Thus the lower density $d = \liminf_{s\to\infty} \frac{n(S)a(C)}{a(S)}$ of the covering satisfies the inequality

$$d \geqslant \frac{a(C)}{\hat{h}(C)}.$$

This completes the proof of our theorem. It should be mentioned that one can further sharpen the bound $\vartheta(C) \geqslant a(C)/\hat{h}(C)$ by considering e.g. for each triangle of the triangulation the sum of the areas of those triangles which are adjacent to the sides of the respective triangle. However, it does not seem to be possible to prove the inequality $\vartheta(C) \geqslant a(C)/h^*(C)$ in this way.

References

[1] R. P. Bambah and C. A. Rogers, *Covering the plane with convex sets*, Journal London Math. Soc. 27 (1952), pp. 304–314.

[2] R. P. Bambah, C. A. Rogers and H. Zassenhaus, *On coverings with convex domains*, Acta Arith. 9 (1964), pp. 191–207.

[3] C. H. Dowker, *On minimum circumscribed polygons*, Bull. Amer. Math. Soc. 50 (1944), pp. 120–122.

[4] L. Fejes Tóth, *Some packing and covering theorems*, Acta Sci. Math. Szeged 12/A (1950), pp. 62–67.

[5] – *Regular figures*, Pergamon Press, Oxford–London–New York–Paris 1964.

[6] W. Kuperberg, *Packing convex bodies in the plane with density greater than 3/4*, Geometriae Dedicata 13 (1982), pp. 149–155.

[7] C. A. Rogers, *The closest packing of convex two-dimensional domains*, Acta Math. 86 (1951), pp. 309–321.

Received on 18.7.1985
and in revised form on 29.4.1986 (1529)

ACTA ARITHMETICA
L (1988)

Some inequalities for the sum of two sets of lattice points

by

Robert D. Stalley (Corvallis, Or.)

1. Introduction. Let J be the set of all nonnegative integers and let Δ be any nonempty set. With $x = \{(\delta, x_\delta)|\ \delta \in \Delta\}$ let

$$J^\Delta = \{x|\ x_\delta \in J \text{ for all } (\delta, x_\delta) \in x\}.$$

If $x \in J^\Delta$ and $x_\delta = 0$ for all $\delta \in \Delta$, we write $x = 0$. For $x, y \in J^\Delta$ let

$$x + y = \{(\delta, x_\delta + y_\delta)|\ \delta \in \Delta\}.$$

Assume that $A, B \subset J^\Delta$, and that $0 \in A$, $A \neq J^\Delta$, and $B \neq \emptyset$, and let $C = A + B = \{a+b|\ a \in A,\ b \in B\}$. We obtain information about how sparse C is. Before describing our results more precisely, we give some more definitions.

If $x, y \in J^\Delta$ then $y - x = \{(\delta, y_\delta - x_\delta)|\ \delta \in \Delta\}$. We write $x \preceq y$ if $y - x \in J^\Delta$, and $x \prec y$ if also $x \neq y$. The family $\mathscr{G}$ consists of all finite nonempty sets $G \subset J^\Delta$ such that $\sum_{x_\delta > 0} 1 < \infty$ if $x \in G$ and also $x \in G$ if $g_1 \prec x \prec g_2$ with $g_1, g_2 \in G$. If $S \cup T \subset J^\Delta$ and T is finite, then $S(T)$ denotes the cardinality of $S \cap T$. We obtain inequalities which give lower bounds for $C(G)$ where $G \in \mathscr{G}$. The main result is Theorem 3 but attention is also called to Theorem 7 which is a companion to a theorem of Kvarda [5].

The family $\mathscr{F}$ is defined as $\{F|\ 0 \in F \in \mathscr{G}\}$. The *Erdős density* α_1 of A is defined by

$$\alpha_1 = \operatorname{glb}\left\{\frac{A(F)-1}{J^\Delta(F)}\ \middle|\ F \in \mathscr{F},\ A(F) < J^\Delta(F)\right\}.$$

This density is first used in Lemma 2. Later we obtain lower bounds for certain "densities" of C one of which is a generalization of this density.

Several more definitions are needed. If $S \subset J^{\Delta}$ then

$$\min S = \{x|\ x \in S,\ \text{if } y \prec x \text{ then } y \notin S\}$$

and

$$\max S = \{x|\ x \in S,\ \text{if } x \prec y \text{ then } y \notin S\}.$$

Also

$$L(x) = \{y|\ 0 \preceq y \preceq x\}, \quad H(x) = \{y|\ x \preceq y\}, \quad \text{and} \quad Q = \bigcup_{x \in B} H(x).$$

If $x \in J^{\Delta}$, $\lambda \in \Delta$, $x_\delta = 0$ for all $\delta \in \Delta \setminus \{\lambda\}$, and $x_\lambda = 1$, then we write $x = \omega_\lambda$. If Δ is finite we sometimes use the special notation $\Delta = \{1, 2, \ldots, n\}$, $J^{\Delta} = I$, and for $x \in J^{\Delta}$,

$$x = \{(\delta, x_\delta)|\ 1 \leqslant \delta \leqslant n\} = (x_1, x_2, \ldots, x_n).$$

Our definition of $S(T)$ differs slightly from that in the literature in that we count the element 0. If Δ is finite and $F \in \mathscr{F}$, then $F \setminus \{0\}$ is called a *fundamental set* in the literature. The density α_1 was first introduced by Erdős [1] for the case in which Δ is a singleton. The construction in the proof of Lemma 2 is that of Kvarda [5].

2. Two lemmas. We prove two lemmas which are needed in the proof of Theorem 3.

LEMMA 1. *A nonempty subset G of a member of $\mathscr{G}$ is also a member of $\mathscr{G}$ if $x \in G$ whenever $g_1 \prec x \prec g_2$ and $g_1, g_2 \in G$.*

Proof. This lemma follows immediately from the definition of $\mathscr{G}$.

Note that $B \subset C \subset Q$ and that in the next four results we also have $G \subset Q$.

LEMMA 2. *If $G \in \mathscr{G}$, $B \cap G \neq \emptyset$, $G \setminus C \neq \emptyset$, and $x \prec y$ for each $x \in B \cap G$ and $y \in G \setminus C$, then*

$$C(G) \geqslant \alpha_1 Q(G) + B(G).$$

Proof. The set G is finite, and if $x \in G$ then $\sum_{x_\delta > 0} 1 < \infty$. Let Δ^* be the finite set of all $\delta \in \Delta$ such that $x_\delta > 0$ for some $x \in G$ and let G^* be the set of elements x in J^{Δ^*} such that $x_\delta = y_\delta$ for all $\delta \in \Delta^*$ where $y \in G$. We redefine G as this set G^* and use the notation introduced earlier for finite dimensional spaces. Now we introduce a lexicographical ordering on $I = J^{\Delta} = J^{\Delta^*}$. For $x, y \in I$ we have $x < y$ if $x_i = y_i$ for $1 \leqslant i < r$ and $x_r < y_r$ for some r where $1 \leqslant r \leqslant n$. Let $e = (e_1, e_2, \ldots, e_n)$ be the lexicographically largest vector in $B \cap G$ such that

$$\sum_{i=1}^{n} e_i = \max\{\sum_{i=1}^{n} x_i|\ x \in B \cap G\}.$$

Let $d = (d_1, d_2, \ldots, d_n)$ be the lexicographically largest vector in $G \setminus C$ such that

$$\sum_{i=1}^{n} d_i = \max\{\sum_{i=1}^{n} y_i|\ y \in G \setminus C\}.$$

Let

$$D = \{d - x|\ x \in B \cap G\} \quad \text{and} \quad E = \{y - e|\ y \in G \setminus C\}.$$

Finally, let

$$\min G = \{\delta_j|\ 1 \leqslant j \leqslant u\}, \quad G_j = \{g|\ g \in G,\ \delta_j \preceq g\},$$

$$G'_j = \{g - \delta_j|\ g \in G_j\}, \quad \text{and} \quad G' = \bigcup_{1 \leqslant j \leqslant u} G'_j.$$

First we prove that

$$D \cup E \subset G' \setminus A. \tag{1}$$

By definition, $D \cap A = \emptyset$ and $E \cap A = \emptyset$. Suppose $d - x \in D$. Then $x \in B \cap G$ and so $x \in G_j$ for some j. Hence, since $d \in G \setminus C$, then $\delta_j \preceq x \prec d$, and so $d \in G_j$. Therefore, $d - \delta_j \in G'_j \in \mathscr{F}$ and $0 \prec d - x \preceq d - \delta_j$, and so $d - x \in G'_j \subset G'$. Thus $D \subset G'$. Similarly, $E \subset G'$.

Next, we prove that

$$D \cap E = \{d - e\}. \tag{2}$$

We have $d - e \in D \cap E$. Assume $v \in D \cap E$. Then $d - x = v = y - e$ where $x \in B \cap G$ and $y \in G \setminus C$. Hence $x + y = e + d$ and so $x_i + y_i = e_i + d_i$, $1 \leqslant i \leqslant n$. Summation over i yields

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} e_i + \sum_{i=1}^{n} d_i.$$

Hence, by the definitions of e and d, we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i \quad \text{and} \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} d_i,$$

and so also $x \leqslant e$ and $y \leqslant d$. Hence

$$x_1 + y_1 \leqslant e_1 + y_1 \leqslant e_1 + d_1 = x_1 + y_1,$$

and so $x_1 = e_1$ and $y_1 = d_1$. We may now proceed similarly to obtain $x_2 = e_2$ and $y_2 = d_2$. Continuing in this manner, we obtain $x = e$ and $y = d$, and so $v = d - e$.

Now, by (1) and (2), we have

$$I(G' \setminus A) \geqslant I(D) + I(E) - 1. \tag{3}$$

By definition,

$$G' = \bigcup_{1 \leqslant j \leqslant u} G'_j \in \mathscr{F}$$

By (3), we have $I(G' \setminus A) > 0$, and so

$$A(G') = I(G') - I(G' \setminus A) < I(G').$$

Hence

(4) $$A(G') - 1 \geqslant \alpha_1 I(G').$$

Suppose $G \notin \mathscr{F}$. Kvarda [5] states and proves "Lemma 1. $Q(S') + 1 \leqslant Q(S)$." In our notation $Q = I \setminus \{0\}$, $S = G$, and $S' = G' \setminus \{0\}$. Thus $Q(S) = I(G)$, $Q(S') = I(G') - 1$, and so

(5) $$I(G') \leqslant I(G).$$

If $G \in \mathscr{F}$, then $G' = G$ and again (5) holds. Hence, by (3), (4), and (5), we have

$$\begin{aligned} C(G) &= I(G) - I(G \setminus C) = I(G) - I(E) \\ &\geqslant I(G) - I(G' \setminus A) + I(D) - 1 \\ &= I(G) - [I(G') - A(G')] + B(G) - 1 \\ &= A(G') - 1 + [I(G) - I(G')] + B(G) \\ &\geqslant \alpha_1 I(G') + \alpha_1 [I(G) - I(G')] + B(G) \\ &= \alpha_1 I(G) + B(G) = \alpha_1 Q(G) + B(G). \end{aligned}$$

3. Fundamental theorems. The main theorem of this paper follows.

THEOREM 3. *Let $G \in \mathscr{G}$. If for each $x \in B \cap G$ there exist $y \in G \setminus C$ and for each $y \in G \setminus C$ there exist $x \in B \cap G$ such that $x \prec y$, then*

(6) $$C(G) \geqslant \alpha_1 Q(G) + B(G).$$

Proof. We use induction on $B(G)$. If $B(G) = 0$ then $G \setminus C = \emptyset$, and so $C(G) = Q(G) \geqslant \alpha_1 Q(G) + B(G)$. Hence, let $B(G) = k \geqslant 1$ and assume the theorem valid for all $G_* \in \mathscr{G}$ such that $B(G_*) < k$. Since $B \cap G \neq \emptyset$, then $G \setminus C \neq \emptyset$. Thus if $G \setminus C \subset \bigcap_{x \in B \cap G} H(x)$, then Lemma 2 gives inequality (6). The case remains where $g \notin H(b)$ for some $g \in G \setminus C$ and some $b \in B \cap G$. We have $C(G \setminus H(b)) < Q(G \setminus H(b))$ and

$$C\big(G \setminus \bigcup_{x \in B \cap G} H(x)\big) = Q\big(G \setminus \bigcup_{x \in B \cap G} H(x)\big).$$

Let P be maximal where $b \in P$, $P \subset B \cap G$, and

$$C\big(G \setminus \bigcup_{x \in P} H(x)\big) < Q\big(G \setminus \bigcup_{x \in P} H(x)\big).$$

Let $H = \bigcup_{x \in P} H(x)$ and $W = H \cap G$. First we prove that inequality (6) holds for W and then that it holds for $G \setminus W$.

We have $W \in \mathscr{G}$ by Lemma 1. Now $P \subset (B \cap G) \cap H = B \cap W$. Suppose $x \in B \cap W$. Then $x \in H$ and so $H \cup H(x) = H$. Thus, since $x \in B \cap G$, we have $x \in P$ by the maximality of P. Hence

(7) $$P = B \cap W.$$

If $x \in B \cap W$ then $x \in B \cap G$, and so there exist $y \in G \setminus C$ where $x \prec y$. Since $x \in H$, then $y \in H$, and so $y \in W \setminus C$. If $y \in W \setminus C$ then $y \in H$, and so $x \prec y$ for some $x \in P = B \cap W$. By (7) we have $B(W) = Q(P) < B(G) = k$, and so

(8) $$C(W) \geqslant \alpha_1 Q(W) + B(W).$$

Next we prove that inequality (6) holds for $G \setminus W$ by using Lemma 2. Since $P \neq B \cap G$, then $W \neq G$ by (7). Hence, since $W \subset G$, we have $G \setminus W \neq \emptyset$. Assume $g_1, g_2 \in G \setminus W$ and $g_1 \prec x \prec g_2$. If $x \in W$, then since $W \subset H$, we have $g_2 \in H \cap G = W$, a contradiction. Hence $x \in G \setminus W$ and so $G \setminus W \in \mathscr{G}$ by Lemma 1. Next, by (7), we have

$$B \cap (G \setminus W) = (B \cap G) \setminus (B \cap W) = (B \cap G) \setminus P \neq \emptyset.$$

Also, since $C(G \setminus H) < Q(C \setminus H)$ and $G \setminus W = G \setminus H$, then $(G \setminus W) \setminus C \neq \emptyset$. Finally, if $x \in B \cap (G \setminus W)$ then $x \in (B \cap G) \setminus P$, and so by the maximality of P we have

(9) $$C\big(G \setminus (H \cup H(x))\big) = Q\big(G \setminus (H \cup H(x))\big).$$

If $y \in (G \setminus W) \setminus C$ then $y \in (G \setminus H) \setminus C = (G \setminus C) \setminus H$. Since $y \in G \setminus C$, then by (9) we have $y \in H \cup H(x)$. Hence, since $y \notin H$, then $y \in H(x)$, and so $x \prec y$. Thus by Lemma 2 we have

(10) $$C(G \setminus W) \geqslant \alpha_1 Q(G \setminus W) + B(G \setminus W).$$

Since $W \subset G$, we obtain inequality (6) by adding inequalities (8) and (10).

Now we prove four related theorems.

THEOREM 4. *If $G \in \mathscr{G}$, then any of the following sets of conditions are sufficient for inequality* (6) *to hold:*

(i) $\min G \subset B$ *and for each $x \in B \cap G$ there exist $y \in G \setminus C$ such that $x \prec y$;*

(ii) $\max G \in G \setminus C$ *and for each $y \in G \setminus C$ there exist $x \in B \cap G$ such that $x \prec y$;*

(iii) $\min G \subset B$ *and* $\max G \subset G \setminus C$;

(iv) *There exists $G_1 \in \mathscr{G}$ such that $G_1 \subset G$, $\min G_1 \subset B$, $\max G_1 \subset G_1 \setminus C$, and $G \setminus G_1 \subset C \setminus B$;*

(v) *There exists* $G_1 \in \mathcal{G}$ *such that* $G \subset G_1$, $\min G_1 \subset B$, $\max G_1 \subset G_1 \setminus C$, *and* $G_1 \setminus G \subset B$.

Proof. Suppose conditions (i) hold and $y \in G \setminus C$. Then $x \preceq y$ for some $x \in \min G$. Hence $x \in B$ and so $x \prec y$. Thus inequality (6) follows from Theorem 3. Conditions (ii) and (iii) may be shown to be sufficient in a similar way. Suppose conditions (iv) hold. Then $C(G_1) \geqslant \alpha_1 Q(G_1) + B(G_1)$ because conditions (iii) hold for G_1. Since $G \setminus G_1 \subset C \setminus B$, then $C(G \setminus G_1) = Q(G \setminus G_1)$ and $B(G \setminus G_1) = 0$, and so $C(G \setminus G_1) \geqslant \alpha_1 Q(G \setminus G_1) + B(G \setminus G_1)$. Thus since $G_1 \subset G$ then we obtain inequality (6) by adding the above two inequalities. Conditions (v) may be shown to be sufficient in a similar way.

THEOREM 5. *Conditions* (iv) *of Theorem* 4 *are equivalent to the hypotheses of Theorem* 3 *if* $G \setminus C \neq \emptyset$ (*or* $B \cap G \neq \emptyset$).

Proof. Suppose conditions (iv) of Theorem 4 hold. If $x \in B \cap G$ then $x \in G_1$, and so $x \preceq y$ for some $y \in \max G_1 \subset G_1 \setminus C \subset G \setminus C$. Hence $x \prec y$. If $y \in G \setminus C$ then $y \in G_1$, and so $x \preceq y$ for some $x \in \min G_1 \subset B \cap G_1 \subset B \cap G$. Again $x \prec y$. Now suppose the hypotheses of Theorem 3 hold and $G \setminus C \neq \emptyset$. Note that $B \cap G \neq \emptyset$ and let

$$G_1 = \Big(\bigcup_{y \in G \setminus C} L(y)\Big) \cap \Big(\bigcup_{x \in B \cap G} H(x)\Big).$$

Then $G_1 \neq \emptyset$ and $G_1 \subset G$, and so by Lemma 1 we have $G_1 \in \mathcal{G}$. It follows from the definition of G_1 that $\min G_1 \subset B$ and $\max G_1 \subset G \setminus C$. If either $x \in B \cap G$ or $x \in G \setminus C$ then $x \in G_1$. Hence $G \setminus G_1 \subset C \setminus B$.

THEOREM 6. *If* $F \in \mathcal{F}$ *and* $\max F \subset F \setminus C$, *then*

$$C(F) \geqslant \alpha_1 Q(F) + B(F).$$

Proof. Since $Q(F \setminus Q) = 0$, then $C(F \setminus Q) = B(F \setminus Q) = 0$, and so

$$C(F \setminus Q) = \alpha_1 Q(F \setminus Q) + B(F \setminus Q). \tag{11}$$

Thus if $F \cap Q = \emptyset$ then $F \setminus Q = F$, and the theorem follows. Hence suppose $G = F \cap Q \neq \emptyset$. Then $G \in \mathcal{G}$ by Lemma 1 since $G \subset F$. Next $\min G \subset \min Q \subset B$. Also $\max G \subset \max F \subset F \setminus C$, and so

$$\max G \subset G \cap (F \setminus C) = (G \cap F) \setminus C = G \setminus C.$$

Thus by Theorem 4 (iii) we have inequality (6). Since $F \setminus Q$ and G form a partition of F, where we note that $F \setminus Q$ may be empty, then addition of inequalities (6) and (11) gives the theorem.

Results similar to Theorem 6 may be obtained in a similar way from Theorem 4 using conditions (i), (iv), or (v).

Theorems 3, 4, and 6 are particularly interesting when $\min B = \{0\}$ or $\min B = \{\omega_\lambda | \ \lambda \in \Delta\}$. In the first case $Q = J^\Delta$. In the second case $Q = J^\Delta \setminus \{0\}$, and so $Q(F) = J^\Delta(F) - 1$ for $F \in \mathcal{F}$ and $Q(G) = J^\Delta(G)$ for $G \in \mathcal{G} \setminus \mathcal{F}$. For example the following theorem follows from Theorem 6.

THEOREM 7. *If* $\Delta = \{1, 2, \ldots, n\}$, $\min B = \{\omega_1, \omega_2, \ldots, \omega_n\}$, $F \in \mathcal{F}$, *and* $\max F \subset F \setminus C$, *then* $C(F) \geqslant \alpha_1 [I(F) - 1] + B(F)$.

If $\Delta = \{1\}$ and $0 \in B$, then Theorem 7 reduces to a theorem of Mann [6]. If $n = 1$, then Theorem 7 reduces to a theorem of Erdős [1]. If $n = 2$, then Theorem 7 reduces to a theorem of Morgali [7]. If n is finite, $0 \in B$, and $G \in \mathcal{F}$, then Theorem 4 (i) reduces to a theorem of Kvarda [5].

4. Density theorems. We prove two density theorems. Suppose $D \subset J^\Delta$, $D \neq \emptyset$, and $R = \bigcup_{x \in D} H(x)$. Then the density of D is defined by

$$d(D) = \operatorname{glb}\left\{\frac{D(F)}{R(F)} \,\middle|\, F \in \mathcal{F},\ \max F \subset R\right\}.$$

Let $d(B) = \beta$ and $d(C) = \gamma$. Note that for the sets B and C we have $R = Q$. For Δ finite $d(D)$ is a density of the author [8] and Freedman [3] if $0 \in D$ and a density of Kvarda [4] if $\min D = \{\omega_\lambda | \ \lambda \in \Delta\}$. (Kvarda's density is Schnirelmann's density if $n = 1$.) Furthermore, $d(D) = 1$ if $D = R$, but $d(D) = 1$ implies $D = R$ only when Δ is finite. The next lemma is needed in the proof of Theorem 9.

LEMMA 8. *If* $\gamma < 1$, *then*

$$\gamma = \operatorname{glb}\left\{\frac{C(F)}{Q(F)} \,\middle|\, F \in \mathcal{F},\quad \max F \subset Q \setminus C\right\}.$$

Proof. Let γ' be the greatest lower bound of the lemma. Since $\gamma \leqslant \gamma'$ is immediate, it remains to prove that $\gamma \geqslant \gamma'$. Suppose $F \in \mathcal{F}$, $\max F \subset Q$, and $C(F) < Q(F)$, and let $M = \bigcup_{x \in F \setminus C} L(x)$. Then since $M \subset F$ and $F \setminus M \subset C$, we have

$$\frac{C(F)}{Q(F)} = \frac{C(M) + C(F \setminus M)}{Q(M) + Q(F \setminus M)} = \frac{C(M) + Q(F \setminus M)}{Q(M) + Q(F \setminus M)} \geqslant \frac{C(M)}{Q(M)}.$$

Hence, since $M \in \mathcal{F}$ and $\max M \subset Q \setminus C$, we have $C(M)/Q(M) \geqslant \gamma'$, and so

$$\gamma = \operatorname{glb}\left\{\frac{C(F)}{Q(F)} \,\middle|\, F \in \mathcal{F},\ \max F \subset Q,\ C(F) < Q(F)\right\} \geqslant \gamma'.$$

THEOREM 9. *If* $\gamma < 1$, *then* $\gamma \geqslant \alpha_1 + \beta$.

Proof. Suppose $F \in \mathcal{F}$ and $\max F \subset Q \setminus C$. Then $C(F) \geqslant \alpha_1 Q(F) + B(F)$ by Theorem 6, and so

$$\frac{C(F)}{Q(F)} \geqslant \alpha_1 + \frac{B(F)}{Q(F)} \geqslant \alpha_1 + \beta$$

since $\max F \subset Q$. The desired inequality now follows from Lemma 8.

Suppose $D \subset J^\Delta$, $D \neq \emptyset$, $D \neq J^\Delta$, and R is defined as before. Then the

Erdős density of D is defined by

$$d_1(D) = \text{glb}\left\{\frac{D(F)-1}{R(F)}\;\middle|\; F\in\mathscr{F},\ \max F\subset R,\ D(F) < R(F)\right\}.$$

Let $d_1(B) = \beta_1$ and $d_1(C) = \gamma_1$. We see that $d_1(A)$ is the Erdős density α_1 of A as defined earlier since for A we have $R = J^{\Delta}$. The following lemma and theorem are proved in the same way that Lemma 8 and Theorem 9 are proved. In the proof of Theorem 11 we need to note also that $B(F) < Q(F)$, which follows from $\max F \subset Q\setminus C \subset Q\setminus B$.

LEMMA 10. *If γ_1 is defined, then*

$$\gamma_1 = \text{glb}\left\{\frac{C(F)-1}{Q(F)}\;\middle|\; F\in\mathscr{F},\ \max F\subset Q\setminus C\right\}.$$

THEOREM 11. *If γ_1 is defined, then $\gamma_1 \geqslant \alpha_1+\beta_1$.*

If $\Delta = \{1\}$ then Theorem 9 reduces to a theorem of the author [8] when $0\in B$, and to a theorem of Erdős [1] when $\min B = \{1\}$. If Δ is finite and $0\in B$, then Theorems 9 and 11 reduce to theorems of Freedman [2], [3]. For the case $\Delta = \{1\}$ and $0\in B$ Theorem 11 is fairly well known. It follows from Theorem 9 that if $\alpha_1+\beta \geqslant 1$ then $\gamma = 1$.

5. Remarks. The density α_1 may be improved in all our results of Sections 2 and 3 by replacing the space J^{Δ} in its definition by the space J^{Δ^*} where as before Δ^* is the set of all $\delta\in\Delta$ such that $x_\delta > 0$ for some $x\in G$.

Nontrivial examples may be easily constructed where the case of equality holds in our theorems.

By an argument using translations we see that our results may be extended to subsets A and B of a translate J_t^{Δ} of J^{Δ} where for each $\delta\in\Delta$ the integers x_δ for all $\{(\delta, x_\delta)|\ \delta\in\Delta\}\in A\cup B$ are bounded below, "$\min A$" is a singleton, there exist $x\in J_t^{\Delta}\setminus A$ such that "$\min A \prec x$", and as before $B\neq\emptyset$. Our results may be further extended in a similar manner by an argument using reflections about the coordinate planes $x_\delta = 0$ where $\delta\in\Delta$.

References

[1] P. Erdős, *On the asymptotic density of the sum of two sequences*, Ann. of Math. 43 (1942), pp. 65–68.

[2] Allen R. Freedman, *A general theory of density in additive number theory*, Ph.D. thesis, Oregon State University, 1965.

[3] — *On the generalization of the modified Schnirelmann density*, J. Number Theory 2 (1970), pp. 97–105.

[4] Betty Kvarda, *On densities of sets of lattice points*, Pacific J. Math. 13 (1963), pp. 611–615.

[5] — *An inequality for the number of elements in a sum of two sets of lattice points*, ibid. 15 (1965), pp. 545–550.

[6] Henry B. Mann, *On the number of integers in the sum of two sets of positive integers*, ibid. 1 (1951), pp. 249–253.

[7] Ronald R. Morgali, *Gaussian integer analogues of two theorems of Mann for positive integers*, M.S. thesis, Oregon State University, 1965.

[8] Robert D. Stalley, *A modified Schnirelmann density*, Pacific J. Math. 5 (1955), pp. 119–124.

DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
Corvallis, Or. 97331
U.S.A.

Received on 26.8.1985
and in revised form on 9.7.1986 (1535)