References

[1] R. Balasubramanian, *An improvement on a theorem of Titchmarsh on the mean square of* $|\zeta(1/2+it)|$, Proc. London Math. Soc. 36 (1978), pp. 540–576.

[2] R. Balasubramanian and K. Ramachandra, *On square full integers* (to appear).

[3] — — *Effective and noneffective results on certain arithmetical functions*, J. Number Theory 12 (1980), pp. 10–19.

[4] — — *Some problems of analytic number theory II*, Studia Sci. Math. Hungarica 14 (1979), pp. 193–202.

[5] — — *Some problems of analytic number theory III*, Hardy Ramanujan J. 4 (1981), pp. 13–40.

[6] — — *Progress towards a conjecture on the mean value of Titchmarsh series, III*, Acta Arith. 45 (1986), pp. 309–318.

[7] P. T. Bateman and E. Grosswald, *On a theorem of Erdős and Szekeres*, Illinois J. Math. 2 (1958), pp. 88–98.

[8] P. Erdős and G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Sci. Math. (Szeged) 7 (1935), pp. 95–102.

[9] A. Ivić, *On the asymptotic formulas for powerful numbers*, Publ. Inst. Math. (Belgrade) 23 (37) (1978), pp. 85–94.

[10] — *On the number of finite non-isomorphic abelian groups in short intervals*, Math. Nachr. 101 (1981), pp. 257–271.

[11] A. Ivić and P. Shiu, *The distribution of powerful integers*, Illinois J. Math. 26 (1982), pp. 576–590.

[12] E. Krätzel, *Zahlen k-ter Art*, Amer. J. Math. 94 (1972), pp. 309–328.

[13] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. 8 (1974), pp. 73–82.

[14] K. Ramachandra, *Some remarks on a theorem of Montgomery and Vaughan*, J. Number Theory 11 (1979), pp. 465–471.

[15] — *Progress towards a conjecture on the mean value of Titchmarsh series*, in: *Recent progress in Analytic Number Theory*, Vol. 1, Academic Press 1981, pp. 303–318.

[16] D. Suryanarayana and R. Sita Rama Chandra Rao, *The distribution of squarefull integers*, Arkiv for Math. 11 (1973), pp. 195–201.

[17] E. C. Titchmarsh, *The theory of Riemann zeta function*, Clarendon Press, Oxford 1951.

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Note to a paper of Bambah, Rogers and Zassenhaus

by

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It is known [7] that the density of a packing of translates of a convex domain C cannot exceed the density of the densest lattice packing of C. It is conjectured ([5], p. 205) that an analogous statement holds for coverings: The density of a covering of the plane with translates of a convex domain C cannot be less than the density of the thinnest lattice covering with C.

For a closed convex domain C let $a(C)$ denote the area of C, $\vartheta(C)$ the infimum of the lower densities of all coverings of the plane by translates of C and $h(C)$ the maximum area of a hexagon inscribed in C. According to a general result of L. Fejes Tóth [4] (see also [1]) we have

$$\vartheta(C) \geqslant \frac{a(C)}{h(C)}.$$

This proves the truth of the above conjecture for centrally symmetric domains. For, if C is centrally symmetric then, by a theorem of Dowker [3], there is a centrally symmetric hexagon of area $h(C)$ inscribed in C. There is a lattice tiling of the plane by translates of this hexagon, and the corresponding translates of C provide a lattice covering with C with density $\vartheta(C) = a(C)/h(C)$.

The proof of the inequality $\vartheta(C) \geqslant a(C)/h(C)$ is based on a construction which associates with each domain from the covering a convex polygon inscribed in the respective domain such that these polygons form a tiling. Carrying out this construction for a lattice covering with C we obtain congruent centro-symmetric hexagons providing a lattice tiling of the plane. It immediately follows that the density of the thinnest lattice covering with C is equal to $a(C)/h^*(C)$, where $h^*(C)$ denotes the supremum of the areas of all centrally symmetric hexagons contained in C. Thus the conjecture above can be reformulated as follows:

Conjecture. *For any convex domain C in the plane we have*

$$\vartheta(C) = \frac{a(C)}{h^*(C)}.$$

We recall another lower bound for $\vartheta(C)$ due to Bambah, Rogers and Zassenhaus [2]. They proved that

$$\vartheta(C) \geqslant \frac{a(C)}{2t(C)},$$

where $t(C)$ denotes the maximum area of a triangle inscribed in C. If C is centrally symmetric then we have $h(C) = h^*(C) = 2t(C)$, so that this bound is also exact. However it was pointed out in [2] that if C lacks a centre of symmetry then neither of the two bounds is best possible, and sometimes one and sometimes the other is stronger. In what follows we shall show that by a slight modification in the proof of Bambah, Rogers and Zassenhaus one obtains a lower bound for $\vartheta(C)$ which is never weaker than the bound $a(C)/h(C)$.

We recall from [6] the definition of a p-hexagon: A *p-hexagon* is a hexagon with a pair of parallel opposite sides of equal length. Opposite sides are which are separated by exactly two other sides in each direction. Pentagons with a pair of parallel sides, quadrilaterals and triangles will be considered to be (degenerate) p-hexagons. We shall prove the following

THEOREM. *Let $\hat{h}(C)$ be the supremum of the areas of all p-hexagons contained in C. Then we have*

$$\vartheta(C) \geqslant \frac{a(C)}{\hat{h}(C)}.$$

The fact that this bound is not weaker than the bound $a(C)/h(C)$ is obvious and it will be clear from the proof that it is a sharpening also of the bound $a(C)/2t(C)$. The main tool to the proof is a construction due to Bambah, Rogers and Zassenhaus which associates with a covering by translates of C a triangulation of the plane the special properties of which are summarized in the following

LEMMA. *Let C be a strictly convex domain and $\mathscr{A}$ a discrete set of points such that the domains $\{C+A\}_{A\in\mathscr{A}}$ cover the plane. Then there exists a triangulation of the plane with vertices at the points of $\mathscr{A}$ such that each triangle can be covered by a translate of $-C$.*

Theorem 4 in [2] has a weaker statement, namely it says only that the triangles have areas not exceeding $t(C)$, but actually the lemma above is proved.

Clearly, it suffices to prove the theorem for strictly convex domains. The general case is settled by approximating C by strictly convex domains containing C. Further we shall suppose that the origin is contained in C. Let $\mathscr{A}$ be a discrete set such that the domains $\{C+A\}_{A\in\mathscr{A}}$ cover the plane, and consider the triangulation $\mathscr{T}$ described in the lemma. We observe that the total area of a pair of triangles from $\mathscr{T}$ sharing an edge is at most $\hat{h}(C)$. For,

let Δ_1 and Δ_2 be triangles from $\mathscr{T}$ with a common edge, and let $\Delta_1' = UVW$ and $\Delta_2' = XYZ$ be translates of Δ_1 and Δ_2, respectively, contained in $-C$ such that the oriented segments UV and XY are parallel and have equal length. Let t be the total area of Δ_1 and Δ_2. The inequality $t \leqslant \hat{h}(C)$ will be shown by constructing a p-hexagon contained in $-C$ whose area is at least t. Let P be the parallel strip bounded by the lines UV and XY. If one of the points W and Z, say W, is contained in P, then the convex hull of $\{U, V, X, Y, Z\}$ is a (degenerate) p-hexagon with the required property. If, on the other hand, both W and Z lie outside of P, then a p-hexagon with the required property is either $UXZYVW$ or $UXWYVZ$, according as Δ_1' and Δ_2' are separated by P, or not.

Let S be a square of side-length s, $\mathscr{A}(S)$ the set of those points A from $\mathscr{A}$ for which $C+A$ is contained in S, and $n(S)$ the cardinality of $\mathscr{A}(S)$. Let C be contained in a square of side-length s' homothetic to S. We suppose that $s > 4s'$. Let S^* and S^{**} be the squares concentric with and homothetic to S with side-lengths $s-2s'$ and $s-4s'$, respectively. From the faces of $\mathscr{T}$ we consider those triangles which are contained in S^*. These triangles are faces of a cell complex $\mathscr{C}$ with f faces, e edges and v vertices.

Let P be an arbitrary point from S^{**}, and let Δ be a triangle from $\mathscr{T}$ containing P. The triangle Δ is contained in a translate of $-C$, which, in turn, is contained in a square of side-length s' concentric with and homothetic to S. It follows that Δ is contained in S^*. Therefore the faces of $\mathscr{C}$ cover S^{**}. Using the assumption that the origin is contained in C, we see in a similar way that the vertices of $\mathscr{C}$ belong to $\mathscr{A}(S)$. Thus we have

$$v \leqslant n(S).$$

Obviously, any connected component of the cell complex $\mathscr{C}$ is simply connected, so that

$$v-e+f \geqslant 1.$$

Since each face of $\mathscr{C}$ is a triangle and each edge in $\mathscr{C}$ belongs to at most two faces, we have

$$3f \leqslant 2e.$$

The last three inequalities imply that

$$e < 3n(S).$$

Adding up the total areas of all pairs of triangles of $\mathscr{T}$ which abut along an edge of $\mathscr{C}$, the obtained sum will be at most $3n(S)\,\hat{h}(C)$. On the other hand, the area of each triangle from $\mathscr{C}$ is counted exactly three times. Since the faces of $\mathscr{C}$ cover S^{**}, this implies that $a(S^{**}) \leqslant n(S)\,\hat{h}(C)$. Therefore we have

$$a(S) = \left(\frac{s}{s-4s'}\right)^2 a(S^{**}) \leqslant \left(\frac{s}{s-4s'}\right)^2 n(S)\,\hat{h}(C).$$

Thus the lower density $d = \liminf_{s\to\infty} \frac{n(S)a(C)}{a(S)}$ of the covering satisfies the inequality

$$d \geqslant \frac{a(C)}{\hat{h}(C)}.$$

This completes the proof of our theorem. It should be mentioned that one can further sharpen the bound $\vartheta(C) \geqslant a(C)/\hat{h}(C)$ by considering e.g. for each triangle of the triangulation the sum of the areas of those triangles which are adjacent to the sides of the respective triangle. However, it does not seem to be possible to prove the inequality $\vartheta(C) \geqslant a(C)/h^*(C)$ in this way.

References

[1] R. P. Bambah and C. A. Rogers, *Covering the plane with convex sets*, Journal London Math. Soc. 27 (1952), pp. 304–314.

[2] R. P. Bambah, C. A. Rogers and H. Zassenhaus, *On coverings with convex domains*, Acta Arith. 9 (1964), pp. 191–207.

[3] C. H. Dowker, *On minimum circumscribed polygons*, Bull. Amer. Math. Soc. 50 (1944), pp. 120–122.

[4] L. Fejes Tóth, *Some packing and covering theorems*, Acta Sci. Math. Szeged 12/A (1950), pp. 62–67.

[5] – *Regular figures*, Pergamon Press, Oxford–London–New York–Paris 1964.

[6] W. Kuperberg, *Packing convex bodies in the plane with density greater than* 3/4, Geometriae Dedicata 13 (1982), pp. 149–155.

[7] C. A. Rogers, *The closest packing of convex two-dimensional domains*, Acta Math. 86 (1951), pp. 309–321.

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Some inequalities for the sum of two sets of lattice points

by

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1. Introduction. Let J be the set of all nonnegative integers and let Δ be any nonempty set. With $x = \{(\delta, x_\delta) | \ \delta \in \Delta\}$ let

$$J^\Delta = \{x | \ x_\delta \in J \text{ for all } (\delta, x_\delta) \in x\}.$$

If $x \in J^\Delta$ and $x_\delta = 0$ for all $\delta \in \Delta$, we write $x = 0$. For $x, y \in J^\Delta$ let

$$x + y = \{(\delta, x_\delta + y_\delta) | \ \delta \in \Delta\}.$$

Assume that $A, B \subset J^\Delta$, and that $0 \in A$, $A \neq J^\Delta$, and $B \neq \emptyset$, and let $C = A + B = \{a + b | \ a \in A, b \in B\}$. We obtain information about how sparse C is. Before describing our results more precisely, we give some more definitions.

If $x, y \in J^\Delta$ then $y - x = \{(\delta, y_\delta - x_\delta) | \ \delta \in \Delta\}$. We write $x \precsim y$ if $y - x \in J^\Delta$, and $x \prec y$ if also $x \neq y$. The family $\mathscr{G}$ consists of all finite nonempty sets $G \subset J^\Delta$ such that $\sum_{x_\delta > 0} 1 < \infty$ if $x \in G$ and also $x \in G$ if $g_1 \prec x \prec g_2$ with $g_1, g_2 \in G$. If $S \cup T \subset J^\Delta$ and T is finite, then $S(T)$ denotes the cardinality of $S \cap T$. We obtain inequalities which give lower bounds for $C(G)$ where $G \in \mathscr{G}$. The main result is Theorem 3 but attention is also called to Theorem 7 which is a companion to a theorem of Kvarda [5].

The family $\mathscr{F}$ is defined as $\{F | \ 0 \in F \in \mathscr{G}\}$. The *Erdős density* α_1 of A is defined by

$$\alpha_1 = \operatorname{glb}\left\{\frac{A(F)-1}{J^\Delta(F)} \,\middle|\, F \in \mathscr{F}, A(F) < J^\Delta(F)\right\}.$$

This density is first used in Lemma 2. Later we obtain lower bounds for certain "densities" of C one of which is a generalization of this density.