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The distribution of divisors of $N!$

by

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Introduction. The divisors of $j!$ have an asymptotic distribution in which the set of prime numbers is embedded. An explicit formula for this distribution is given which leads to a sequence of splines converging to its density. Approximation, rate of convergence, and large deviations are also considered.

Results. Let X_j be a random variable uniformly distributed over the set $\{\log d\colon d|j!\}$ and let F_j be the normalized (to have expectation zero and variance one) distribution function for X_j. Let p_i be the ith prime and let χ_I be the indicator of the interval I.

THEOREM 1. *The sequence F_j converges completely to the distribution ψ having density ϱ represented by the infinite convolution*

$$\varrho = X_1 * X_2 * \ldots,$$

where

$$X_i = (2\xi_i)^{-1}\chi_{[-\xi_i,\xi_i]},$$

$$\xi_i = \frac{\sigma\log p_i}{p_i-1},$$

$$\sigma = \left(\frac{1}{3}\sum_p\left(\frac{\log p}{p-1}\right)^2\right)^{-1/2}.$$

Moreover, $\sup_x|F_j(x)-\psi(x)| \ll_{\varepsilon} j^{-1/3+\varepsilon}$ *for any* $\varepsilon>0$.

Let $\varrho_N = X_1 * \ldots * X_N$. The sequence of splines $\{\varrho_N\}_1^\infty$ converges to ϱ and is given explicitly by

THEOREM 2. *For* $N>1$,

$$\varrho_N = A_N\sum_{j=1}^{2^{N-1}} s_N(j)\big(\langle\theta_N(j)+x\rangle^{N-1}+\langle\theta_N(j)-x\rangle^{N-1}\big),$$

where

$$A_N = \frac{1}{4(N-1)!}\prod_{j=1}^{N} \xi_j^{-1},$$

$$s_N(j) = 2^{1-N}(-1)^{0\sum_{<i<N}\varepsilon_i(j)},$$

$$\theta_N(j) = \xi_i + \sum_{0<i<N} (-1)^{\varepsilon_i(j)}\xi_{i+1},$$

$$\varepsilon_i(j) = [2\{j2^{-i}\}],$$

$$\langle x\rangle^M = x^M \operatorname{sgn}(x).$$

Since logarithms of primes are linearly independent over Q, the nodes of the spline ϱ_N consist of the 2^N distinct points $\{\pm\theta_N(j)\}_1^{2^{N-1}}$. Moreover, ϱ_N is supported on an interval symmetric about the origin of length $\ll \log N$. Since the complexity of ϱ_N makes it difficult to calculate, the representations

$$\hat{\psi}(t) = \prod_{k>0}\frac{\sin(\xi_k t)}{\xi_k t} = \exp\Big\{-\sum_{n>0}\frac{4^n B_n}{(2n)(2n)!}\sum_{k>0}(\xi_k t)^{2n}\Big\}$$

(see [2]) for the characteristic function of ψ were used instead to obtain the following result which is stated as a conjecture (this computation has not been independently verified).

CONJECTURE. *Let* $f(x) = \exp(-.954-.434x^2-.011x^4)$. *Then*

$$|f(x)-\varrho(x)| < .001 \quad \text{for} \quad |x| < 3.$$

The probability distribution ψ concentrates mass about the origin more so than does the Gaussian distribution. We have the following result as $x\to\infty$:

THEOREM 3. *Both* $1-\psi(x)$ *and* $\varrho(x)$ *are* $o(\exp(-e^{(\sigma^{-1}-\varepsilon)x}))$ *for any* $\varepsilon > 0$.

Demonstrations.

Proof of Theorem 1. Define $\mu_n(j)$ for positive integer n by

$$\mu_n = \Big(12^{-1}\sum_{p^\alpha \| j!}\big((\alpha+1)^n-1\big)(\log p)^n\Big)^{1/n}.$$

The convergence of F_j to ψ follows from Theorem 1 of [2] which in the present case reduces to:

THEOREM. *A necessary and sufficient condition for* F_j *to converge to a distribution* ψ *is that for each n the limits* $a_n = \lim_{j\to\infty}\mu_{2n}/\mu_2$ *exist. In this case* $\hat{\psi}$ *is entire and is represented in the disk* $|z| < 1/4$ *by*

$$\hat{\psi}(z) = \exp\Big(-\sum_{n>0}\frac{6B_n}{n(2n)!}(a_n z)^{2n}\Big),$$

where

$$B_n = 4n\int_0^\infty (e^{2\pi t}-1)^{-1} t^{2n-1}\,dt$$

are the Bernoulli numbers.

The limits a_n are computed from the prime factorization

$$j! = p_1^{\alpha_1}\dots p_k^{\alpha_k}, \quad \text{where} \quad \alpha_i = \sum_{l>0}[jp_i^{-l}].$$

It follows that

$$\alpha_i = \frac{j}{p_i-1} + O\Big(\frac{\log j}{\log p_i}\Big), \tag{1.1}$$

and, for any $\varepsilon > 0$,

$$(\mu_{2n}(j))^{2n} = \frac{j^{2n}}{12}\sum_p\Big(\frac{\log p}{p-1}\Big)^{2n}(1+O_\varepsilon(j^{\varepsilon-1}))^{2n}. \tag{1.2}$$

Hence the limit distribution ψ exists, and $\hat{\psi}$ is represented near the origin by

$$\hat{\psi}(t) = \exp\Big\{-\sum_{n>0}\frac{4^n B_n}{(2n)(2n)!}\sum_{k>0}(\xi_k t)^{2n}\Big\}$$

$$= \exp\Big\{\sum_{k>0}\int_0^{\xi_k|t|}\cot x - x^{-1}\,dx\Big\}$$

$$= \prod_{k>0}\frac{\sin(\xi_k t)}{\xi_k t}.$$

Therefore, $\varrho = X_1 * X_2 * \dots$

To estimate the rate of convergence of F_j to ψ, we use the Berry–Esseen inequality [1]:

For all $T > 0$,

$$\sup_x |F_j(x)-\psi(x)| \ll \frac{1}{T} + \int_{-T}^{T}\frac{|\hat{F}_j(t)-\hat{\psi}(t)|}{|t|}\,dt. \tag{1.3}$$

We will use the following representations of $\hat{F}_j$:

$$\hat{F}_j(t) = \exp\Big\{-\sum_{n=1}^{\infty}\frac{6B_n}{n(2n)!}(\mu_{2n}/\mu_2)^{2n}t^{2n}\Big\} = \prod_{i=1}^{k} R_i(t),$$

where

$$R_i(t) = \frac{\sin\left(\frac{t}{2\mu_2}(\alpha_i+1)\log p_i\right)}{(\alpha_i+1)\sin\left(\frac{t}{2\mu_2}\log p_i\right)},$$

see [2]. From (1.1) and (1.2) follows the estimate

(1.4) $$R_i(t) = \frac{\sin(\xi_i t)}{\xi_i t} + O_\varepsilon\left(\frac{p_i}{j^{1-\varepsilon}\log p_i} + \frac{t^2\log^2 p_1}{j^{2-\varepsilon}}\right).$$

Let $0 < \eta < 1/2$ and suppose $1 \ll t \leqslant j^\eta$. Since $\xi_i \asymp i^{-1}$, Sterling's formula applied to the product representations of $\hat{F}_j$ and $\hat{\psi}$ gives

$$|\hat{F}_j(t)| + |\hat{\psi}(t)| \ll \sqrt{t}e^{-t}.$$

Assuming the further condition that $t^2 = O(j^\eta)$, and using

$$(a\sin x)^{-1}\sin ax = 1 + O\big((1+a^2)x^2\big),$$

we have

$$\prod_{i=j^\eta}^{k} R_i(t) = 1 + O_\varepsilon(T^2 j^{-\eta}).$$

Since the tail of the product representing $\hat{\psi}$ is similarly small, (1.4) applied tc the product representation for $\hat{F}_j$ allows us to conclude:

$$|\hat{F}_j(t) - \hat{\psi}(t)| \ll \sqrt{t}e^{-t} \quad \text{for} \quad 1 \ll t \leqslant j^\eta,$$

$$|\hat{F}_j(t) - \hat{\psi}(t)| \ll_\varepsilon \frac{t^{5/2}e^{-t}}{j^\eta} + j^{\varepsilon+2\eta-1} \quad \text{for} \quad 1 \ll t^2 = O(j^\eta).$$

We base our estimate of $|\hat{F}_j - \hat{\psi}|$ for small values of t on (1.4). Using (1.2) we have

$$(\mu_{2n}/\mu_2)^{2n} = \frac{4^n}{12}\sum_{k>0}\big(\xi_k(1+O_\varepsilon(j^{\varepsilon-1}))\big)^{2n},$$

hence

$$\hat{F}_j(t) = \hat{\psi}\big(t(1+O(j^{\varepsilon-1}))\big).$$

Since $\hat{\psi}'(t)$ is Lipschitz for small t, and since $\hat{\psi}'(0) = 0$, we obtain

$$|\hat{F}_j(t) - \hat{\psi}(t)| \ll t^2 j^{\varepsilon-1} \quad \text{for} \quad |t| < \tfrac{1}{4}.$$

Using our estimates of $|\hat{F}_j - \hat{\psi}|$ in (1.3) with $T = j^\eta$ and $\eta = \frac{1}{3}+\varepsilon$ completes the proof. ∎

Proof of Theorem 2. Let $t_N(x) = \left(-\frac{d}{dx}\right)^N \cos x$. Since

$$\prod_{j=1}^{N}\sin\xi_j = \sum_{j=1}^{2^{N-1}} s_N(j)\,t_N(\theta_N(j)),$$

Fourier inversion gives

(2.1) $$\varrho_N(x) = \lim_{\varepsilon\to 0}\frac{4(N-1)!\,A_N}{\pi}\sum_{j=1}^{2^{N-1}} s_N(j)\int_{\varepsilon}^{\infty}\cos(ux)\,t_N(\theta_N(j)u)\frac{du}{u^N}.$$

Assume $N > 1$ is odd, say $N = 2n+1$. Let $\eta^+ = |\theta_N(j)+x|$ and $\eta^- = |\theta_N(j)-x|$. Then the integral in (2.1) is

(2.2) $$\left(\langle\theta_N(j)+x\rangle^{N-1}\int_{\eta^+\varepsilon}^{\infty}\frac{\sin u}{2u^{2n+1}}du + \langle\theta_N(j)-x\rangle^{N-1}\int_{\eta^-\varepsilon}^{\infty}\frac{\sin u}{2u^{2n+1}}du\right)(-1)^n.$$

Since

$$\int\frac{\sin u}{u^{2n+1}}du = \left(H_n(u) + \frac{1}{(2n)!}\int\frac{\sin u}{u}du\right)(-1)^n$$

where

$$H_n(u) = \frac{1}{u(2n)!}\sum_{k=0}^{n-1}\frac{(-1)^k(2k)!}{u^{2k}}\left(\cos u + \frac{2k+1}{u}\sin u\right),$$

(2.2) becomes

$$\langle\theta_N(j)+x\rangle^{N-1}\left(\frac{\pi}{4(2n)!} + o(1) - H_n(\eta^+\varepsilon)\right)$$
$$+ \langle\theta_N(j)-x\rangle^{N-1}\left(\frac{\pi}{4(2n)!} + o(1) - H_n(\eta^-\varepsilon)\right).$$

Therefore ϱ_N is represented by

(2.3) $$A_N\sum_{j=1}^{2^{N-1}} s_N(j)\big(\langle\theta_N(j)+x\rangle^{N-1} + \langle\theta_N(j)-x\rangle^{N-1}\big)$$
$$-\lim_{\varepsilon\to 0}\frac{4(N-1)!\,A_N}{\pi}\sum_{j=1}^{2^{N-1}} s_n(j)\big(\langle\theta_N(j)+x\rangle^{N-1}H_n(\eta^+\varepsilon)$$
$$+ \langle\theta_N(j)-x\rangle^{N-1}H_n(\eta^-\varepsilon)\big).$$

Now view the ξ_i as indeterminates so that ϱ_N is a function of the ξ_i and of x. Notice that if ξ_i and x are algebraic, then both $\varrho_N(x)$ and the first term of

(2.3) are algebraic. Since π is transcendental, it follows that the limit as $\varepsilon \to$ of the sum in the second term of (2.3) is either transcendental or zero. Since $H_n(u)$ is meromorphic, with rational coefficients in its Laurent expansion about zero, this limit must therefore be zero. ■

Before proving Theorem 3, we establish the following

LEMMA. *For any* $\varepsilon > 0$, $\hat{\varrho}(iy) \ll_{\varepsilon} \exp((\sigma+\varepsilon)\, y \log y)$ *as* $y \to \infty$.

Proof of the Lemma. We have

$$\hat{\varrho}(iy) \ll \prod_{k=1}^{\infty} (\xi_k y)^{-1} \sinh(\xi_k y) = \exp\Big(\sum_{k=1}^{\infty} \log\big((\xi_k y)^{-1} \sinh(\xi_k y)\big)\Big).$$

Since $x^{-1} \sinh x < e^x$ for $x \geqslant \frac{1}{2}$, and since $\xi_k \sim \sigma k^{-1}$, we have

$$\sum_{k \leqslant 3\sigma y/2} \log\big((\xi_k y)^{-1} \sinh(\xi_k y)\big) \leqslant (\sigma+\varepsilon)\, y \log y,$$

for any $\varepsilon > 0$ provided y is sufficiently large (depending on ε). If $|x| < 1$, the $\log\left(\frac{\sinh x}{x}\right) \ll x^2$, so that

$$\sum_{k > 3\sigma y/2} \big((\xi_k y)^{-1} \sinh(\xi_k y)\big) \ll y.$$

Combining these estimates completes the proof of the lemma. ■

Proof of Theorem 3. Let $f(x) = \chi_{[y,\infty]}(x)\, e^{-\alpha(y)x}$, $g(x) = e^{\alpha(y)x} \varrho(x)$, $h_\lambda(x) = \frac{1}{2\lambda} \chi_{[-\lambda,\lambda]}(x)$, where $\alpha(y)$ is a function to be chosen later, and λ is positive parameter. It follows that

$$1-\psi(y) = \int f * h_\lambda(x)\, g(x)\, dx + \int g(x)\big(f(x) - f * h_\lambda(x)\big)\, dx,$$

which by the Parseval identity applied to the first integral is

$$\int \left(\frac{1}{\sqrt{2\pi}} \frac{e^{-\alpha(y)y+ity}}{\alpha(y)-it}\right)\left(\frac{1}{\sqrt{2\pi}} \frac{\sin(t\lambda)}{t\lambda}\right) \hat{\varrho}\big(t+i\alpha(y)\big)\, dt + \int e^{\alpha(y)x} \varrho(x)\, H(x)\, dx$$

where

$$H(x) = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \big(f(x) - f(x-t)\big)\, dt.$$

Note that

$$H(x) \begin{cases} = 0 & \text{if} \quad x < y-\lambda, \\ = e^{-\alpha(y)x}\left(1 - \dfrac{\sinh(\lambda\alpha(y))}{\lambda\alpha(y)}\right) & \text{if} \quad x > y+\lambda, \\ \ll e^{-\alpha(y)(y-\lambda)} & \text{if} \quad y-\lambda \leqslant x \leqslant y+\lambda. \end{cases}$$

Assuming that $\alpha(y) \to \infty$ as $y \to \infty$, $\lambda = o(\alpha(y)^{-1})$, and estimating $\hat{\varrho}(i\alpha(y))$ by the lemma produces

$$1-\psi(y) \ll_{\varepsilon} \lambda^{-1} \exp\big(-\alpha(y)\big(y - (\sigma+\varepsilon)\log\alpha(y)\big)\big) + \lambda\varrho(y-\lambda).$$

Choosing $\alpha(y) = \exp\big((\sigma+\varepsilon)^{-1} y - 1\big)$, and noting that $\varrho(y+1) < 1-\psi(y)$, we obtain

$$\varrho(y+1) \ll_{\varepsilon_0} \lambda^{-1} \exp\big(-\exp((\sigma^{-1}-\varepsilon_0)\, y)\big) + \lambda\varrho(y-\lambda),$$

for any $\varepsilon_0 > 0$. Choosing

$$\lambda = \exp\big(-\tfrac{1}{2}\exp((\sigma^{-1}-\varepsilon_0)\, y)\big)$$

completes the proof since $\varepsilon_0 > 0$ was arbitrary. ■

References

[1] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. II, Wiley, New York 1966.

[2] M. Vose, *Limit theorems for divisor distributions*, Proc. Amer. Math. Soc. 95, Number 4, (1985), pp. 505–511.

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