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Disjoint covering systems with precisely one multiple modulus *

by

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1. Introduction. A *disjoint covering system* $(R_1, \ldots, R_t)$, $t > 1$, is a partition of the integers into residue sets

$$R_i = \{k \in Z\colon\ k \equiv a_i (\mathrm{mod}\ n_i)\}, \qquad 1 \leqslant i \leqslant t.$$

Two obvious necessary conditions for this to occur are

$$(1) \qquad \sum_{i=1}^{t} n_i^{-1} = 1 \quad \text{and} \quad (n_i, n_j) > 1, \quad 1 \leqslant i, j \leqslant t.$$

One of the earliest results about such systems is that the moduli n_i cannot all be distinct. (See [2].) Znám [9] and Newman [3] independently proved that the largest modulus, n, must be repeated at least $p(n)$ times, where $p(n)$ denotes the least prime divisor of n. Thus if we order the moduli

$$n_1 \leqslant n_2 \leqslant \ldots \leqslant n_t$$

then necessarily

$$n_{t-j+1} = n_{t-j+2} = \ldots = n_t, \qquad j = p(n_t).$$

Berger, Felzenbaum and Fraenkel [1] give a geometric proof of this fact, and of the extension discovered by Porubský [5] to any *maximal* modulus n, maximal in the sense of division. Thus if

$$(2) \qquad n_k \nmid n_i, \quad k+1 \leqslant i \leqslant t,$$

then

$$(3) \qquad n_{k-j+1} = n_{k-j+2} = \ldots = n_k, \qquad j = p(n_k).$$

The fact that some moduli of a disjoint covering system must be repeated leads one naturally to enquire about systems with precisely one multiple modulus. According to what we have just said the repeated modulus

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must be the largest, and so

(4) $$n_1 < n_2 < \ldots < n_{t-m+1} = n_{t-m+2} = \ldots = n_t.$$

Furthermore on account of (2), (3) it follows that

$$n_i | n_t, \quad 1 \leqslant i \leqslant t.$$

The case $m = t$ is always possible, but we shall refer to it as the *trivial* system.

Any disjoint covering system with moduli $(n_1, \ldots, n_t)$ can always be modified to one with moduli $(2, 2n_1, \ldots, 2n_t)$ by a procedure we refer to as the 2-*add*. In this procedure the a_i and n_i are all doubled, and the residue set R of all odd numbers is annexed. Conversely, if $n_1 = 2$ then all the moduli must be even, and when $t \geqslant 3$ the system can be modified to one with moduli $(\frac{1}{2}n_2, \ldots, \frac{1}{2}n_t)$ by a procedure we refer to as the 2-*drop*. In this procedure the set R_1 is discarded, the n_i are all halved and the a_i are all replaced by either $\frac{1}{2}a_i$ or $\frac{1}{2}(a_i+1)$ depending on their parity. Referring back to (4), then, this means that we can always assume, after repeated application of the 2-drop procedure, that $n_1 \geqslant 3$ or that the system is the trivial one $m = t = 2$. Thus if we characterize the disjoint covering systems with

(5) $$3 \leqslant n_1 < n_2 < \ldots < n_{t-m+1} = n_{t-m+2} = \ldots = n_t$$

then the more general systems satisfying (4) will simply be those obtained from the ones satisfying (5), together with the trivial system $m = t = 2$, by repeated application of the 2-add procedure. This elementary observation reduces for us the number of possible systems from infinite to finite, as will shortly be seen.

Stein [7] showed that no disjoint covering system satisfies (5) with $m = 2$. Thus the only systems satisfying (4) with $m = 2$ are those obtained from the trivial one, $m = t = 2$, by the 2-add procedure; namely,

$$n_i = 2^i, \quad 1 \leqslant i \leqslant t-1 \quad \text{and} \quad n_t = 2^{t-1}.$$

Znám [8] showed that the only system satisfying (5) with $m = 3$ is the trivial one. Porubský [4] showed that the only nontrivial system satisfying (5) with $m = 4$ has parameters

$$n_1 = 3, \quad n_5 = 6.$$

Furthermore for $m = 5$ there is no such nontrivial system. Thus we can summarize

SUMMARY. *The only nontrivial disjoint covering system satisfying* (5) *with* $m \leqslant 5$ *is the one with* $m = 4$, $n_1 = 3$, $n_5 = 6$.

Porubský further conjectures that the only nontrivial systems satisfying (5) with $m = 7$ are

$$n_1 = 4, n_2 = 6, n_9 = 12 \quad \text{and} \quad n_1 = 3, n_2 = 6, n_3 = 9, n_{10} = 18.$$

Our main result is

THEOREM I. *The only nontrivial disjoint covering systems satisfying* (5) *with* $m \leqslant 9$ *are*

$$m = 4, \quad n_1 = 3, \quad n_5 = 6;$$

$$m = 6 \quad \begin{cases} n_1 = 4, & n_7 = 8, \\ n_1 = 3, & n_7 = 9, \\ n_1 = 3, & n_2 = 6, \quad n_8 = 12; \end{cases}$$

$$m = 7 \quad \begin{cases} n_1 = 4, & n_2 = 6, \quad n_9 = 12, \\ n_1 = 3, & n_2 = 6, \quad n_3 = 9, \ n_{10} = 18; \end{cases}$$

$$m = 8 \quad \begin{cases} n_1 = 5, & n_9 = 10, \\ n_1 = 3, & n_9 = 12; \end{cases}$$

$$m = 9 \quad \begin{cases} n_1 = 4, & n_{10} = 12, \\ n_1 = 3, & n_2 = 6, \quad n_{11} = 18, \\ n_1 = 4, & n_2 = 6, \quad n_3 = 8, \quad n_4 = 12, \ n_{13} = 24, \\ n_1 = 3, & n_2 = 6, \quad n_3 = 9, \quad n_4 = 12, \quad n_5 = 18, \ n_{14} = 36. \end{cases}$$

All of these systems are readily constructed, and so the main result here is the assertion that there are no others.

We find it more convenient to formulate this theorem for *coset partitions* of σ_n, the (additive) cyclic group $\{0, 1, \ldots, n-1\}$ where $n = n_t$. In general $(R_1, \ldots, R_t)$ is a disjoint covering system if and only if $(R_1 \cap \sigma_n, \ldots, R_t \cap \sigma_n)$ is a coset partition of σ_n, where n is a common multiple of $n_1, \ldots, n_t$. In this setting $|R_i \cap \sigma_n| = n/n_i$, so the equivalent to Theorem I concerns coset partitions containing precisely m singletons, all the other cosets having distinct orders. Specifically we consider coset partitions $(K_1, \ldots, K_t)$ of σ_n with

(6) $$n/3 \geqslant |K_1| > |K_2| > \ldots > |K_{t-m+1}| = |K_{t-m+2}| = \ldots = |K_t| = 1.$$

Again, if $m = t$ we say the partition is trivial. We can now rephrase Theorem I as

THEOREM II. *The only nontrivial coset partitions of* σ_n *satisfying* (6) *with* $m \leqslant 9$ *are*

$$m = 4, \quad n = 6, \quad |K_1| = 2;$$

$$m = 6 \quad \begin{cases} n = 8, & |K_1| = 2, \\ n = 9, & |K_1| = 3, \\ n = 12, & |K_1| = 4, \quad |K_2| = 2; \end{cases}$$

$$\begin{cases} n = 12, & |K_1| = 3, \quad |K_2| = 2, \\ n = 18, & |K_1| = 6, \quad |K_2| = 3, \ |K_3| = 2; \end{cases}$$

$$m = 8 \quad \begin{cases} n = 10, \ |K_1| = 2, \\ n = 12, \ |K_1| = 4; \end{cases}$$

$$m = 9 \quad \begin{cases} n = 12, \ |K_1| = 3, \\ n = 18, \ |K_1| = 6, \quad |K_2| = 3, \\ n = 24, \ |K_1| = 6, \quad |K_2| = 4, \ |K_3| = 3, \ |K_4| = 2, \\ n = 36, \ |K_1| = 12, \ |K_2| = 6, \ |K_3| = 4, \ |K_4| = 3, \ |K_5| = 2. \end{cases}$$

In terms of coset partitions (1) becomes

$$[|K_i|, |K_j|] < n. \tag{7}$$

On account of this condition Theorem II is a consequent of the seemingly weaker

THEOREM III. *If* σ_n *admits a nontrivial coset partition satisfying* (6) *with* $m \leqslant 9$ *then* $n = 6, 8, 9, 10, 12, 18, 24, 36$.

Indeed if we tally, for each such n, all sets of divisors which lie between 2 and $n/3$, no two of which have l.c.m. n, and the sum of which lies between $n-2$ and $n-9$, then we get precisely the list in Theorem II. Thus 12, for example, has divisors 2, 3, 4 in the range 2 through 4. Since $[3, 4] = 12$ we need only consider the sets $\{4, 2\}$, $\{3, 2\}$, $\{4\}$, $\{3\}$, $\{2\}$. We discard the last set, as its sum does not lie in the range 3 through 10. The remaining four sets correspond to the four partitions with $n = 12$ in Theorem II. Similarly 24 has divisors 2, 3, 4, 6, 8 in the range 2 through 8. Since $[3, 8] = [6, 8] = 24$ we need only consider subsets of $\{2, 4, 8\}$ and $\{2, 3, 4, 6\}$. The first set has sum outside the range 15 through 22, as does any proper subset of the second set. So the only alternative here is $\{2, 3, 4, 6\}$.

In Section 3 we prove Theorem III, thereby establishing Theorem I. We emphasize again that from Theorem I one obtains all disjoint covering systems satisfying (4) with $m \leqslant 9$ by including the trivial ones and applying the 2-add procedure repeatedly. In this way Theorem I gives the complete characterization of all disjoint covering systems with precisely one multiple modulus, the multiplicity of which does not exceed nine. Our proof reproduces the Stein, Znám and Porubský results for $m = 2$, $m = 3$ and $m = 4, 5$ respectively, and validates the Porubský conjecture. A nice general survey of disjoint covering systems is Porubský ([6], Chap. 2).

2. Parallelotope partitions. We introduce certain sets of integer lattice points. For

$$b_i \in N, \quad b_i \geqslant 2, \quad 1 \leqslant i \leqslant n \tag{1}$$

the set

$$\mathscr{P} = \{c = (c_1, \ldots, c_n) \in Z^n\colon\ 0 \leqslant c_i < b_i,\ 1 \leqslant i \leqslant n\}$$

is called the $(n; \boldsymbol{b})$-*parallelotope*, $\boldsymbol{b} = (b_1, \ldots, b_n)$. If $b_1 = b_2 = \ldots = b_n = b$ then it is called the $(n; b)$-*cube*. Let $T \subset \{1, \ldots, n\}$. A *T-cell*, $\mathscr{K}$, of $\mathscr{P}$ is any set

$$\mathscr{K} = \{\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathscr{P}\colon\ c_i = u_i,\ \forall i \notin T\}$$

where $\boldsymbol{u} \in \mathscr{P}$. T is said to be the *index set* for $\mathscr{K}$,

$$T = \operatorname{index}(\mathscr{K}).$$

(This is well-defined on account of (1).) Let $\hat{\mathscr{P}} \subset \mathscr{P}$ be the $(n; \hat{\boldsymbol{b}})$-parallelotope, and suppose

$$\hat{b}_i = b_i, \quad \forall i \in T.$$

Then either

$$\mathscr{K} \subset \hat{\mathscr{P}} \quad \text{or} \quad \mathscr{K} \cap \hat{\mathscr{P}} = \emptyset. \tag{2}$$

A partition $\mathscr{T} = (\mathscr{K}_1, \ldots, \mathscr{K}_t)$ of $\mathscr{P}$ into cells is called a *cell partition* of $\mathscr{P}$. If $\hat{\mathscr{P}} \subset \mathscr{P}$ is the $(n; \hat{\boldsymbol{b}})$-parallelotope set

$$\mathscr{T}(\hat{\mathscr{P}}) = \{\mathscr{K} \in \mathscr{T}\colon\ \mathscr{K} \cap \hat{\mathscr{P}} \neq \emptyset\}$$

and

$$\hat{\mathscr{T}}(\hat{\mathscr{P}}) = \{\mathscr{K} \cap \hat{\mathscr{P}}\colon\ \mathscr{K} \in \mathscr{T}(\hat{\mathscr{P}})\}.$$

Observe that $\hat{\mathscr{T}}(\hat{\mathscr{P}})$ is a cell partition of $\hat{\mathscr{P}}$.

In what follows we shall use the following fact. Suppose $x_1, \ldots, x_n \geqslant 0$ are non-negative numbers, $(B_1, \ldots, B_k)$ is a partition of $\{1, \ldots, n\}$ and

$$\sum_{i \in B_1} x_i = \sum_{i \in B_2} x_i = \ldots = \sum_{i \in B_k} x_i.$$

Then

$$\sum_{i=1}^{n} x_i \geqslant k \max_{1 \leqslant i \leqslant n} x_i. \tag{3}$$

Let $s(k)$ denote the sum of all (positive) divisors of k.

PROPOSITION I. *Let* $\mathscr{T} = (\mathscr{K}_1, \ldots, \mathscr{K}_t)$ *be a cell partition of the* $(n; \boldsymbol{b})$-*parallelotope* $\mathscr{P}$, $t > 1$. *Then for each* $\mathscr{K} \in \mathscr{T}$

$$(b-1)|\mathscr{K}| \leqslant \sum_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \neq \mathscr{K}}} |\mathscr{K}_i|,$$

where

$$b = \min(b_i\colon\ i \notin \operatorname{index}(\mathscr{K})).$$

Proof. Let $\hat{\mathscr{P}}$ be the $(n; \hat{\boldsymbol{b}})$-parallelotope, where

$$\hat{b}_i = \begin{cases} b_i, & i \in \operatorname{index}(\mathscr{K}), \\ b, & i \notin \operatorname{index}(\mathscr{K}). \end{cases}$$

By translating we can assume without loss of generality that $\mathscr{K} \subset \hat{\mathscr{P}}$. Set

$$\mathscr{S}_k = \{c = (c_1, \ldots, c_n) \in \hat{\mathscr{P}}: \sum_{i \notin \operatorname{index}(\mathscr{K})} c_i \equiv k \pmod b\}, \quad 0 \leqslant k < b.$$

For any cell $\mathscr{L} \subset \mathscr{P}$ with $\operatorname{index}(\mathscr{L}) \not\subset \operatorname{index}(\mathscr{K})$

$$|\hat{\mathscr{L}} \cap \mathscr{S}_k| = \frac{|\hat{\mathscr{L}}|}{b}, \quad 0 \leqslant k < b,$$

where $\hat{\mathscr{L}} = \mathscr{L} \cap \hat{\mathscr{P}}$. Furthermore if $\operatorname{index}(\mathscr{L}) \subset \operatorname{index}(\mathscr{K})$ and $\mathscr{L} \cap \hat{\mathscr{P}} \neq \emptyset$ then, according to (2), $\mathscr{L}$ lies entirely within one set $\mathscr{S}_k$. Thus

$$\sum_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \subset \mathscr{S}_0}} |\mathscr{K}_i| = \sum_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \subset \mathscr{S}_1}} |\mathscr{K}_i| = \ldots = \sum_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \subset \mathscr{S}_{b-1}}} |\mathscr{K}_i|.$$

From (3) follows then

$$\sum_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \subset \hat{\mathscr{P}}}} |\mathscr{K}_i| \geqslant b \max_{\substack{\operatorname{index}(\mathscr{K}_i) \subset \operatorname{index}(\mathscr{K}) \\ \mathscr{K}_i \subset \hat{\mathscr{P}}}} |\mathscr{K}_i| \geqslant b|\mathscr{K}|. \ \blacksquare$$

PROPOSITION II. *Let $\mathscr{P}$ be the $(l_1+l_2+1;\, b)$-parallelotope, where*

$$b_i = \begin{cases} 2, & 1 \leqslant i \leqslant l_1, \\ 3, & l_1+1 \leqslant i \leqslant l_1+l_2, \\ p, & i = l_1+l_2+1, \end{cases}$$

and $p \geqslant 5$ is prime. We allow the possibility $l_1 = 0$, but l_2 must be positive. Let $\mathscr{T} = (\mathscr{K}_1, \ldots, \mathscr{K}_t)$ be a cell partition of $\mathscr{P}$ containing at most 9 singletons, and suppose there exists $\mathscr{K} \in \mathscr{T}$ with

(4) $$3 \nmid |\mathscr{K}|, \quad p \,\big|\, |\mathscr{K}|.$$

Then there exist $i \neq j$ with

(5) $$3 \nmid |\mathscr{K}_i|, \quad |\mathscr{K}_i| = |\mathscr{K}_j| > 1.$$

Proof. Let $n = l_1+l_2+1$. We introduce two partitions of $\mathscr{P}$. The first is defined by

$$\mathscr{S}_k = \{c = (c_1, \ldots, c_n) \in \mathscr{P}: \sum_{i=l_1+1}^{l_1+l_2} c_i \equiv k \pmod 3\}, \quad 0 \leqslant k \leqslant 2.$$

The second is defined by

$$\mathscr{P}_j = \{c = (c_1, \ldots, c_n) \in \mathscr{P}: c_n = j\}, \quad 0 \leqslant j < p.$$

For any cell $\mathscr{L} \subset \mathscr{P}$ with $3 \,\big|\, |\mathscr{L}|$ we have

$$|\mathscr{L} \cap \mathscr{P}_j \cap \mathscr{S}_k| = \tfrac{1}{3}|\mathscr{L} \cap \mathscr{P}_j|, \quad 0 \leqslant j < p,\ 0 \leqslant k \leqslant 2.$$

Furthermore if $3 \nmid |\mathscr{L}|$ then $\mathscr{L}$ lies entirely within one set $\mathscr{S}_k$. Thus

$$\sum_{\substack{3 \nmid |\mathscr{K}_i| \\ \mathscr{K}_i \subset \mathscr{S}_0}} |\mathscr{K}_i \cap \mathscr{P}_j| = \sum_{\substack{3 \nmid |\mathscr{K}_i| \\ \mathscr{K}_i \subset \mathscr{S}_1}} |\mathscr{K}_i \cap \mathscr{P}_j| = \sum_{\substack{3 \nmid |\mathscr{K}_i| \\ \mathscr{K}_i \subset \mathscr{S}_2}} |\mathscr{K}_i \cap \mathscr{P}_j|, \quad 0 \leqslant j < p.$$

From (3) follows then

(6) $$\sum_{3 \nmid |\mathscr{K}_i|} |\mathscr{K}_i| \geqslant 3 \sum_{j=0}^{p-1} \max_{3 \nmid |\mathscr{K}_i|} |\mathscr{K}_i \cap \mathscr{P}_j|.$$

Let $\mathscr{K}^* \in \mathscr{T}$ be such that

$$|\mathscr{K}^*| = 2^{m_1},$$

and m_1 is maximal such. Similarly let $\mathscr{K}^{**} \in \mathscr{T}$ be such that

$$|\mathscr{K}^{**}| = 2^{m_2} p,$$

and m_2 is maximal such. (It exists by virtue of (4). Note that m_1 or m_2 may be zero.) For any cell $\mathscr{L} \subset \mathscr{P}$ with $p \,\big|\, |\mathscr{L}|$

$$|\mathscr{L} \cap \mathscr{P}_j| = \frac{|\mathscr{L}|}{p}, \quad 0 \leqslant j < p.$$

Furthermore if $p \nmid |\mathscr{L}|$ then $\mathscr{L}$ lies entirely within one set $\mathscr{P}_j$. Thus $\mathscr{K}^*$ lies entirely within one set $\mathscr{P}_j$, which we can assume to be $\mathscr{P}_0$. Then

(7) $$\sum_{j=0}^{p-1} \max_{3 \nmid |\mathscr{K}_i|} |\mathscr{K}_i \cap \mathscr{P}_j| \geqslant |\mathscr{K}^* \cap \mathscr{P}_0| + \sum_{j=1}^{p-1} |\mathscr{K}^{**} \cap \mathscr{P}_j|$$
$$= |\mathscr{K}^*| + \frac{p-1}{p}|\mathscr{K}^{**}| = 2^{m_1} + (p-1)\,2^{m_2}.$$

Assume now that there do not exist $i \neq j$ satisfying (5). Then all the non-singletons $\mathscr{K}_i$ with $3 \nmid |\mathscr{K}_i|$ have distinct cardinalities. Thus

$$\sum_{3 \nmid |\mathscr{K}_i|} |\mathscr{K}_i| \leqslant s(2^{m_1}) + s(2^{m_2} p) + 7 = (2^{m_1+1}-1) + (2^{m_2}-1)(p+1) + 7$$
$$< 3\,[2^{m_1} + 2^{m_2}(p-1)],$$

which, together with (7), contradicts (6). $\blacksquare$

PROPOSITION III. *Let $\mathscr{P}$ be the $(l_1+l_2+1;\, b)$-parallelotope, where*

$$b_i = \begin{cases} 2, & 1 \leqslant i \leqslant l_1, \\ p, & i = l_1+l_2+1, \end{cases}$$

$p \geqslant 5$ is prime and

$$b_i \geqslant 3, \quad l_1+1 \leqslant i \leqslant l_1+l_2.$$

We allow the possibility $l_1 = 0$ but l_2 must be positive. Let $\mathscr{T} = (\mathscr{K}_1, \ldots, \mathscr{K}_t)$

be a cell partition of $\mathscr{P}$ *containing at most* 9 *singletons, and suppose there exists* $\mathscr{K}\in\mathscr{T}$ *with*

(8) $$\operatorname{index}(\mathscr{K})\cap\{l_1+1,\,l_1+2,\,\ldots,\,l_1+l_2+1\}=\{l_1+l_2+1\}.$$

Then there exist $i\neq j$ *with*

(9) $$|\mathscr{K}_i|=|\mathscr{K}_j|>1.$$

Proof. Let $\hat{\mathscr{P}}$ be the $(l_1+l_2+1;\,\hat{b})$-parallelotope, where

$$\hat{b}_i=\begin{cases}b_i, & 1\leqslant i\leqslant l_1 \text{ or } i=l_1+l_2+1,\\ 3, & l_1+1\leqslant i\leqslant l_1+l_2.\end{cases}$$

By translating we can assume that $\mathscr{K}$ satisfying (8) lies in $\hat{\mathscr{P}}$. Clearly $\mathscr{K}$ satisfies (4), and thus we can apply Proposition II to the cell partition $\mathscr{T}(\hat{\mathscr{P}})$ of $\hat{\mathscr{P}}$. Accordingly there exists $i\neq j$ with

$$3\nmid|\mathscr{K}_i\cap\hat{\mathscr{P}}|,\qquad |\mathscr{K}_i\cap\hat{\mathscr{P}}|=|\mathscr{K}_j\cap\hat{\mathscr{P}}|>1.$$

From (2) follows that $\mathscr{K}_i,\ \mathscr{K}_j\subset\hat{\mathscr{P}}$, and we thus obtain (9). ∎

3. Coset partitions. Let $\sigma=\sigma_n$ where n has the prime factorization

(1) $$n=\prod_{j=1}^{l}p_j^{s_j},\qquad p_1<p_2<\ldots<p_l.$$

For $j\in\{1,\ldots,l\}$ let $\mathscr{R}_j$ be the $(s_j;\,p_j)$-cube. Let $\mathscr{P}=\mathscr{P}_n$ be the parallelotope

$$\mathscr{P}=\mathscr{R}_1\times\mathscr{R}_2\times\ldots\times\mathscr{R}_l.$$

Define the *parallelotope function*

$$\varphi=\varphi_n\colon\ \sigma\to\mathscr{P}$$

as follows. Given $k\in\sigma$ and $j\in\{1,\ldots,l\}$ let $\varphi^{(j)}(k)=\boldsymbol{b}^{(j)}=(b_1^{(j)},\ldots,b_{s_j}^{(j)})\in\mathscr{R}_j$ be the s_j-tuple of p_j-ary coefficients for $k \pmod{p_j^{s_j}}$. That is

$$\varphi^{(j)}(k)=\boldsymbol{b}^{(j)}\Leftrightarrow k \pmod{p_j^{s_j}}=\sum_{i=1}^{s_j}b_i^{(j)}p_j^{s_j-i}.$$

Then set

$$\varphi(k)=\big(\varphi^{(1)}(k),\,\ldots,\,\varphi^{(l)}(k)\big).$$

The following result is from [1].

Proposition I. *φ is bijective, and if K is a coset of σ with*

$$|K|=\prod_{j=1}^{l}p_j^{r_j},$$

then $\varphi(K)$ is a cell of $\mathscr{P}$ with index set

$$T=\operatorname{index}(\varphi(K))=\bigcup_{j=1}^{l}\Big(\sum_{i<j}s_j+\{1,\,\ldots,\,r_j\}\Big).$$

Proposition II. *Let $(K_1,\ldots,K_t)$ be a coset partition of σ. For any $K=K_k$*

$$\left[p\left(\frac{n}{|K|}\right)-1\right]|K|\leqslant\sum_{\substack{|K_i|\,|\,|K|\\ K_i\neq K}}|K_i|.$$

Proof. $(\varphi(K_1),\ldots,\varphi(K_t))$ is a cell partition of $\mathscr{P}$. Thus by Proposition 2.I

$$(b-1)\,|\varphi(K)|\leqslant\sum_{\substack{|K_i|\,|\,|K|\\ K_i\neq K}}|\varphi(K_i)|$$

where

$$b=\min\big(p_j\colon\ p_j^{s_j}\nmid|K|\big)=\min\left(p_j\colon\ p_j\,\Big|\,\frac{n}{|K|}\right).\ ∎$$

Proposition III. *Let n have the prime factorization* (1) *with*

(2) $$l\geqslant 2,\qquad s_l=1,\qquad p_{l-1}>2.$$

Let $(K_1,\ldots,K_t)$ be a coset partition of σ_n containing at most 9 *singletons, and suppose some $K=K_k$ satisfies*

(3) $$p_l\,\big|\,|K|,\qquad p_j\nmid|K|,\qquad \forall j\colon\ 2<p_j<p_l.$$

Then for some $i\neq j$

$$|K_i|=|K_j|>1.$$

Proof. $(\varphi(K_1),\ldots,\varphi(K_t))$ is a coset partition of $\mathscr{P}$, and if K satisfies (3) then

$$\{\bar{s}+1,\,\ldots,\,n\}\cap\operatorname{index}(\varphi(K))=\{n\},$$

where $\bar{s}=s_1$ or 0 depending on whether $p_1=2$ or not, respectively. According to Proposition 2.III then, for some $i\neq j$

$$|\varphi(K_i)|=|\varphi(K_j)|>1.\ ∎$$

Let $P(n)$ denote the greatest prime divisor of n.

Lemma IV. *Let $n\geqslant 2$. If*

(4) $$n-s\left(\frac{n}{P(n)}\right)\leqslant 8$$

then either

$$n \leqslant 10 \quad or \quad n = 12, 18, 36 \quad or \quad n = 2^s.$$

Proof. Let n have the prime factorization (1). Then

$$(5)\quad n-s\left(\frac{n}{P(n)}\right) = n-\left(\prod_{j=1}^{l-1}\frac{p_j^{s_j+1}-1}{p_j-1}\right)\frac{p_l^{s_l}-1}{p_l-1}$$
$$= \frac{n}{p_l}\left[p_l-\left(\prod_{j=1}^{l-1}\frac{p_j-p_j^{-s_j}}{p_j-1}\right)\frac{p_l-p_l^{-s_l+1}}{p_l-1}\right] > \frac{n}{p_l}\left(p_l-\prod_{j=1}^{l}\frac{p_j}{p_j-1}\right).$$

Let $2 = q_1, q_2, \ldots$ be the consecutive enumeration of the primes. Then

$$q_k-\prod_{j=1}^{k}\frac{q_j}{q_j-1}$$

is non-decreasing in k. (Use induction on k.) Thus if $p_l = 5$ then

$$p_l-\prod_{j=1}^{l}\frac{p_j}{p_j-1} \geqslant p_l-\prod_{q_j\leqslant p_l}\frac{q_j}{q_j-1} = \frac{5}{4};$$

and if $p_l = 7$ then

$$p_l-\prod_{j=1}^{l}\frac{p_j}{p_j-1} \geqslant \frac{21}{8};$$

and if $p_l \geqslant 11$ then

$$p_l-\prod_{j=1}^{l}\frac{p_j}{p_j-1} \geqslant \frac{99}{16}.$$

Thus it follows from (5) that

(a) if $p_l = 5$ and $\frac{n}{p_l} \geqslant 7$ then $n-s\left(\frac{n}{P(n)}\right) > 8$;

(b) if $p_l = 7$ and $\frac{n}{p_l} \geqslant 4$ then $n-s\left(\frac{n}{P(n)}\right) > 8$;

(c) if $p_l \geqslant 11$ and $\frac{n}{p_l} \geqslant 2$ then $n-s\left(\frac{n}{P(n)}\right) > 8$.

Using (a)–(c) we see that if (4) holds then the only possibilities for n are

$$n = 10, 14, 15, 20, 21, 25, 30 \text{ or } n = p \geqslant 5 \text{ or } n = 2^{s_1}3^{s_2}.$$

We rule out $n = 14, 15, 20, 21, 25, 30$ and $n = p \geqslant 11$.

Suppose $n = 2^{s_1}3^{s_2}$, $s_2 > 0$. Then

$$n-s\left(\frac{n}{3}\right) = 2^{s_1}+\left(\frac{3^{s_2}-1}{2}\right).$$

Thus (4) holds if and only if $s_1, s_2 \leqslant 2$. ∎

Proof of Theorem 1.III. Let $(K_1, \ldots, K_t)$ be a nontrivial coset partition of σ_n satisfying (1.6) with $m \leqslant 9$. Since $2 \leqslant |K_1| \leqslant n/3$ it is clear that $n \geqslant 6$ and n cannot be prime. Furthermore $n = 2^s$, $s \geqslant 4$ is impossible since for such an n

$$s\left(\frac{n}{4}\right)+8 < n.$$

Thus it suffices to show that n must satisfy the conclusion of Lemma IV, or else $n = 24$.

Suppose not. That is, suppose $n \neq 24$ and n does not satisfy the conclusion of Lemma IV. Let n have the prime factorization (1). Then

$$(6)\quad n-s\left(\frac{n}{p_l}\right) > 8.$$

Set

$$\mathscr{C} = \{K_i\colon\ p_l^{s_l} \mid |K_i|\}.$$

It follows from (6) that $\mathscr{C} \neq \emptyset$. Let $K \in \mathscr{C}$ be of minimal order in the sense of division; that is,

$$K_i \in \mathscr{C}, \quad |K_i| \mid |K| \ \Rightarrow\ K_i = K.$$

Thus if $|K_i| \mid |K|$, $K_i \neq K$, then in fact $|K_i| \,\Big|\, \frac{|K|}{p_l}$. According to Proposition II, then

$$(m-1)+s\left(\frac{|K|}{p_l}\right) \geqslant |K|.$$

Thus $|K|$ must satisfy the conditions of Lemma IV. Now the only way $|K|$ can satisfy these conditions without n satisfying them is

$$n = 2^{s_1}3^{s_2},\ s_2 = 1, 2 \quad \text{or} \quad n = 2^s p,\ s \geqslant 1,\ p = 5, 7 \quad \text{or} \quad (2) \text{ holds}.$$

In this latter case $|K|$ has to be divisible by a prime $p \geqslant 5$, so it is clear from the list of possibilities in Lemma IV that $|K| = 2^r p$, $r = 0, 1$ and thus K satisfies (3). So we rule this case out by appealing to Proposition III.

Case (i): $n = 2^{s_1}3^{s_2}$, $s_2 = 1, 2$. Since $[2^{s_1}, |K|] = n$ it follows from (1.7) that $|K_i| \,\Big|\, \frac{n}{2}$, $1 \leqslant i \leqslant t$. Now

$$s\left(\frac{n}{2}\right)-\frac{n}{2}+8 \geqslant n \ \Leftrightarrow\ n = 3, 6, 12, 24.$$

Case (ii): $n = 2^s p$, $s \geqslant 1$, $p = 5, 7$. Again, since $[2^s, |K|] = n$ it follows that $|K_i| \,\Big|\, \frac{n}{2}$, $1 \leqslant i \leqslant t$. But

$$s\left(\frac{n}{2}\right) - \frac{n}{2} + 8 < n. \ \blacksquare$$

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An effective order of Hecke–Landau zeta functions near the line $\sigma = 1$. I

by

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1. Let K be an algebraic number field of finite degree n and absolute value of the discriminant equal to d. Denote by $\mathfrak{f}$ a given nonzero integral ideal of the ring of algebraic integers R_K. Let $\chi(C)$ be a Dirichlet character of the abelian group of ideal classes C (mod $\mathfrak{f}$) in the "narrow" sense.

Denote by $\zeta_K(s, \chi)$, $s = \sigma + it$, the Hecke–Landau zeta function associated to χ, defined for $\sigma > 1$ by the series

$$\zeta_K(s, \chi) = \sum_{\mathfrak{a} \in R_K} \chi(\mathfrak{a}) N\mathfrak{a}^{-s}$$

where $\mathfrak{a}$ runs through integral ideals of K and $\chi(\mathfrak{a})$ is the usual extension of $\chi(C)$ (see [5], def. X and LVI).

Basing on some estimates connected with the applications of I. M. Vinogradov's methods to the theory of Hecke–Landau zeta functions we shall prove the following theorems.

Theorem 1. *For* $1 - 1/(n+1) \leqslant \sigma \leqslant 1$, $t \geqslant 1.1$, *the following inequality holds:*

$$|\zeta_K(\sigma + it, \chi)| \leqslant A_1 N\mathfrak{f}^{1-\sigma} t^{A_2(1-\sigma)^{3/2}} \ln^{2/3} t + A_3 N\mathfrak{f}^{1-\sigma} \ln N\mathfrak{f} \tag{1.1}$$

where $A_1 = \exp(c_1 \sqrt{d}\, D n^5)$, $A_2 = 14 \cdot 10^3 n^{2.5}(n+2)$, $A_3 = \sqrt{d} \ln^{2n} d \cdot n^{c_2 n}$, c_1, c_2 *are pure numerical constants and* $D = \left(\frac{5 \ln d}{2(n-1)}\right)^{n-1} < d$ *denotes the constant from Siegel's theorem on the fundamental system of units* (see [10]).

For the Riemann zeta-function the strongest estimate of the form (1.1) is due to H. E. Richert [8] and for the Dedekind zeta-function to W. Staś [12].

Theorem 1 permits us to exhibit zero-free regions for $\zeta_K(s, \chi)$ such that the dependence of the shape of the regions on the parameters of K and χ is explicit.

As an application of (1.1) we get the following

Theorem 2 (compare [2] and [3]). *There exists a positive constant*