s+1) + \log 2 + 1)$$
$$+2\beta_{v_1}(1+2\delta ns)\log C + 2\delta n\log|T|_{v_1} + (1+2\delta ns)\log^+|\theta|_{v_1})D$$
$$+\log|Q|_{v_1}\} + \{2\beta_v(2\delta n\log(s+1)+\log 2+1) + (\beta_v(1+2\delta ns)$$
$$+\alpha_v(1+n^{-1}-3\delta n))\log C + 2\delta n\log|T|_v$$
$$+(\delta_1 - 1)(1+n^{-1}-3\delta n)\log h(\theta)\}D + \log|Q|_v$$
$$\leqslant \log(n+1) + \log h(H) + \log h(Q) + \{6\delta n\log(s+1) + 3(\log 2 + 1)$$
$$+(2(1+2\delta ns) + \alpha_v(1+n^{-1}-3\delta n))\log C + 2\delta n\log h(T) - \mu_1\log h(\theta)\}D$$
$$\leqslant \log A + (A - \mu_1\log h(\theta))D + \log h(H).$$

We now choose D in such a way that

$$(10)\qquad (D-1)(\mu_1\log h(\theta) - A) \leqslant \log h(H) + \log A < D(\mu_1\log h(\theta) - A)$$

assuming $h(H) > C_1$, where

$$(11)\qquad \log C_1 = (\mu_1\log h(\theta) - A)\max\left(N,\ (1+(n+1)(s+1)/2)/\delta,\ n/(1-3\delta n^2(s+1))\right)$$

(this implies that D is sufficiently large to satisfy all the assumptions of our lemmas). Thus the truth of (9) follows.

By using (7), (8) and (9), we obtain

$$\log|l_v(\theta)|_v \geqslant -\log(n+1) - \sum_{v_1\neq v}\log^+\max_i|H_i|_{v_1} - \log|r_{D+ks}Q^{\langle k\rangle}(\theta)|_v$$
$$-\sum_{v_1\neq v}\log\max\left(|r_{D+ks}Q^{\langle k\rangle}(\theta)|_{v_1},\ \max_{1\leqslant i\leqslant n}|r_{D+ks}P_i^{\langle k\rangle}(\theta)|_{v_1}\right).$$

By deduction similar to the above, we obtain, by Lemmas 4 and 5 and the inequalities (10), the estimate

$$\log|l_v(\theta)|_v \geqslant -(6\delta n\log(s+1) + 3(\log 2+1)$$
$$+2(1+2\delta ns)\log C + 2\delta n\log h(T)$$
$$+(1+2\delta ns)\log h(\theta))D - \log(n+1) - \log h(Q)$$
$$-\log h(H) + \log^+\max_i|H_i|_v$$
$$\geqslant -\frac{(\mu_1 + 1 + 2\delta ns)\log h(\theta)}{\mu_1\log h(\theta) - A}\log h(H) - B$$
$$+\log^+\max_i|H_i|_v.$$

Thus Theorem A is proved.

We can now give a proof for the Theorem itself. We choose

$$\delta_1 = \frac{u\varepsilon}{(n+1)(n+\varepsilon)},\qquad \delta = \frac{(1-u)\varepsilon}{4(2s+3)n(n+\varepsilon)(n+1+(1+u)\varepsilon/2)}.$$

Then

$$(n+1+(1+u)\varepsilon/2)\mu_1 - \mu_1 - 1 - 2\delta ns$$
$$> (n+(1+u)\varepsilon/2)(n^{-1} - \delta_1(1+n^{-1}) - \delta n(2s+3)) - 1 - 2\delta ns$$
$$\geqslant (1-u)\varepsilon/4(n+\varepsilon) > 0.$$

By taking

$$(12)\qquad \lambda = \frac{4(n+\varepsilon)(n+1+(1+u)\varepsilon/2)A}{(1-u)\varepsilon},$$

the condition $\log h(\theta) > \lambda$ implies

$$(n+1+(1+u)\varepsilon/2)A < ((n+1+(1+u)\varepsilon/2)\mu_1 - \mu_1 - 1 - 2\delta ns)\log h(\theta).$$

This gives $\mu_1 \log h(\theta) > A$ and

$$(\mu_1+1+2\delta ns)\log h(\theta)/(\mu_1 \log h(\theta)-A) < n+1+(1+u)\varepsilon/2.$$

By Theorem A, we now have

$$\log|l_v(\theta)|_v > -(n+1+(1+u)\varepsilon/2)\log h(H)-B+\log^+ \max_i |H_i|_v$$

for all $h(H) > C_1$. Let us choose

$$C_0 = \max(C_1,\ 2B/(1-u)\varepsilon). \tag{13}$$

Then, for all $h(H) > C_0$,

$$\log|l_v(\theta)|_v > -(n+1+\varepsilon)\log h(H)+\log^+ \max_i |H_i|_v,$$

which proves our Theorem.

The Theorem implies the linear independence of the numbers 1, $y_1(\theta), \ldots, y_n(\theta)$ over K. Namely, if $l_v(\theta) = 0$, then we must have $h(H) < C_0$. Using the properties of the absolute height we obtain (suppose $H_i \neq 0$)

$$h(F) \leqslant h(FH_i)\,h(1/H_i) = h(FH_i)\,h(H_i),$$

where $F = [C_0^2+1]$ is a natural number. Hence,

$$h(FH) \geqslant h(FH_i) \geqslant h(F)/h(H_i) \geqslant F/h(H) > F/C_0 > C_0.$$

Thus the equality $Fl_v(\theta) = 0$ contradicts the Theorem. This implies $l_v(\theta) \neq 0$.

5. Proof of the corollaries. First we prove Corollary 1. Since $K = Q$ and $v|\infty$, we have

$$|a/b|_v = |a/b|, \qquad h(a/b) = \max(|a|,\ |b|) = |b|.$$

If we take $c_0 = e^{u\varepsilon\lambda}$, then our assumptions imply

$$\log h(a/b) > \lambda, \qquad \log|a/b| < \left(\frac{u\varepsilon}{(n+1)(n+\varepsilon)}-1\right)\log h(a/b).$$

Further, we notice that $\log h(H) = \log H$ and $\log^+ \max_i |H_i|_v = \log H$. Thus Corollary 1 immediately follows from our Theorem.

The proof of Corollary 2 is analogous to the above. We simply note that in this case $\log^+ \max_i |H_i|_v = 0$.

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