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Singularities of analytic functions in a differential ring

by

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1. Introduction. In previous papers [4], [5] we characterized differential rings $\mathscr{R}$ of functions of a complex variable and functions of a non-Archimedean variable. In this paper we prove that these functions have no essential singularities.

The ring $\mathscr{R}$ is closed under differentiation and if $\mathscr{R}_0$ is a subring of $\mathscr{R}$ we can define the ring $\mathscr{L} = \mathscr{R}_0[D]$ of linear differential operators with coefficients in $\mathscr{R}_0$ and consider $\mathscr{R}$ an $\mathscr{L}$-module.

DEFINITION 1.1. The elements $f_1, f_2, \ldots, f_n$ of $\mathscr{R}$ are *linearly dependent over* $\mathscr{L}$ if there exist $L_1, \ldots, L_n \in \mathscr{L}$, not all 0, so that $L_1 f_1 + \ldots + L_n f_n = 0$ and *linearly independent over* $\mathscr{L}$ otherwise.

The *dimension of* $\mathscr{R}$ *over* $\mathscr{L}$ is the maximum number of linearly independent elements of $\mathscr{R}$ over $\mathscr{L}$.

Let $\mathscr{L} = C[D]$ denote the *ring of differential operators with constant coefficients*.

Let $\mathscr{E}(f)$ denote the set of *essential singularities of* f.

Let $\mathscr{E}(\mathscr{R}) = \bigcup_{f \in \mathscr{R}} \mathscr{E}(f)$.

2. Functions of a complex variable. In this section we prove the result for differential rings of analytic functions of one complex variable.

DEFINITION 2.1. Let $\mathscr{R}$ be a *differential ring* of single valued functions which are analytic except on a denumerable set $S = \{z_n\colon n = 1, 2, \ldots\}$ with no limit points.

THEOREM 2.2. *If $\mathscr{R}$ is finite dimensional over $\mathscr{L}$, then* $\mathscr{E}(\mathscr{R}) = \emptyset$.

Since the conclusion is based on the properties of the individual elements, we may restrict attention to the differential subring

$$\langle f \rangle = C[f, f', f'', \ldots]$$

generated by an element of $\mathscr{R}$. Without loss of generality we may assume that f has a singularity at $z_1 = 0$.

LEMMA 2.3. *If $f \in \mathscr{R}$ is such that $\langle f \rangle$ is finite dimensional over $\mathscr{L} = \boldsymbol{C}[D]$, then the singularity at $z_1 = 0$ is not an essential singularity.*

LEMMA 2.4. *Let f be analytic for $0 < |z| \leqslant R$ with a singularity at $z = 0$ and let, as usual,*

$$M(r, f) = \max_{|z| = r} |f(z)|.$$

Then for every $\delta > 0$ we have

$$M(r, f^{(K)}) < K!\, M\big(r - M(r, f)^{-\delta}, f\big)^{1+\delta K}$$

if r is sufficiently small.

Proof. We have

$$f^{(K)}(z) = \frac{K!}{2\pi i} \int\limits_{|\zeta - z| = \varrho} \frac{f(\zeta)}{(\zeta - z)^{K+1}}\, d\zeta$$

provided $\varrho < |z|$. If we choose z so that $|z| = r$ and $|f^{(K)}(z)| = M(r, f^{(K)})$ we have

$$M(r, f^{(K)}) = |f^{(K)}(z)| = \left| \frac{K!}{2\pi i} \int\limits_{|\zeta - z| = \varrho} \frac{f(\zeta)}{(\zeta - z)^{K+1}}\, d\zeta \right|.$$

Now since $f(z)$ has a singularity at $z = 0$ we have, for sufficiently small r, $M(r - \varrho, f) \geqslant |f(\zeta)|$ for all ζ on the circle $|\zeta - z| = \varrho$ so

$$M(r, f^{(K)}) \leqslant \frac{K!}{2\pi} 2\pi\varrho \frac{M(r - \varrho, f)}{\varrho^{K+1}} = K! \frac{M(r - \varrho, f)}{\varrho^{K}}.$$

Set $\varrho = M(r, f)^{-\delta}$ and for small r we have

$$M(r, f^{(K)}) \leqslant K!\, M\big(r - M(r, f)^{-\delta}, f\big)^{1+\delta K} \left[\frac{M(r, f)}{M\big(r - M(r, f)^{-\delta}, f\big)} \right]^{K\delta}$$
$$\leqslant K!\, M\big(r - M(r, f)^{-\delta}, f\big)^{1+\delta K}.$$

Proof of Lemma 2.3. If $\langle f \rangle$ is finite dimensional over $\mathscr{L}$, let n be the least positive integer so that there exists an $L_n \in \mathscr{L}^*$ with

(2.1) $$L_n(f^n) = L_1 f + L_2(f^2) + \ldots + L_{n-1}(f^{n-1}) = g$$

where $L_1, L_2, \ldots, L_{n-1} \in \mathscr{L}$ and $\mathscr{L}^* = \mathscr{L} \setminus \{0\}$.

If $n = 1$, f^n is an exponential polynomial and so f^n and hence f is bounded as $z \to 0$.

If $n > 1$, write

$$L_n = (D - \lambda_1) \ldots (D - \lambda_m) = (D - \lambda_1)^{m_1} \ldots (D - \lambda_k)^{m_k}$$

where $\lambda_1, \ldots, \lambda_k$ are distinct. If we set $g = P(f, f', \ldots, f^{(m)})$ we get

(2.2) $$f^n = L_n^{-1} g$$
$$= h + e^{\lambda_1 z} \int\limits_a^z e^{(\lambda_2 - \lambda_1) z_1} dz_1 \int\limits_a^{z_1} e^{(\lambda_3 - \lambda_2) z_2} dz_2 \ldots \int\limits_a^{z_{m-1}} e^{-\lambda_m z_m} g(z_m)\, dz_m,$$

where $0 < |a| < |z| < R$, and $L_n h = 0$. Hence there exist constants (all generically denoted by c) with

$$M(r, f)^n = M(r, f^n) < c + cM(r, g)$$

unless $g = 0$, in which case f^n, and hence f, satisfies the lemma.

However, we can estimate $M(r, g)$ directly from the definition of g in (2.1) to get

$$M(r, g) < c \max_{\substack{1 \leqslant i \leqslant n-1 \\ j \leqslant N}} M(r, D^j f^i) \quad \text{where} \quad N = \max_{1 \leqslant i \leqslant n-1} \deg L_i.$$

Thus by Lemma 2.4 we get

(2.3) $$M(r, g) < c \max_{1 \leqslant i \leqslant n-1} M\big(r - M(r, f^i)^{-\delta}, f^i\big)^{1+N\delta}$$
$$= c \max_{1 \leqslant i \leqslant n-1} M\big(r - M(r, f^i)^{-\delta}, f\big)^{i(1+N\delta)}$$
$$< cM\big(r - M(r, f)^{-\delta}, f\big)^{n-1+nN\delta}.$$

If we choose $\delta = 1/2nN$ and substitute in (2.3) we get

$$M(r, f)^n < c + cM\big(r - M(r, f)^{-\delta}, f\big)^{n-1/2} < M\big(r - M(r, f)^{-\delta}, f\big)^{n-1/4}$$

for sufficiently small r. Hence

(2.4) $$M\big(r - M(r, f)^{-\delta}, f\big) > M(r, f)^{\frac{n}{n-1/4}} > M(r, f)^{1+\frac{1}{4n}}.$$

Now if the lemma does not hold, there exists arbitrarily small r for which

(2.5) $$c/r^c < M(r, f)^{1/2}.$$

Now choose r so that $r < 1$ and an inequality

(2.6) $$M(r, f)^{\delta} > \frac{1}{r} + \frac{1}{1-r}$$

slightly stronger than (2.5) holds, and so that

(2.7) $$M(r, f)^{\frac{\delta}{4n}} > 1/r^2.$$

We now take $r = r_0, r_1, r_2, \ldots, r_s, \ldots$ successively where

$$(2.8)\quad r_{s+1} = r_s - M(r_s, f)^{-\delta} > r - \sum_{k=0}^{s} (r^{2k+1} - r^{2k+2}) > r - \frac{r}{1+r} = \frac{r^2}{1+r} > 0$$

and

$$(2.9)\quad M(r_{s+1}, f)^{\delta} > r^{-2} M(r_s, f)^{\delta} > r^{-2s-2} M(r, f)^{\delta}.$$

We get these properties by induction as follows: If $s = 0$, (2.8) and (2.9) are true since by (2.6)

$$r_1 = r - M(r, f)^{-\delta} > r - (r - r^2)$$

and

$$M(r_1, f) = M\big(r - M(r, f)^{-\delta}, f\big) > M(r, f) M(r, f)^{1/(4n)} > (1/r^2)^{1/\delta} M(r, f)$$

by (2.7). Now assume (2.8) and (2.9) for s. Then

$$r_{s+1} = r_s - M(r_s, f)^{-\delta} > r - \sum_{k=0}^{s-1} (r^{2k+1} - r^{2k+2}) - r^{2s} M(r, f)^{-\delta}$$

$$> r - \sum_{k=0}^{s-1} (r^{2k+1} - r^{2k+2}) - r^{2s}(r - r^2)$$

by (2.9) for s and (2.6). Hence

$$r_{s+1} > r - \sum_{k=0}^{s} (r^{2k+1} - r^{2k+2}) > r - \frac{r}{1+r} = \frac{r^2}{1+r} > 0.$$

Also

$$\begin{aligned} M(r_{s+1}, f) &> M(r_s, f)^{1+1/(4n)} && \text{(by (2.4))} \\ &= M(r_s, f) M(r_s, f)^{1/(4n)} \\ &> M(r_s, f)(1/r)^{2/\delta} && \text{(by (2.7))} \\ &> r^{-2s/\delta} r^{-2/\delta} M(r, f) && \text{(by (2.8))} \\ &= r^{-(2s+2)/\delta} M(r, f) \end{aligned}$$

which completes the proof of (2.8) and (2.9). However (2.9) implies

$$M\left(\frac{r^2}{1+r}, f\right) > M(r_s, f) > \left(\frac{1}{r}\right)^{2s} M(r, f)$$

for all s, which is impossible.

In other words, $c/r^c < M(r, f)^{1/2}$ cannot hold for all r small enough to satisfy

$$M\big(r - M(r, f)^{-\delta}, f\big) > M(r, f)^{1+1/(4n)}.$$

3. Functions of a non-Archimedean variable. In this section we prove the result for functions of a non-Archimedean variable. The domain of our functions is an algebraically closed field of characteristic 0 with a non-Archimedean valuation and complete with respect to that valuation. Functions analytic in a region are represented by power series or Laurent series that converge for all values of the variable in the region.

DEFINITION 3.1. Let $\mathscr{R}$ be a *differential ring* of functions, analytic and single valued, of one non-Archimedean variable x, except at $x_0 = 0, x_1, x_2, \ldots$ without limit points.

THEOREM 3.2. *If $\mathscr{R}$ is finite dimensional over $\mathscr{L}$, then $\mathscr{E}(\mathscr{R}) = \emptyset$.*

Once again, we can restrict our effort to consideration of a single function f.

LEMMA 3.3. *If $f \in \mathscr{R}$ is such that $\langle f \rangle$ is finite dimensional over $\mathscr{L} = C[D]$, then the singularity at $x_0 = 0$ is not an essential singularity.*

DEFINITION 3.4. Let $f(x) = \sum_{n=-\infty}^{\infty} c_n x^n \in \mathscr{R}$ be analytic in $0 < r \leqslant a$, then

$$M(r, f) = \sup\{|f(x)|:\ r \leqslant |x| \leqslant a\},$$

$$m(r, f) = \max\{|c_n| r^n:\ n = 0, \pm 1, \pm 2, \ldots\} = \max\{|c_n|/r^{|n|}:\ n = -1, -2, \ldots\}$$

for small r.

The *degree of* $m(r, f)$ is the integer n for which the value of $m(r, f)$ is taken on (written $\deg m(r, f)$).

The function $\mu(\varrho)$ where $\mu = \log M(r, f)$ and $\varrho = \log r$ is the *maximum modulus diagram* of f.

LEMMA 3.5. *For small r, we have*

$$M(r, f) = m(r, f).$$

The proof as well as other useful definitions is given in [1].

Thus the maximum modulus diagram of f is a convex polygonal curve with negative integral slopes for all small r, and

$$(3.1)\quad \left|\frac{d\mu}{d\varrho}\right| = \deg m\left(\frac{1}{r}, f\right).$$

LEMMA 3.6. *For small r and all $f \in \mathscr{R}$ analytic in $0 < r \leqslant a$,*

$$M(r, f') \leqslant r^{-1} M(r, f).$$

Proof. By Lemma 3.5 we have

$$M(r, f') = m(r, f') = \max\{|n||c_n| r^{n-1}:\ n = -1, -2, -3, \ldots\}$$

$$\leqslant r^{-1} m(r, f) = r^{-1} M(r, f).$$

Proof of Lemma 3.3. Assume that f is an element of $\mathcal{R}$ with $x_0 = 0$ an essential singularity of f. Since $\mathcal{R}$ is finite dimensional over $\mathcal{L}$, f must satisfy a differential equation of the form

$$L(f^n) = L_1 f + L_2(f^2) + \ldots + L_{n-1}(f^{n-1}) = g$$

where $L_1, L_2, \ldots, L_{n-1} \in \mathcal{L}$ and $L \in \mathcal{L}^*$.

Hence if

$$L = c_m(D-\alpha_1)\ldots(D-\alpha_m) = c_m(D-\alpha_1)^{m_1}\ldots(D-\alpha_k)^{m_k}$$

where $\alpha_1, \ldots, \alpha_k$ are distinct, we have

$$\begin{aligned} (3.2)\quad f^n &= L^{-1} g \\ &= P_1(x)e^{\alpha_1 x} + \ldots + P_k(x)e^{\alpha_k x} \\ &\quad + e^{\alpha_1 x}\int_x e^{(\alpha_2-\alpha_1)x_1}\int_{x_1} e^{(\alpha_3-\alpha_2)x_2}\ldots\int_{x_{m-1}} e^{-\alpha_m x_m}(1/c_m)g(x_m)\,dx_m\ldots dx_1 \end{aligned}$$

where $P_i(x)$ is a polynomial of degree at most $m_i - 1$.

Let $g(x) = \sum_{j=-\infty}^{\infty} a_j x^j$ and estimate $M(r, f^n)$ using (3.2) and the non-Archimedean property, $\max|a+b| \leqslant \max(|a|, |b|)$. We find that, for small r, the terms with negative exponents dominate and the polynomial and exponential terms are bounded so that for sufficiently small $|x| = r$ we have

$$(3.3)\qquad \begin{aligned} |c_m|\, m(r, f^n) &\leqslant \max_{j=m+1,m+2,\ldots} \frac{|a_{-j}|\, r^{-j+m}}{|(-j+1)\ldots(-j+m-1)|} \\ &\leqslant m(r, g)\, r^m (j_0-1)^m \end{aligned}$$

for a fixed integer j_0 depending on r. Since we are inverting the differential operator L_n applied to f^n, the integer $j = -j_0$ for which this maximum occurs is the same integer $-j_0$ for which the value $m(r, f)$ is taken on. That is, j_0 is $-\deg m(r, f^n)$. We are now able to rewrite (3.3) as

$$|c_m|\, M(r, f)^n = |c_m|\, m(r, f^n) \leqslant m(r, g)\, r^m (j_0-1)^m \leqslant M(r, g)\left(-\frac{d\mu}{d\varrho}\right)^m$$

if $r < 1$. Hence

$$(3.4)\qquad -\frac{d\mu}{d\varrho} \geqslant \left(\frac{|c_m|\, M(r, f)^n}{M(r, g)}\right)^{1/m}.$$

However, we can estimate $m(r, g)$ directly from the definition of g to get

$$(3.5)\qquad M(r, g) = C \max_{\substack{1 \leqslant i \leqslant n-1 \\ j \leqslant N}} M(r, D^j f^i)$$

where

$$N = \max_{1 \leqslant i \leqslant n-1} \deg L_i$$

and

$$C = \max_{1 \leqslant i \leqslant n-1} \text{ of } |\text{the coefficients of the } L_i|.$$

Thus by Lemma 3.6

$$M(r, g) \leqslant C r^{-N} M(r, f)^{n-1}$$

and putting this in (3.4) gives

$$-\frac{dM}{dr}\frac{r}{M} = -\frac{d\mu}{d\varrho} \geqslant C_1\left(\frac{M(r, f)^n r^N}{M(r, f)^{n-1}}\right)^{1/m} = C_1 M(r, f)^{1/m} r^{N/m}$$

so we obtain

$$\frac{dM}{M} \geqslant -C_1 M(r, f)^{1/m} r^{N/m}\frac{dr}{r}$$

if we assume r to be decreasing.

Since 0 is an essential singularity of f,

$$M(r, f)^\delta > 1/r$$

for any $\delta > 0$ if r is sufficiently small. Hence

$$-C_1\frac{dr}{r} \leqslant \frac{dM}{M^{1+1/m} r^{N/m}} < \frac{dM}{M^{1+1/m-N\delta/m}}$$

and if we choose δ so that $N\delta < 1$ we have, since M increases as r decreases,

$$\infty > \int_{M_0}^{\infty} \frac{dM}{M^{1+1/m-N\delta/m}} > \int_{r=r_0}^{r=0} -C_1\frac{dr}{r} = \infty,$$

which is absurd, and the proof of Theorem 3.2 is complete.

4. Concluding remarks. In this paper we have only considered the case for which the subring $\mathcal{R}_0$ is the ring of complex constants. It might be useful to consider other choices for $\mathcal{R}_0$ in both the complex and the non-Archimedean cases. In both theorems we have taken $\mathcal{R}$ to be finite dimensional over $\mathcal{L}$ as an hypothesis. It might also be interesting to look for conditions which would give $\mathcal{R}$ this property. Alternate choices for the ring $\mathcal{R}$ seem to be more limited but rings of functions of several variables could have theorems of a similar type.

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Editor's note. The proofsheets sent to the second author have not returned in time, thus the paper has been printed without author's correction. In Lemma 2.4 it is tacitly assumed that $M(r, f)^{-\delta} < r$.

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Nouvelles caractérisations des nombres de Pisot et de Salem

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1. Introduction, rappels. Soit S l'ensemble des nombres de Pisot, c'est-à-dire l'ensemble des entiers algébriques supérieurs à 1 dont tous les conjugués (autres que lui-même) ont un module strictement inférieur à 1 et soit T l'ensemble des nombres de Salem, c'est-à-dire l'ensemble des entiers algébriques supérieurs à 1 dont tous les conjugués (autres que lui-même) ont un module inférieur ou égal à 1, l'un au moins étant de module 1.

Si θ est un élément de S ou T, λ un entier algébrique de $Q(\theta)$, s désigne le degré de θ et l'on note:

$$\theta^{(i)}, \quad i = 2, \ldots, s,$$
$$\lambda^{(i)}, \quad i = 2, \ldots, s,$$

les conjugués respectifs de θ et λ (autres qu'eux-mêmes) alors le nombre

$$\lambda\theta^n + \sum_{i=2}^{s} \lambda^{(i)} \theta^{(i)n}$$

est un entier rationnel. Ainsi, si l'on note pour x réel, $\|x\|$ la distance de x à l'entier le plus voisin, on a, pour $\theta \in S$, à partir d'un certain rang:

$$\|\lambda\theta^n\| = \Big|\sum_{i=2}^{s} \lambda^{(i)} \theta^{(i)n}\Big|$$

et la suite $(\|\lambda\theta^n\|)$ tend vers zéro comme une progression géométrique. Si λ est un élément quelconque de $Q(\theta)$, alors il existe $l \in N$ tel que $l\lambda$ soit entier algébrique, et la suite $(\|\lambda\theta^n\|)$ a, au maximum, l valeurs d'adhérence toutes rationnelles (les suites extraites convergent vers ces valeurs d'adhérence comme des progressions géométriques).

Réciproquement, Pisot a montré [4] que, pour un réel $\theta > 1$, l'existence d'un réel λ non nul tel que soit réalisée l'une ou l'autre des conditions:

$$(1.1) \qquad \sum_{n=0}^{\infty} \|\lambda\theta^n\|^2 < +\infty$$