Case (ii): $n = 2^s p$, $s \geqslant 1$, $p = 5, 7$. Again, since $[2^s, |K|] = n$ it follows that $|K_i| \,\Big|\, \frac{n}{2}$, $1 \leqslant i \leqslant t$. But

$$s\left(\frac{n}{2}\right) - \frac{n}{2} + 8 < n. \quad \blacksquare$$

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An effective order of Hecke–Landau zeta functions near the line $\sigma = 1$. I

by

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1. Let K be an algebraic number field of finite degree n and absolute value of the discriminant equal to d. Denote by $\mathfrak{f}$ a given nonzero integral ideal of the ring of algebraic integers R_K. Let $\chi(C)$ be a Dirichlet character of the abelian group of ideal classes C (mod $\mathfrak{f}$) in the "narrow" sense.

Denote by $\zeta_K(s, \chi)$, $s = \sigma + it$, the Hecke–Landau zeta function associated to χ, defined for $\sigma > 1$ by the series

$$\zeta_K(s, \chi) = \sum_{\mathfrak{a} \in R_K} \chi(\mathfrak{a}) N\mathfrak{a}^{-s}$$

where $\mathfrak{a}$ runs through integral ideals of K and $\chi(\mathfrak{a})$ is the usual extension of $\chi(C)$ (see [5], def. X and LVI).

Basing on some estimates connected with the applications of I. M. Vinogradov's methods to the theory of Hecke–Landau zeta functions we shall prove the following theorems.

THEOREM 1. *For* $1 - 1/(n+1) \leqslant \sigma \leqslant 1$, $t \geqslant 1.1$, *the following inequality holds*:

$$(1.1) \qquad |\zeta_K(\sigma + it, \chi)| \leqslant A_1 N\mathfrak{f}^{1-\sigma} t^{A_2(1-\sigma)^{3/2}} \ln^{2/3} t + A_3 N\mathfrak{f}^{1-\sigma} \ln N\mathfrak{f}$$

where $A_1 = \exp(c_1 \sqrt{d} D n^5)$, $A_2 = 14 \cdot 10^3 n^{2.5}(n+2)$, $A_3 = \sqrt{d} \ln^{2n} d \cdot n^{c_2 n}$, c_1, c_2 *are pure numerical constants and* $D = \left(\frac{5 \ln d}{2(n-1)}\right)^{n-1} < d$ *denotes the constant from Siegel's theorem on the fundamental system of units* (see [10]).

For the Riemann zeta-function the strongest estimate of the form (1.1) is due to H. E. Richert [8] and for the Dedekind zeta-function to W. Staś [12].

Theorem 1 permits us to exhibit zero-free regions for $\zeta_K(s, \chi)$ such that the dependence of the shape of the regions on the parameters of K and χ is explicit.

As an application of (1.1) we get the following

THEOREM 2 (compare [2] and [3]). *There exists a positive constant*

$c_3 > 1$ *independent of* K *and* χ *such that in the region*

$$\sigma \geqslant 1 - \frac{1}{c_3 \max\{\ln N\mathfrak{f},\ A_4 \ln^{2/3}(|t|+3)(\ln\ln(|t|+3))^{1/3}\}}, \quad -\infty < t < \infty, \tag{1.2}$$

where $A_4 = \sqrt{d}\,Dn^5$ *the function* $\zeta_K(\sigma+it, \chi)$ *has no zeros except for the hypothetical real simple zero of* $\zeta_K(s, \chi_1)$, χ_1 *real.*

The methods used to prove Theorems 1 and 2 are generalizations of the Kubilius–Sokolovskii method used by A. V. Sokolovskii to obtain a zero-free region for the Dedekind zeta function (see [11]). The main difficulty of the present paper was to estimate some special trigonometric sums connected with $\zeta_K(s, \chi)$ by expressions which do not depend on the norm of the ideal $\mathfrak{f}$ (Lemmas 4, 5, 6).

2. For $\operatorname{Re} s > 1$ the Hecke–Landau zeta function is equal to

$$\zeta_K(s, \chi) = \sum_{C(\mathfrak{f})} \chi(C) \sum_{\mathfrak{a}\in C(\mathfrak{f})} N\mathfrak{a}^{-s}$$

where the inner sum is taken over all ideals of R_K belonging to an ideal class C (mod $\mathfrak{f}$) and the outer sum is taken over all $h(\mathfrak{f})$ ideal classes. It is well known that $\zeta_K(s, \chi)$ has an analytic continuation as a meromorphic function having at most one pole. This pole is a simple pole located at $s = 1$, and is present only when χ is trivial.

It is easy to verify that

$$f_{C(\mathfrak{f})}(s) = \sum_{\mathfrak{a}\in C(\mathfrak{f})} N\mathfrak{a}^{-s} = N\mathfrak{b}^s \sum_{\substack{\xi\in\mathfrak{b} \\ \xi\equiv 1 \pmod{\mathfrak{f}} \\ \xi\succ 0}}' |N\xi|^{-s}$$

where the last sum is taken over a system of pairwise not associated totally positive algebraic integers (all of the real embeddings of K into the complex field are positive) belonging to the ideal $\mathfrak{b}$ from the inverse class $C^{-1}(\mathfrak{f})$ and congruent to 1 modulo $\mathfrak{f}$.

In the following we can assume that

$$N\mathfrak{b} < 2^{n+1} n^{n/2} d N\mathfrak{f}. \tag{2.1}$$

This follows from Rieger's estimate (see [9]) which we state in a slightly completed version, because its original version depends in an undetermined way on n.

Since $\mathfrak{b}$ is relatively prime to $\mathfrak{f}$ the conditions $\xi \equiv 0 \pmod{\mathfrak{b}}$ and $\xi \equiv 1 \pmod{\mathfrak{f}}$ are equivalent to the condition $\xi \equiv \xi_0 \pmod{\mathfrak{bf}}$ where $\xi_0 \equiv 0 \pmod{\mathfrak{b}}$ and $\xi_0 \equiv 1 \pmod{\mathfrak{f}}$. We choose ξ_0 so that

$$|\xi_0^{(i)}| \leqslant \sqrt{d}\, n^{n+1} (N\mathfrak{bf})^{1/n} \quad \text{for} \quad i = 1, \ldots, n \tag{2.2}$$

(see [2], Lemma 4). Then

$$f_{C(\mathfrak{f})}(s) = (N\mathfrak{b})^s \sum_{\substack{\xi\equiv\xi_0 \pmod{\mathfrak{bf}} \\ \xi\succ 0}}' |N\xi|^{-s} \tag{2.3}$$

where the sum is taken over a complete system of pairwise not associated totally positive algebraic integers congruent to ξ_0 modulo $\mathfrak{bf}$.

Let $\alpha_1, \ldots, \alpha_n$ form a basis for $\mathfrak{bf}$ such that

$$A_5^{-(n-1)} (N\mathfrak{bf})^{1/n} \leqslant |\alpha_k^{(i)}| \leqslant A_5 (N\mathfrak{bf})^{1/n} \tag{2.4}$$

where $A_5 = \sqrt{d}\, n^n$ and $k, i = 1, \ldots, n$. The estimate (2.4) is a corollary from Mahler's theorem (see [7] and [1], Lemma 2). Then each algebraic integer $\xi \equiv \xi_0 \pmod{\mathfrak{bf}}$ can be written in a unique way as a sum

$$\xi = a_1\alpha_1 + \ldots + a_n\alpha_n + \xi_0$$

with rational integral coefficients $a_1, \ldots, a_n$.

Let $\beta \to \beta^{(j)}$ $(1 \leqslant j \leqslant n)$ denote the embeddings of K into the complex field, ordered so that the first r_1 are real and the jth $(j > r_1)$ and $(j+r_2)$-th are complex-conjugate. Then each algebraic number β can be considered as an element of the n-dimensional real space R^n,

$$x(\beta) = (x_1, \ldots, x_{r_1}, y_1, z_1, \ldots, y_{r_2}, z_{r_2}),$$

where $x_i = \beta^{(i)}$ for $1 \leqslant i \leqslant r_1$ and $\beta^{(r_1+j)} = y_j + iz_j$ for $1 \leqslant j \leqslant r_2$.

Denote by $\mathfrak{M}$ the n-dimensional lattice in R^n formed by images of algebraic integers from the ideal $\mathfrak{bf}$ and denote by V the fundamental domain of K. Then the summation in (2.3) reduces to the summation over rational integers $a_1, \ldots, a_n$ such that $x(\xi-\xi_0) \in \mathfrak{M} \cap V$. We get

$$f_{C(\mathfrak{f})}(s) = N\mathfrak{b}^s \sum_{\substack{a_1 \\ x(\xi-\xi_0)\in\mathfrak{M}\cap V \\ \xi\succ 0}} \cdots \sum_{a_n} |Nx(\xi)|^{-s}. \tag{2.5}$$

Denote by $\bar{V}$ the set which we get by multiplying the elements of V by the images of all roots of unity belonging to K. Then we can write the series (2.5) as follows:

$$f_{C(\mathfrak{f})}(s) = \frac{1}{m} N\mathfrak{b}^s \sum_{\substack{a_1 \\ x(\xi-\xi_0)\in\mathfrak{M}\cap\bar{V} \\ \xi\succ 0}} \cdots \sum_{a_n} |Nx(\xi)|^{-\sigma} e^{2\pi i F(a_1,\ldots,a_n)} \tag{2.6}$$

where m denotes the number of roots of unity belonging to K and

$$F(a_1, \ldots, a_n) = -\frac{t}{2\pi} \ln |Nx(\xi)|.$$

Now, for any ideal $\mathfrak{a}$ we define the set $K_{\mathfrak{a}}^X$ in the n-dimensional real space R^n as follows (see [11], p. 324):

$$K_{\mathfrak{a}}^X = \{(u_1, \ldots, u_n) \in R^n:\ \max_{1 \leqslant i \leqslant n} |u_i| \leqslant x,\ x(u) \in \bar{V}\} \tag{2.7}$$

and $x(u) = u_1 x(\alpha_1) + \ldots + u_n x(\alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ form a basis for $\mathfrak{a}$ satisfying (2.4).

3. The proof of Theorem 1 will rest on the following lemmas.

LEMMA 1. *For each ideal $\mathfrak{a}$ of K there exists an integral basis $\alpha_1, \ldots, \alpha_n$ such that for any point $(u_1, \ldots, u_n)$ of $K_{\mathfrak{a}}^{2X} \setminus K_{\mathfrak{a}}^X$*

$$A_6 (N\mathfrak{a})^{1/n} X < |u_1 \alpha_1^{(i)} + \ldots + u_n \alpha_n^{(i)}| < 2A_5\, n (N\mathfrak{a})^{1/n} X \tag{3.1}$$

where $i = 1, \ldots, n$ and $A_6 = \exp(-3\sqrt{d}\, Dn^4)$, $A_5 = \sqrt{d}\, n^n$ as in (2.4).

Proof (compare [11], Lemma 1). Owing to (2.4) the estimate from above is obvious. Now, we consider the system of n linear equations

$$u^{(l)} = u_1 \alpha_1^{(l)} + \ldots + u_n \alpha_n^{(l)}$$

where $l = 1, \ldots, n$. By Cramer's rule

$$u_i = c_{1i} u^{(1)} + \ldots + c_{ni} u^{(n)}$$

and $c_{ki} = D_{ki}/D_0$ where $|D_0| = |\det_{1 \leqslant i,k \leqslant n} [\alpha_i^{(k)}]| = \sqrt{d}\, N\mathfrak{a}$.

By (2.4) and by Hadamard's inequality we have

$$|D_{ki}| \leqslant (n-1)^{(n-1)/2} A_5^{n-1} (N\mathfrak{a})^{(n-1)/n}.$$

Hence

$$|c_{ki}| \leqslant n^{n^2-1} d^{n/2-1} (N\mathfrak{a})^{-1/n}.$$

Putting $|u^{(i)}| \leqslant A_7 X$, where $A_7 = n^{-n^2} d^{-n/2+1} (N\mathfrak{a})^{1/n}$, $i = 1, \ldots, n$, we get $|u_i| \leqslant X$. This means that all solutions $(u_1, \ldots, u_n)$ of the system of inequalities $|u^{(i)}| < A_7 X$, $i = 1, \ldots, n$, belong to $K_{\mathfrak{a}}^X$ is $x(u) \in \bar{V}$.

Hence for any $(u_1, \ldots, u_n) \in K_{\mathfrak{a}}^{2X} \setminus K_{\mathfrak{a}}^X$ there exists j $(1 \leqslant j \leqslant n)$ such that

$$|u_1 \alpha_1^{(j)} + \ldots + u_n \alpha_n^{(j)}| \geqslant A_7 X. \tag{3.2}$$

Furthermore, for each $u = (u_1, \ldots, u_n)$ belonging to V we have

$$\ln |u^{(i)}| = \frac{1}{n} \ln |Nu| + \sum_{k=1}^{r} \xi_k \ln |\varepsilon_k^{(i)}|$$

where $i = 1, \ldots, n$, $0 \leqslant \xi_k < 1$ and $\varepsilon_1, \ldots, \varepsilon_r$ are fundamental units of K, satisfying Siegel's theorem (see [10]):

$$\left|\ln |\varepsilon_k^{(j)}|\right| < \tfrac{3}{2} \sqrt{d}\, Dn^2, \quad k = 1, \ldots, r,\ j = 1, \ldots, n.$$

Hence putting $i = j$ we get

$$|u^{(j)}| = |Nu|^{1/n} \prod_{k=1}^{r} |\varepsilon_k^{(j)}|^{\xi_k} \leqslant |Nu|^{1/n} \exp(\tfrac{3}{2} \sqrt{d}\, Dn^3).$$

Now owing to (3.2) we obtain $|Nu| \geqslant A_7^n \exp(-\frac{3}{2} \sqrt{d}\, Dn^4) X^n$ and finally

$$|u^{(i)}| = \frac{|Nu|}{\prod_{\substack{k=1 \\ k \neq i}}^{n} |u^{(k)}|} \geqslant \exp(-3\sqrt{d}\, Dn^4) (N\mathfrak{a})^{1/n} X.$$

This completes the proof of Lemma 1.

COROLLARY. *For each ideal $\mathfrak{a}$ of K there exists an integral basis $\alpha_1, \ldots, \alpha_n$ such that for any point $(u_1, \ldots, u_n)$ of $K_{\mathfrak{a}}^{2X} \setminus K_{\mathfrak{a}}^X$ and for any α_0 from the class C modulo $\mathfrak{a}$ satisfying* (2.2) *and for $X \geqslant A_6^{-1} \sqrt{d}\, n^{n+1}$ we have the inequality*

$$\tfrac{1}{2} A_6 (N\mathfrak{a})^{1/n} X < |u_1 \alpha_1^{(i)} + \ldots + u_n \alpha_n^{(i)} + \alpha_0^{(i)}| < (2nA_5 + 1)(N\mathfrak{a})^{1/n} X \tag{3.3}$$

for $i = 1, \ldots, n$.

Using (3.1), (3.3) and Turán's second main theorem we obtain the next two lemmas.

LEMMA 2. *If*

$$F(u) = -\frac{t}{2\pi} \ln |Nx(u)|$$

then for any $u = (u_1, \ldots, u_n) \in K_{\mathfrak{b}\mathfrak{f}}^{2X} \setminus K_{\mathfrak{b}\mathfrak{f}}^X$ and $X \geqslant A_5 A_6^{-1}$

$$\left| \frac{\partial^m F}{\partial u_i^m} \right| \leqslant A_8^m (m-1)!\, |t|\, X^{-m} \tag{3.4}$$

where $A_8 = \exp(4\sqrt{d}\, Dn^4)$.

LEMMA 3. *Let i be a natural number, $1 \leqslant i \leqslant n$, and let $X \geqslant A_5 A_6^{-1}$. Then for arbitrary fixed u_j $(1 \leqslant j \leqslant n,\ j \neq i)$ such that $(u_1, \ldots, u_n)$ belongs to $K_{\mathfrak{b}\mathfrak{f}}^{2X} \setminus K_{\mathfrak{b}\mathfrak{f}}^X$ and for any rational integer $m_1 \geqslant 1$ we can divide the interval in which u_i is determined into at most $A_9^{m_1+n+1} = \exp(5\sqrt{d}\, Dn^4)$ subintervals in such a way that for each subinterval there exists m, $m_1 + 1 \leqslant m \leqslant m + n$, such that*

$$\left| \frac{\partial^m F}{\partial u_i^m} \right| \geqslant A_{10}^m (m-1)!\, |t|\, X^{-m} \tag{3.5}$$

with $A_{10} = d^{-1/2} n^{-2n}$ for every point of the subinterval.

LEMMA 4. *Let*

$$1 < t^{1/(n+2)} \leqslant X < A_6^{-1} t^{(n+1)/n}$$

and write

$$m_1 = \left[11 \frac{n+2}{n} \frac{\ln t}{\ln X} \right].$$

Then

$$(3.6) \qquad |S_i| = \Bigl| \sum_{\substack{a < a_i \leqslant a' \\ (a_1, \ldots, a_n) \in K_{\mathfrak{bf}}^{2X} \setminus K_{\mathfrak{bf}}^{X}}} e^{2\pi i F(a_1, \ldots, a_n)} \Bigr| \leqslant 4 A_{11} X^{1 - \frac{1}{10^6 n^4 m_1^2}}$$

where $A_{11} = \exp(10^{-5} \sqrt{d}\, D)$.

Proof. We use Vinogradov's theorem (see [13], p. 210) with $\tau = 10^{-3} n^{-1}$, Lemma 3 and apply the method presented by Sokolovskii in [11], Lemma 5.

Lemma 5. *Let* $1 < X < t^{1/(n+2)}$ *and write* $m = \left[\frac{\ln t}{\ln X} \right] + 1$. *Then*

$$(3.7) \qquad |S_i| = \Bigl| \sum_{\substack{a < a_i \leqslant a' \\ (a_1, \ldots, a_n) \in K_{\mathfrak{bf}}^{2X} \setminus K_{\mathfrak{bf}}^{X}}} e^{2\pi i F(a_1, \ldots, a_n)} \Bigr| \leqslant 4 A_{12} X^{1 - \frac{1}{10^7 n^4 m^2}}$$

where $A_{12} = (3^{-1} 10^{-5} n^{-1} \sqrt{d}\, D)$.

The proof of this lemma uses essentially the same method as that presented in [1], Lemma 11. We use Vinogradov's theorem (see [14], p. 55) and Lemmas 2 and 3.

The next lemma is a simple corollary of Lemmas 4 and 5.

Lemma 6. *Let* $1 < X < A_6^{-1} t^{(n+1)/n}$, $t > 1$ *and write*

$$m_1 = \left[11 \frac{n+2}{n} \frac{\ln t}{\ln X} \right].$$

Then

$$(3.8) \qquad |S_i| \leqslant A_{11} X^{1 - \frac{1}{10^6 n^4 m_1^2}}, \quad \textit{where} \quad A_{11} = \exp(10^{-5} \sqrt{d}\, D).$$

Since the estimate (3.8) does not depend on the norm of the ideal $\mathfrak{f}$, our main objective is attained.

The next two lemmas are Landau's theorems in which the dependence of the constants on the degree n of the field K is explicit.

Lemma 7 (Landau [6]). *If* $0 < \vartheta < 1$, *then*

$$(3.9) \qquad \sum_{\substack{\mathfrak{a} \in R_K \\ N\mathfrak{a} \leqslant x}} N\mathfrak{a}^{-\vartheta} \leqslant \frac{2-\vartheta}{1-\vartheta} A_{13} X^{1-\vartheta}$$

where $A_{13} = n^{c_4^n} d^{1/(n+1)} \ln^{n-1} d$ *and* c_4 *is a numerical constant.*

Lemma 8 (Landau [6]). *For any nonzero ideal* $\mathfrak{f}$ *from* R_K, *if* χ *modulo* $\mathfrak{f}$ *is not the principal character, i.e.,* $\chi \neq \chi_0$, *then*

$$(3.10) \qquad |H(x, \chi)| = \Bigl| \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a}) \Bigr| \leqslant A_{14} X^{1 - \frac{2}{n+1}}$$

where $A_{14} = n^{c_5^n} (dN\mathfrak{f})^{1/(n+1)} \ln^n (dN\mathfrak{f})$ *and* c_5 *is a numerical constant.*

Denote by $\mu(\mathfrak{b})$ the generalized Möbius function and write $\alpha_K = \operatorname{res}_{s=1} \zeta_K(s)$. If χ is the principal character, $\chi = \chi_0$, then

$$(3.11) \qquad \Bigl| H(x, \chi_0) - \alpha_K X \sum_{\mathfrak{b} | \mathfrak{f}} \frac{\mu(\mathfrak{b})}{N\mathfrak{b}} \Bigr| \leqslant A_{15} X^{1 - \frac{2}{n+1}}$$

where $A_{15} = n^{c_6^n} d^{2/(n+1)} \ln^{2n} d$ and c_6 is a numerical constant.

Lemma 9. *In the region* $\sigma \geqslant 1 - 1/(n+1)$, $t > 1$,

$$(3.12) \qquad \Bigl| \zeta_K(s, \chi) - \sum_{1 \leqslant m \leqslant B_1 t^{n+1}} F(m, \chi) m^{-s} \Bigr| \leqslant c_7^n \ln^{n-1} d + B_2$$

where $B_1 B_2^{n+1} = n^{c_8^{n(n+1)}} d^2 \ln^{2n(n+1)} d \cdot N\mathfrak{f} \ln^{n(n+1)} (N\mathfrak{f} + 1)$ *and* c_7, c_8 *are pure numerical constants,* $F(m, \chi) = \sum_{N\mathfrak{a} = m} \chi(\mathfrak{a})$.

Proof. Let

$$g(\chi) = \begin{cases} 0 & \text{for} \quad \chi \neq \chi_0, \\ \alpha_K \sum_{\mathfrak{b} | \mathfrak{f}} \frac{\mu(\mathfrak{b})}{N\mathfrak{b}} & \text{for} \quad \chi = \chi_0. \end{cases}$$

Then by partial summation we obtain for $\sigma > 1$

$$\sum_{m=x}^{\infty} F(m, \chi) m^{-s} + g(\chi) \frac{x^{1-s}}{1-s} = -\frac{H(x, \chi) - g(\chi) x}{x^s} + s \int_x^{\infty} \frac{H(u, \chi) - g(\chi) u}{u^{1+s}} du.$$

Putting $x = B_1 t^{n+1}$, by Lemma 8 we get the estimate (3.12) in the region $\sigma \geqslant 1 - \frac{1}{n+1}$, $t > 1$.

4. Proof of Theorem 1. Setting

$$B_1 = n^{c_9^{n^2}} d^2 \ln^{2n(n+1)} d \cdot N\mathfrak{f} \ln^{n(n+1)} (N\mathfrak{f} + 1),$$

from Lemma 9 we get the following estimate in the region $1 - 1/(n+1) \leqslant \sigma \leqslant 1$, $t > 1$:

$$(4.1) \qquad |\zeta_K(s, \chi)| \leqslant \Bigl| \sum_{1 \leqslant m \leqslant A_{16} N\mathfrak{f} \ln^{n(n+1)} (N\mathfrak{f}+1) t^{n+1} = B_3 t^{n+1}} F(m, \chi) m^{-s} \Bigr| + c_{10}^n \ln^{n-1} d$$

where $A_{16} = n^{c_9^{n^2}} d^2 \ln^{2n(n+1)} d$.

Write $K_i = K_{\mathfrak{bf}}^{2^i t_0}$, where $t_0 = A_5 A_6^{-1} \exp(\ln^{2/3} t)$, so that $X = t_0$ satisfies the assumption of (3.3).

We have

$$(4.2)\qquad |\zeta_K(s,\chi)| \leqslant \Big|\sum_{1\leqslant m\leqslant A_{17}N\mathfrak{f}e^{n\ln^{2/3}t}=B_4e^{n\ln^{2/3}t}} F(m,\chi)m^{-s}\Big| + \Big|\sum_{B_4e^{n\ln^{2/3}t}<m\leqslant B_3t^{n+1}} F(m,\chi)m^{-s}\Big| + c_{10}^n\ln^{n-1}d = S_1+S_2+c_{10}^n\ln^{n-1}d$$

where $A_{17} = 2^{-n}n^{n(n+1)}d^{n/2}$. The second sum is taken over m which do not belong to K_0. The first sum in (4.2) is estimated trivially by partial summation using Lemma 8:

$$(4.3)\qquad |S_1| \leqslant \sum_{m\leqslant B_4e^{n\ln^{2/3}t}} F(m)m^{1-\sigma}m^{-1} \leqslant (n^{c_{11}^n}\ln^{2n}d)(n^{n+1}d^{n/2})^{1-\sigma}(N\mathfrak{f})^{1-\sigma}e^{n(1-\sigma)\ln^{2/3}t}(\ln^{2/3}t+\ln N\mathfrak{f}).$$

The second sum in (4.2) is estimated as follows. We have by (2.6)

$$S_2 = \frac{1}{m}\sum_{C(\mathfrak{f})}\chi(C)N\mathfrak{b}_j^s \sum_{\substack{a_1\ \ a_n \\ x(\xi-\xi_0)\in\mathfrak{M}\cap\bar{V} \\ B_4N\mathfrak{b}_je^{n\ln^{2/3}t}<|N\xi|\leqslant B_3N\mathfrak{b}_jt^{n+1} \\ \xi\succ 0}}\cdots\sum |Nx(\xi)|^{-s}$$

where $\mathfrak{b}_j$ are ideals belonging to the inverse class $C^{-1}(\mathfrak{f})$ and chosen in the same way as in Section 2, $N\mathfrak{b}_j < 2^{n+1}n^{n/2}dN\mathfrak{f}$.

Hence we obtain the estimate

$$(4.4)\qquad |S_2| \leqslant \frac{1}{m}\sum_{j=1}^{h(\mathfrak{f})}(N\mathfrak{b}_j)^{\sigma}\sum_{i=1}^{i_0}\Big|\sum_{\substack{(a_1,\ldots,a_n)\in K_i\setminus K_{i-1} \\ B_4N\mathfrak{b}_je^{n\ln^{2/3}t}<|N\xi|\leqslant B_3N\mathfrak{b}_jt^{n+1} \\ \xi\succ 0}} |Nx(\xi)|^{-s}\Big|$$

where i_0 can be estimated by using Lemma 1. For the class number $h(\mathfrak{f})$ we use the simplest estimate

$$h(\mathfrak{f}) \leqslant 2^{r_1}N\mathfrak{f}h \leqslant c_{12}\sqrt{d}\ln^{n-1}d\cdot 2^nN\mathfrak{f} \qquad \text{(see [4])}.$$

Now we write the outer sum in (4.4) as follows:

$$(4.5)\qquad \sum_{\substack{(a_1,\ldots,a_n)\in K_i\setminus K_{i-1} \\ \xi\succ 0 \\ B_4N\mathfrak{b}_je^{n\ln^{2/3}t}<|Nx(\xi)|\leqslant B_3N\mathfrak{b}_jt^{n+1}}} |Nx(\xi)|^{-s} = \sum_{k=1}^{2n}\sum_{\substack{(a_1,\ldots,a_n)\in B_{ki} \\ \xi\succ 0 \\ B_4N\mathfrak{b}_je^{n\ln^{2/3}t}<|Nx(\xi)|\leqslant B_3N\mathfrak{b}_jt^{n+1}}} |Nx(\xi)|^{-s} = \sum_{k=1}^{2n} S_{ki}$$

where B_{ki} is the set of points $(a_1,\ldots,a_n)\in K_i\setminus K_{i-1}$ such that

$$\begin{array}{ll}
2^{i-1}t_0 < a_k \leqslant 2^it_0 & -2^it_0 \leqslant a_{k-n} < -2^{i-1}t_0 \\
-a_k < a_1 \leqslant a_k & a_{k-n} < a_1 \leqslant -a_{k-n} \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
-a_k < a_{k-1} \leqslant a_k & a_{k-n} < a_{k-n-1} \leqslant -a_{k-n} \\
-a_k \leqslant a_{k+1} < a_k & a_{k-n} \leqslant a_{k-n+1} < -a_{k-n} \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
-a_k \leqslant a_n < a_k & a_{k-n} \leqslant a_n < -a_{k-n} \\
k = 1,\ldots,n & k = n+1,\ldots,2n
\end{array}$$

Now, we have to split the sum S_{ki} into at most three sums in such a way that only one contains all totally positive ξ's and the remaining two sums contain all non-totally positive ξ's. This can be done since if α is totally positive and $\alpha+\beta$ is not totally positive then $\alpha+2\beta$, $\alpha+3\beta$, ... are not totally positive either.

Writing $m_1 = \left[11\dfrac{n+2}{n}\dfrac{\ln t}{\ln X}\right]$ where $X = 2^{i-1}t_0$, we get by partial summation using Lemmas 1 and 6

$$|S_{1i}| \leqslant m2^{n+4}n^2A_5^{n(1-\sigma)}A_6^{-n}A_{11}(N\mathfrak{b}_j\mathfrak{f})^{-\sigma}e^{n(1-\sigma)\ln^{2/3}t}2^{\varphi(i)}2^{-(2A_{18}/\ln^{2/3}t)i}$$

where $A_{18} = (10^6\,11^2(n+2)^2n^2)^{-1}$ and for $1\leqslant i\leqslant i_0$

$$\varphi(i) = n(1-\sigma)i - \frac{A_{18}}{4\ln^2t}i^3 - \frac{A_{18}}{\ln^{2/3}t}i^2 \leqslant \frac{2}{\sqrt{3A_{18}}}(n(1-\sigma))^{3/2}\ln t.$$

For the remaining S_{ki}, $k = 2,\ldots,2n$ we get similar estimates and summing over i, since

$$\sum_{i=1}^{i_0}2^{-(2A_{18}/\ln^{2/3}t)i} \leqslant \sum_{i=1}^{\infty}2^{-(2A_{18}/\ln^{2/3}t)i} \leqslant \frac{\ln^{2/3}t}{2\ln 2A_{18}},$$

we obtain by (4.4)

$$(4.6)\qquad |S_2| \leqslant \exp(c_{13}\sqrt{d}Dn^5)(N\mathfrak{f})^{1-\sigma}t^{n(1-\sigma)/\ln^{2/3}t}\,t^{13.75\cdot 10^3n(n+2)(n(1-\sigma))^{3/2}}\ln^{2/3}t.$$

By (4.2), (4.3) and (4.6) we get the following estimate in the region $t > 1.1$, $1-\dfrac{1}{n\ln^{2/3}t} \leqslant \sigma \leqslant 1$:

(4.7) $$|\zeta_K(s,\chi)| \leqslant \exp(c_{14}\sqrt{d}\,Dn^5)(N\mathfrak{f})^{1-\sigma}\ln^{2/3} t + n^{c_{15}^n}\ln^{2n} d\sqrt{d}(N\mathfrak{f})^{1-\sigma}\ln(N\mathfrak{f}+1),$$

and in the region $1-\frac{1}{n+1} \leqslant \sigma < 1-\frac{1}{n\ln^{2/3} t}$, $t > 1.1$:

(4.8) $$|\zeta_K(s,\chi)| \leqslant \exp(c_{16}\sqrt{d}\,Dn^5)(N\mathfrak{f})^{1-\sigma}t^{14\cdot 10^3 n(n+2)(n(1-\sigma))^{3/2}}\ln^{2/3} t$$

because in this region, by Lemma 7,

$$|S_1| \leqslant n^{c_{17}^n}\ln^{n-1} d\sqrt{d}\,e^{n(1-\sigma)\ln^{2/3}t}(N\mathfrak{f})^{1-\sigma}\ln^{2/3} t.$$

(4.7) and (4.8) prove Theorem 1.

The method of the proof of Theorem 2 is standard. Our starting point is the well-known inequality

$$3\frac{\zeta'}{\zeta}(\sigma,\chi_0)+4\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it,\chi)+\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+2it,\chi^2) \leqslant 0$$

valid for $\sigma > 1$. After cumbersome calculations we obtain Theorem 2 just as the Theorem in [1].

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