Case (ii): $n = 2^p$, $p \geq 1$, $p = 5$, 7. Again, since $[2^p, [K]] = n$ it follows that $|K| \leq n/2$, $1 \leq i \leq t$. But

$$s \left( \frac{n}{2} \right) - \frac{n}{2} + 8 < n.$$  

References


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An effective order of Hecke–Landau zeta functions near the line $\sigma = 1$. I

by

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Let $K$ be an algebraic number field of finite degree $n$ and absolute value of the discriminant equal to $d$. Denote by $\nu$ a given nonzero integral ideal of the ring of algebraic integers $R_K$. Let $\chi(C)$ be a Dirichlet character of the abelian group of ideal classes $C$ (mod $\nu$) in the "narrow" sense.

Denote by $\zeta_K(s, \chi)$, $s = \sigma + it$, the Hecke–Landau zeta function associated to $\chi$, defined for $\sigma > 1$ by the series

$$\zeta_K(s, \chi) = \sum_{\nu \mid \text{ideals of } K} \chi(\nu) N\nu^{-s}$$

where $\nu$ runs through integral ideals of $K$ and $\chi(\nu)$ is the usual extension of $\chi(C)$ (see [5], def. X and LVI).

Basing on some estimates connected with the applications of I. M. Vinogradov's methods to the theory of Hecke–Landau zeta functions we shall prove the following theorems.

Theorem 1. For $1 - 1/(n+1) \leq \sigma \leq 1$, $t \geq 1.1$, the following inequality holds:

$$|\zeta_K(\sigma + it, \chi)| \leq A_1 N^{1/2} t^{1/2} N^{1/2} \ln^{1/2} t + A_3 N^{1/2} \ln N$$

where $A_1 = \exp(c_1 \sqrt{d \ln^2 t})$, $A_2 = 14 \cdot 10^3 n^{2.5} (n+2)$, $A_3 = \sqrt{d \ln^{1/2} d \cdot n^{2.5}}$, $c_1$, $c_2$ are pure numerical constants and $D = \left( \frac{5 \ln d}{2(n-1)} \right)^{n-1}$ denotes the constant from Siegel's theorem on the fundamental system of units (see [10]).

For the Riemann zeta-function the strongest estimate of the form (1.1) is due to H. E. Richert [8] and for the Dedekind zeta-function to W. Staâ [12].

Theorem 1 permits us to exhibit zero-free regions for $\zeta_K(s, \chi)$ such that the dependence of the shape of the regions on the parameters of $K$ and $\chi$ is explicit.

As an application of (1.1) we get the following

Theorem 2 (compare [2] and [3]). There exists a positive constant
c_3 > 1 \text{ independent of } K \text{ and } \chi \text{ such that in the region}

\begin{equation}
\sigma \geq 1 - \frac{1}{c_3 \max \left[ \ln |N\xi|, A_4 \ln \frac{1}{|t| + 3} (\ln (|t| + 3))^{1/3} \right]}, \quad -\infty < t < \infty,
\end{equation}

where \( A_4 = \sqrt{d} Dn^5 \) the function \( \zeta_K(\sigma + it, \chi) \) has no zeros except for the hypothetical real simple zero of \( \zeta_K(s, \chi_1) \), \( \chi_1 \) real.

The methods used to prove Theorems 1 and 2 are generalizations of the Kubilius–Sokolovskii method used by A. V. Sokolovskii to obtain a zero-free region for the Dedekind zeta function (see [11]). The main difficulty of the present paper was to estimate some special trigonometric sums connected with \( \zeta_K(s, \chi) \) by expressions which do not depend on the norm of the ideal \( \mathfrak{f} \) (Lemmas 4, 5, 6).

2. For \( \Re \sigma > 1 \) the Hecke–Landau zeta function is equal to

\[ \zeta_K(s, \chi) = \sum_{C(\mathfrak{f})} N^{\sigma - s} = N^b \sum_{\xi \equiv \xi_0 \pmod{b\mathfrak{f}}} |N\xi|^{-s}, \]

where the inner sum is taken over all ideals of \( R_K \) belonging to an ideal class \( C \pmod{\mathfrak{f}} \) and the outer sum is taken over all \( h(\mathfrak{f}) \) ideal classes. It is well known that \( \zeta_K(s, \chi) \) has an analytic continuation as a meromorphic function having at most one pole. This pole is a simple pole located at \( s = 1 \), and is present only when \( \chi \) is trivial.

It is easy to verify that

\[ f_{C(\mathfrak{f})}(s) = \sum_{\mathfrak{a}} N^{\sigma - s} = N^b \sum_{\xi \equiv \xi_0 \pmod{b\mathfrak{f}}} |N\xi|^{-s}, \]

where the last sum is taken over a system of pairwise not associated totally positive algebraic integers (all of the real embeddings of \( K \) into the complex field are positive) belonging to the ideal \( \mathfrak{b} \) from the inverse class \( C^{-1} \) (\( \mathfrak{f} \)) and congruent to 1 modulo \( \mathfrak{f} \).

In the following we can assume that

\begin{equation}
N^b < 2^{n+1} n^{n/2} dN, \quad (2.1)
\end{equation}

This follows from Rieger's estimate (see [9]) which we state in a slightly completed version, because its original version depends in an undetermined way on \( n \).

Since \( b \) is relatively prime to \( \mathfrak{f} \) the conditions \( \xi \equiv 0 \pmod{b} \) and \( \xi \equiv 1 \pmod{\mathfrak{f}} \) are equivalent to the condition \( \xi \equiv \xi_0 \pmod{b\mathfrak{f}} \) where \( \xi_0 \equiv 0 \pmod{b} \) and \( \xi_0 \equiv 1 \pmod{\mathfrak{f}} \). We choose \( \xi_0 \) so that

\begin{equation}
|\xi_0^n| \leq \sqrt{d} n^{n+1} (Nb)^{1/n} \quad \text{for} \quad l = 1, \ldots, n \quad (2.2)
\end{equation}

(see [2], Lemma 4). Then

\begin{equation}
f_{C(\mathfrak{f})}(s) = (Nb)^s \sum_{\xi \equiv \xi_0 \pmod{b\mathfrak{f}}} |N\xi|^{-s}, \quad (2.3)
\end{equation}

where the sum is taken over a complete system of pairwise not associated totally positive algebraic integers congruent to \( \xi_0 \) modulo \( b\mathfrak{f} \).

Let \( \alpha_1, \ldots, \alpha_n \) form a basis for \( \mathfrak{b} \) such that

\begin{equation}
A_5^{(2k-1)(Nb)^k} \leq |\xi_0^n| \leq A_5 (Nb)^{k/n} \quad (2.4)
\end{equation}

where \( A_5 = \sqrt{d} n^k \) and \( k, i = 1, \ldots, n \). The estimate (2.4) is a corollary from Mahler's theorem (see [7] and [1], Lemma 2). Then each algebraic integer \( \xi \equiv c_0 \pmod{b\mathfrak{f}} \) can be written in a unique way as a sum

\[ \xi = a_1 \alpha_1 + \ldots + a_n \alpha_n + \xi_0 \]

with rational integral coefficients \( a_1, \ldots, a_n \).

Let \( \beta = \beta^0 (1 \leq j \leq n) \) denote the embeddings of \( K \) into the complex field, ordered so that the first \( r_1 \) are real and the \( j \)th \( (j > r_1) \) and \( (j + r_2) \)-th are complex-conjugate. Then each algebraic number \( \beta \) can be considered as an element of the \( n \)-dimensional real space \( \mathbb{R}^n \).

\[ x(\beta) = (x_1, x_2, x_3, \ldots, x_n), \quad (2.5) \]

where \( x_i = \beta^j \) for \( 1 \leq i \leq r_1 \) and \( \beta^{r_1+j} = x_i + i x_j \) for \( 1 \leq j \leq r_2 \).

Denote by \( \mathcal{M} \) the \( n \)-dimensional lattice in \( \mathbb{R}^n \) formed by images of algebraic integers from the ideal \( b\mathfrak{f} \) and denote by \( V \) the fundamental domain of \( K \). Then the summation in (2.3) reduces to the summation over rational integers \( a_1, \ldots, a_n \) such that \( x(x_i - \xi_0) \in \mathcal{M} \cap V \). We get

\begin{equation}
f_{C(\mathfrak{f})}(s) = Nb^s \sum_{a_1, a_n} \sum_{x_i - \xi_0 \in \mathcal{M} \cap V} |N x_i|^{-s} \quad (2.6)
\end{equation}

Denote by \( \tilde{F} \) the set which we get by multiplying the elements of \( V \) by the images of all roots of unity belonging to \( K \). Then we can write the series (2.5) as follows:

\begin{equation}
f_{C(\mathfrak{f})}(s) = \frac{1}{m} Nb^s \sum_{a_1, a_n} \sum_{x_i - \xi_0 \in \mathcal{M} \cap V} |N x_i|^{-s} e^{2\pi i (a_1, \ldots, a_n)} \quad (2.7)
\end{equation}

where \( m \) denotes the number of roots of unity belonging to \( K \) and

\[ F(a_1, \ldots, a_n) = -\frac{1}{2\pi} \ln |N x_i| \cdot \]
Now, for any ideal \( a \) we define the set \( K^*_a \) in the \( n \)-dimensional real space \( R^n \) as follows (see [11], p. 342):

\[
K^*_a = \{ (u_1, \ldots, u_n) \in R^n : \max_{1 \leq i \leq \xi_n} u_i \leq x_i, \ x(u) \in \bar{V} \}
\]

and \( x(u) = u_1 x_1 + \ldots + u_n x_n \), where \( x_1, \ldots, x_n \) form a basis for \( a \) satisfying (2.4).

3. The proof of Theorem 1 will rest on the following lemmas.

**Lemma 1.** For each ideal \( a \) of \( K \) there exists an integral basis \( x_1, \ldots, x_n \) such that for any point \( (u_1, \ldots, u_n) \) of \( K^*_a \) \( K^*_a \)

\[
A_5(Na)^{1/n} X < |u_1 x_1^0 + \ldots + u_n x_n^0| < A_5 n (Na)^{1/n} X
\]

where \( i = 1, \ldots, n \) and \( A_5 = \exp(-3 \sqrt{d} N^n) \). \( A_5 = \sqrt{d} n^n \) as in (2.4).

**Proof.** (compare [11], Lemma 1). Owing to (2.4) the estimate from above is obvious. Now, we consider the system of \( n \) linear equations

\[
u_l = u_1 a_1^0 + \ldots + u_n a_n^0
\]

where \( l = 1, \ldots, n \). By Cramer’s rule

\[u_i = c_{i1} + \ldots + c_{il} u_l^0\]

and \( c_{il} = D_{il}/D_0 \) where \( |D_0| = \det [a_l^0] = \sqrt{d} Na \).

By (2.4) and by Hadamard’s inequality we have

\[|D_{il}| < (n-1)^{\sqrt{n-1}} A_5^{-1} (Na)^{1/n} X\]

Hence

\[|c_{il}| < n^{\sqrt{n-1}} A_5^{-1} (Na)^{-1/n} \]

Putting \( |u_l^0| < A_7 X \), where \( A_7 = n^{-\sqrt{n-1}} d^{-\sqrt{n-1}} (Na)^{-1/n} \), \( i = 1, \ldots, n \), we get \( |u_l^0| < X \). This means that all solutions \( (u_1, \ldots, u_n) \) of the system of inequalities \( |u_l^0| < A_7 X \) \( i = 1, \ldots, n \), belong to \( K^*_a \) and \( x(u) \in \bar{V} \).

Hence for any \( (u_1, \ldots, u_n) \in K^*_a \) there exists \( 1 \leq j \leq n \) such that

\[|u_j x_j^0 + \ldots + u_n x_n^0| \geq A_7 X\]

Furthermore, for each \( u = (u_1, \ldots, u_n) \) belonging to \( V \) we have

\[|\ln|u_l^0|| = \frac{1}{n} \ln|Nu| + \sum_{k=1}^r c_k \ln |e_k^0|
\]

where \( i = 1, \ldots, n \), \( 0 < \xi_k < 1 \) and \( e_1, \ldots, e_r \) are fundamental units of \( K \), satisfying Siegel’s theorem (see [10]):

\[|\ln |e_k^0|| < \frac{3}{2} \sqrt{d} Dn, \quad k = 1, \ldots, r, j \leq 1, \ldots, n.
\]

Hence putting \( i = j \) we get

\[|u_l^0| = |Nu|^{1/n} \prod_{k=1}^r |c_k^0|^{1/k} \leq |Nu|^{1/n} \exp(\frac{3}{2} \sqrt{d} Dn^3).
\]

Now owing to (3.2) we obtain \( |Nu| \geq A_7^2 \exp(-\frac{3}{2} \sqrt{d} Dn^3) X^{-n} \) and finally

\[|u_l^0| = \frac{|Nu|}{\prod_{k=1}^r |u_l^0|} \geq \exp(-3 \sqrt{d} Dn^3) (Na)^{1/n} X.
\]

This completes the proof of Lemma 1.

**Corollary.** For each ideal \( a \) of \( K \) there exists an integral basis \( x_1, \ldots, x_n \) such that for any point \( (u_1, \ldots, u_n) \) of \( K^*_a \) \( K^*_a \) and for any \( a_0 \) from the class \( C \) modulo \( a \) satisfying (2.2) and for \( X \geq A_5^{-1} \sqrt{d} n^{n+1} \) we have the inequality

\[\frac{1}{n} A_6(Na)^{1/n} X < |u_1 a_1^0 + \ldots + u_n a_n^0 + a_0^0| < (2nA_5 + 1)|Na|^{1/n} X
\]

for \( i = 1, \ldots, n \).

Using (3.1), (3.3) and Turán’s second main theorem we obtain the next two lemmas.

**Lemma 2.** If

\[F(u) = \frac{i}{2 \pi} \ln|Nu(x)|
\]

then for any \( u = (u_1, \ldots, u_n) \in K^*_a \) \( K^*_a \) \( \sum_{i=1}^n \frac{\partial F}{\partial u_i} \mid_{u} \mid X^{-m} \]

\[\mid \frac{\partial^n F}{\partial u_i^n} \mid \leq A_6^m (m-1)! |u| X^{-m}
\]

where \( A_6 = \exp(4 \sqrt{d} Dn^3) \).

**Lemma 3.** Let \( i \) be a natural number, \( 1 \leq i \leq n \), and let \( X \geq A_5 A_6^{-1} \).

Then for arbitrary fixed \( u_j \) \( 1 \leq j \leq n \), \( j \neq i \) such that \( (u_1, \ldots, u_n) \) belongs to \( K^*_a \) \( \sum_{i=1}^n \frac{\partial F}{\partial u_i} \mid_{u} \mid X^{-m} \) and for any rational integer \( m_1 \geq 1 \) we can divide the interval in which \( u_i \) is determined into at most \( A_6^{m_1+1} + 1 = \exp(5 \sqrt{d} Dn^3) \) subintervals in such a way that for each subinterval there exists \( m, m_1 + 1 \leq m \leq m+n \), such that

\[\mid \frac{\partial^n F}{\partial u_i^n} \mid \leq A_6^m (m-1)! |u| X^{-m}
\]

with \( A_{10} = d^{-1/2} n^{-2} \) for every point of the subinterval.

**Lemma 4.** Let

\[1 \leq \xi^{(n+1)} \leq X < A_6^{-1} d^{(n+1)/n} \]
and write

\[ m_t = \left\lfloor \frac{11 n + 2 \ln t}{n \ln X} \right\rfloor. \]

Then

\[ |S| = \left| \sum_{a < q \leq r} e^{2\pi i \rho(a_1, \ldots, a_d)} \right| \leq 4 A_{11} X^{1 - \frac{1}{10n^2\ln^2 X}}, \]

where \( A_{11} = \exp(10^{-5} \sqrt{d} D) \).

**Proof.** We use Vinogradov's theorem (see [13], p. 210) with \( \tau = 10^{-3} n^{-1} \), Lemma 3 and apply the method presented by Sokolovskii in [11], Lemma 5.

**Lemma 5.** Let \( 1 < X < t^{1/(n+2)} \) and write \( m = \left\lfloor \frac{\ln t}{\ln X} \right\rfloor + 1 \). Then

\[ |S| = \left| \sum_{a < q \leq r} e^{2\pi i \rho(a_1, \ldots, a_d)} \right| \leq 4 A_{12} X^{\frac{1}{10n^2\ln^2 X}}, \]

where \( A_{12} = (3^{-1} 10^{-5} n^{-1} \sqrt{d} D) \).

The proof of this lemma uses essentially the same method as that presented in [1], Lemma 11. We use Vinogradov's theorem (see [14], p. 55) and Lemmas 2 and 3.

The next lemma is a simple corollary of Lemmas 4 and 5.

**Lemma 6.** Let \( 1 < X < A_{20}^{1/2} \) and \( t > 1 \) and write

\[ m_t = \left\lfloor \frac{11 n + 2 \ln t}{n \ln X} \right\rfloor. \]

Then

\[ |S| \leq A_{11} X^{1 - \frac{1}{10n^2\ln^2 X}}, \]

where \( A_{11} = \exp(10^{-5} \sqrt{d} D) \).

Since the estimate (3.8) does not depend on the norm of the ideal \( \mathfrak{f} \), our main objective is attained.

The next two lemmas are Landau's theorems in which the dependence of the constants on the degree \( n \) of the field \( K \) is explicit.

**Lemma 7 (Landau [6]).** If \( 0 < \beta < 1 \), then

\[ \sum_{a \in \mathcal{O}_K} \mathcal{N} a^{-\beta} \leq 2 - \frac{\beta}{\frac{1}{8}} A_{13} X^{1 - \beta} \]

where \( A_{13} = n^{6n} d^{3(n+1)} \ln^{n+1} d \) and \( c_4 \) is a numerical constant.

**Lemma 8 (Landau [6]).** For any nonzero ideal \( \mathfrak{f} \) from \( \mathcal{R}_K \), if \( \chi \) modulo \( \mathfrak{f} \) is not the principal character, i.e., \( \chi \neq \chi_0 \), then

\[ |H(x, \chi)| = \left| \sum_{a \in \mathcal{O}_K} \frac{\chi(a)}{a} \right| \leq A_{13} X^{1 - \frac{2}{n+1}} \]

where \( A_{13} = n^{6n} d^{3(n+1)} \ln^{n+1} d \) and \( c_5 \) is a numerical constant.

Denote by \( \mu(b) \) the generalized Möbius function and write \( \alpha_{\chi} = \text{res}_{\chi = 1} \chi(s) \). If \( \chi \) is the principal character, \( \chi = \chi_0 \), then

\[ |H(x, \chi_0) - \alpha_{\chi_0} X \sum_{b \in \mathcal{O}_K} \frac{\mu(b)}{|b|} | \leq A_{15} X^{1 - \frac{2}{n+1}} \]

where \( A_{15} = n^{6n} d^{2(n+1)} \ln^{n+1} d \) and \( c_6 \) is a numerical constant.

**Lemma 9.** In the region \( \sigma \geq 1 - 1/(n+1) \), \( t > 1 \),

\[ |g_{\chi}(s, \chi) - \sum_{1 \leq m \leq B \chi^{1+1}} F(m, \chi)| | \leq \gamma \ln^{n+1} d + B_2 \]

where \( B_1 B_2 \gamma^{1+1} = n^{6n} d^{2(n+1)} \ln^{2(n+1)} d \cdot N \ln^{n+1} d + c_7, c_8 \) are pure numerical constants, \( F, \chi = \sum_{N=1}^{\infty} \chi(s) \).

**Proof.** Let

\[ g(s) = \begin{cases} 0 & \text{for } \chi \neq \chi_0, \\ \alpha_{\chi} \left( \sum_{b \in \mathcal{O}_K} \frac{\mu(b)}{|b|} \right) & \text{for } \chi = \chi_0. \end{cases} \]

Then by partial summation we obtain for \( \sigma > 1 \)

\[ \sum_{m=2}^{\infty} \frac{F(m, \chi)|m|^{-s} + g(s)|m|^{1-s}}{\ln m} = -\frac{H(x, \chi) - g(s)}{s^2} + \int_{x}^{\infty} \frac{H(u, \chi) - g(s)}{u^{1+s}} \, du. \]

Putting \( x = B_1 \gamma^{1+1} \), by Lemma 8 we get the estimate (3.12) in the region \( \sigma \geq 1 - \frac{1}{n+1} \), \( t > 1 \).

**4. Proof of Theorem 1.** Setting

\[ B_1 = n^{6n} d^{2(n+1)} \ln^{2(n+1)} d \cdot N \ln^{n+1} d + c_7, \]

from Lemma 9 we get the following estimate in the region \( 1 - 1/(n+1) \leq \sigma \leq 1, t > 1 \):

\[ |g_{\chi}(s, \chi) - \sum_{1 \leq m \leq B \chi^{1+1}} F(m, \chi)| | \leq c_{7} \ln^{n+1} d \]

where \( A_{16} = n^{6n} d^{2(n+1)} \ln^{2(n+1)} d \).

Write \( K_1 = K_2 \chi_0 \), where \( \chi_0 = A_3 A_0 \exp(\ln^{2/3} t) \), so that \( X = \chi_0 \) satisfies the assumption of (3.3).
We have
\begin{equation}
(4.2) \quad \left| K_k(s, \chi) \right| \leq \sum_{|m| < A_0^{10^{10}} \leq B_0^{10^{10}}} F(m, \chi) m^{-s} + \sum_{B_0^{10^{10}} \leq m \leq B_0^{10^{10} + 1}} F(m, \chi) m^{-s} + c_{10} \ln n^{-1} d
\end{equation}
where \( A_1 = 2^{-n} n^{2n+1} d^{n^2} \). The second sum is taken over \( m \) which do not belong to \( K_0 \). The first sum in (4.2) is estimated trivially by partial summation using Lemma 8:
\begin{equation}
(4.3) \quad |S_1| \leq \sum_{m \in B_0^{10^{10}} \leq B_0^{10^{10} + 1}} F(m) m^{-s} m^{-1} \\
\leq (n^{-1} \ln 2^{10}) (\ln 2^{10}) (\ln 2^{10})^{-1} \cdot (N \ln 2^{10}) (\ln 2^{10})^{-1} (\ln 2^{10}) (\ln \ln 2^{10})
\end{equation}
The second sum in (4.2) is estimated as follows. We have by (2.6)
\begin{equation}
S_2 = \frac{1}{m} \sum_{C_{10}} \sum_{b_j \in K_{10}} (N b_j \equiv b_j) \sum_{\alpha_1 \leq \alpha_2} \sum_{b_j \in B_0^{10^{10}} \leq B_0^{10^{10} + 1}} |N x(\xi)|^{-s}
\end{equation}
where \( b_j \) are ideals belonging to the inverse class \( C_{10}^{-1} (f) \) and chosen in the same way as in Section 2, \( N b_j \leq 2^{-n^2} n^2 d N f \).

Hence we obtain the estimate
\begin{equation}
(4.4) \quad |S_2| \leq \frac{1}{m} \sum_{f = 1}^{k_0} (N b_j \equiv b_j) \sum_{i = 1}^{l_0} |N x(\xi)|^{-s}
\end{equation}
where \( i_0 \) can be estimated by using Lemma 1. For the class number \( h(f) \) we use the simplest estimate
\begin{equation}
h(f) \leq 2^{2} N b_j \leq c_{12} \sqrt{d \ln n^{-1} d \cdot 2^{n}} N f \tag{see [4].}
\end{equation}

Now we write the outer sum in (4.4) as follows:
\begin{equation}
(4.5) \quad \sum_{(\alpha_1, \ldots, \alpha_{n-1}, k_{T-1}) \leq K_{T-1}} |N x(\xi)|^{-s}
\end{equation}
we obtain by (4.4)
\begin{equation}
|S_2| \leq \exp(c_{12} \sqrt{d b_j} (N b_j)^{-s} t^{(n^2 - 1) / 2} \ln 2^{10} t^{13.75 + 23(n+2)(1-s) / 3} \ln 2^{10} t)
\end{equation}

By (4.2), (4.3) and (4.6) we get the following estimate in the region
\begin{equation}
t > 1.1, 1 - \frac{1}{n \ln 2^{10} t} \leq \sigma \leq 1:
\end{equation}

where \( B_{H} \) is the set of points \( (a_1, \ldots, a_n) \in K_n \setminus K_{n-1} \) such that
\begin{align*}
2^{-1} t_0 < a_0 < 2^{1} t_0 & \quad -2^{1} t_0 < a_{n-1} < -2^{-1} t_0 \\
-a_0 < a_1 < a_0 & \quad a_{k-1} < a_k < a_{k-n} < -a_{k-n} \\
\cdots \cdots \cdots & \cdots \cdots \\
-a_0 < a_{k-1} < a_k & \quad a_{k-1} < a_k < a_{k-n} < -a_{k-n} \\
-a_0 < a_{k-1} < a_k & \quad a_{k-1} < a_k < a_{k-n} < -a_{k-n} \\
\cdots \cdots \cdots & \cdots \cdots \\
-a_0 < a_{k-1} < a_k & \quad a_{k-1} < a_k < a_{k-n} < -a_{k-n} \\
-k = 1, \ldots, n & \quad k = n+1, \ldots, 2n
\end{align*}
(4.7) \[ \zeta_K(s, \chi) \leq \exp(c_{14} \sqrt{d}Dn^{\delta})(N\bar{f})^{1-\sigma}\ln n^{2/3} t \]

\[ + n^{1/8} \ln n \sqrt{d}Dn^{\delta}(N\bar{f})^{1-\sigma}\ln(n\bar{f}+1), \]

and in the region \( 1 - \frac{1}{n+1} \leq \sigma \leq 1 - \frac{1}{n\ln n^{2/3}} \), \( t > 1.1 \):

(4.8) \[ \zeta_K(s, \chi) \leq \exp(c_{16} \sqrt{d}Dn^{\delta})(N\bar{f})^{1-\sigma} t^{1/4} \cdot 10^{3n(5+2\sigma(3-\sigma))3/2} \ln n^{2/3} t \]

because in this region, by Lemma 7,

\[ |S_1| \leq n^{1/2} \ln^{-1} d \sqrt{d} e^{\delta(1-\sigma) n^{2/3}} (N\bar{f})^{1-\sigma} \ln n^{2/3} t. \]

(4.7) and (4.8) prove Theorem 1.

The method of the proof of Theorem 2 is standard. Our starting point is the well-known inequality

\[ 3 \zeta'(\sigma, \chi_0) + 4 \text{Re} \zeta'(\sigma + it, \chi) + \text{Re} \zeta'(\sigma + 2it, \chi^2) \leq 0 \]

valid for \( \sigma > 1 \). After cumbersome calculations we obtain Theorem 2 just as the Theorem in [1].

References