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An extension of a result of C. J. Smyth to polynomials in several variables

by

A. Bazylewicz (Warszawa)

H. L. Montgomery and A. Schinzel in their paper [2] have asked whether for every positive integer d there exists a number $C(d) < 1$ with the following property:

Every polynomial $F(z_1, \ldots, z_n)$ with integral coefficients of the total degree d such that $F(0, \ldots, 0) = 1$ has a zero $(\varrho_1, \ldots, \varrho_n)$ satisfying $\max_{1 \leqslant i \leqslant n} |\varrho_i| \leqslant C(d)$.

We present here a solution of this problem.

By analogy with polynomials in one variable, a polynomial $F \in C[z_1, \ldots, z_n]$ is called *reciprocal* if

$$F(z_1, \ldots, z_n) = \text{const}\, z_1^{d_1} \ldots z_n^{d_n} F(z_1^{-1}, \ldots, z_n^{-1})$$

where d_i is the degree of F with respect to z_i.

Our principal result is the following:

Theorem 1. *Let $F \in Z[z_1, \ldots, z_n]$ be a non-reciprocal polynomial such that $F(0, \ldots, 0) = 1$.*

Then F has a zero $(\varrho_1, \ldots, \varrho_n)$ satisfying

$$\max_{1 \leqslant i \leqslant n} |\varrho_i| \leqslant \theta_0^{-1/d}$$

where d is the total degree of F and θ_0 is the least Pisot–Vijayaraghavan number.

For $n = 1$ this result has been obtained by C. J. Smyth [4] as a corollary to his theorem on the Mahler measure.

For the proof of the theorem we need another definition.

Definition. A polynomial $F \in C[z_1, \ldots, z_n]$ of degree d_i with respect to z_i is called *self-inversive* if there exists a $c \in C$ such that

$$z_1^{d_1} \ldots z_n^{d_n} \bar{F}(z_1^{-1}, \ldots, z_n^{-1}) = cF$$

where the bar denotes the complex conjugation.

Clearly, for $F \in R[z_1, \ldots, z_n]$ the notions of reciprocal and self-inversive polynomials coincide.

The proof is based on three lemmata.

Lemma 1. *Let $G \in C[z_1, \ldots, z_n]$ be of degree d_i with respect to z_i, $1 \leqslant i \leqslant n$, $E \subset C$, $\operatorname{card} E > \max d_i$. If for all $(\varrho_1, \ldots, \varrho_n) \in E^n$ we have $G(\varrho_1, \ldots, \varrho_n) = 0$, then $G = 0$.*

Proof. See Lemma 8 in [3].

Lemma 2. *Let E be the set of all primitive roots of unity of order not divisible by 3. If $F \in C[z_1, \ldots, z_n]$, $F(0, \ldots, 0] = 1$ and for all $\varepsilon \in E^n$ the polynomial $F_\varepsilon(z) = F(\varepsilon_1 z, \ldots, \varepsilon_n z)$ is self-inversive then also $F(z_1, \ldots, z_n)$ is self-inversive.*

Proof. Let F be of total degree d and of degree d_i with respect to z_i. We have

$$F_\varepsilon(z) = \sum_{i=0}^{d} z^i F_i(\varepsilon_1, \ldots, \varepsilon_n) \tag{1}$$

where F_i is the sum of all terms of degree i in F.

The assumption that $F_\varepsilon(z)$ is self-inversive means that

$$z^d \bar{F}_\varepsilon(z^{-1}) = c(\varepsilon) F_\varepsilon(z), \quad \text{where} \quad c(\varepsilon) \in C, \text{ if } F_d(\varepsilon_1, \ldots, \varepsilon_n) \neq 0.$$

This condition together with (1) implies that

$$z^d \sum_{i=0}^{d} \bar{F}_i(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1}) z^{-i} = c(\varepsilon) \sum_{i=0}^{d} F_i(\varepsilon_1, \ldots, \varepsilon_n) z^i,$$

$$F_d(\varepsilon_1, \ldots, \varepsilon_n)\big(\bar{F}_d(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1}) - c(\varepsilon) F(0, \ldots, 0)\big) = 0,$$

$$F_d(\varepsilon_1, \ldots, \varepsilon_n)\big(\bar{F}_{d-1}(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1}) - c(\varepsilon) F_1(\varepsilon_1, \ldots, \varepsilon_n)\big) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_d(\varepsilon_1, \ldots, \varepsilon_n)\big(\bar{F}_0(0, \ldots, 0) - c(\varepsilon) F_d(\varepsilon_1, \ldots, \varepsilon_n)\big) = 0$$

for all $\varepsilon \in E^n$. The polynomials

$$F_d(z_1, \ldots, z_n) \times \Big(\big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}_{d-j}(z_1^{-1}, \ldots, z_n^{-1}) - \big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}_d(z_1^{-1}, \ldots, z_n^{-1}) F_j(z_1, \ldots, z_n)\Big)$$

vanish for $j = 0, 1, \ldots, d$ in all points $\varepsilon \in E^n$.

In virtue of Lemma 1 we get

$$\big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}_{d-j}(z_1^{-1}, \ldots, z_n^{-1}) = \big(\prod_{i=1}^{n} z^{d_i}\big) \bar{F}_d(z_1^{-1}, \ldots, z_n^{-1}) F_j(z_1, \ldots, z_n),$$

because $F_d(z_1, \ldots, z_n) \neq 0$, hence, by (1),

$$\big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}(z_1^{-1}, \ldots, z_n^{-1}) = \big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}_d(z_1^{-1}, \ldots, z_n^{-1}) F(z_1, \ldots, z_n). \tag{2}$$

The polynomial on the left-hand side is with respect to z_i of degree at most d_i, i.e., the degree of F, hence

$$\big(\prod_{i=1}^{n} z_i^{d_i}\big) \bar{F}_d(z_1^{-1}, \ldots, z_n^{-1}) \in C$$

and (2) shows that F is self-inversive.

Lemma 3. *Let K be a complex multiplication field, $\sqrt{3} \notin K$ and let Σ be the set of all injections of K into C. Let P be a monic non self-inversive polynomial with coefficients being integers of K and $P(0) \neq 0$.*

If for $\sigma \in \Sigma$

$$P^\sigma(z) = \prod_{i=1}^{\deg P} (z - \alpha_{\sigma i})$$

then

$$\max_{\sigma \in \Sigma} \prod_{i=1}^{\deg P} \max(1, |\alpha_{\sigma i}|) \geqslant \theta_0$$

where θ_0 is the least Pisot–Vijayaraghavan number.

Proof. See Theorem 1 in [1].

Proof of Theorem 1. By Lemma 2 there exists an $\varepsilon \in E^n$ such that $F_\varepsilon(z)$ is not self-inversive.

By Lemma 3, with $P = z^d F_\varepsilon(z^{-1})$, there exists a $z_0 \in C$ such that

$$F_\varepsilon(z_0^{-1}) = 0 \quad \text{and} \quad |z_0| \geqslant \theta_0^{1/d}.$$

Hence, by the definition of F_ε,

$$F(\varepsilon_1 z_0^{-1}, \ldots, \varepsilon_n z_0^{-1}) = 0 \quad \text{and} \quad \max_{1 \leqslant i \leqslant n} |\varepsilon_i z_0^{-1}| \leqslant \theta_0^{-1/d}.$$

Remark. The above proof clearly works for all non self-inversive polynomials $F \in K[z_1, \ldots, z_n]$ with integral coefficients and $F(0, \ldots, 0) = 1$.

Theorem 2. *If $F \in Z[z_1, \ldots, z_n]$ of the total degree d satisfies $F(0, \ldots, 0) = 1$, then there exist $C(d) < 1$ and $(\varrho_1, \ldots, \varrho_n) \in C^n$ such that $F(\varrho_1, \ldots, \varrho_n) = 0$ and $\max|\varrho_i| \leqslant C(d)$.*

Proof. In virtue of Theorem 1 we can assume that F is reciprocal. By the definition of a reciprocal polynomial, $F(0, \ldots, 0) = 1$ implies that the term $z_1^{d_1} \ldots z_n^{d_n}$ occurs in F, so $d = d_1 + \ldots + d_n$. Hence F really depends only on d variables at most and Corollary 1 of [2] gives our result.

We note that

$$C(d) = 1 - \frac{c_0 (\log d)^3}{d^{d+6} e^{3d^2+4d}}$$

where the constant c_0 is absolute.

References

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