The distribution of divisors of $N!$

by

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Introduction. The divisors of $j!$ have an asymptotic distribution in which the set of prime numbers is embedded. An explicit formula for this distribution is given which leads to a sequence of splines converging to its density. Approximation, rate of convergence, and large deviations are also considered.

Results. Let $X_j$ be a random variable uniformly distributed over the set \{log $d$: $d|j!$\} and let $F_j$ be the normalized (to have expectation zero and variance one) distribution function for $X_j$. Let $p_i$ be the $i$th prime and let $\chi_i$ be the indicator of the interval $I_i$.

Theorem 1. The sequence $F_j$ converges completely to the distribution $\psi$ having density $\varrho$ represented by the infinite convolution

$$\varrho = X_1 * X_2 * \ldots,$$

where

$$X_i = (2\pi)^{-1} \chi_{\{\cdot \leq \log p_i\}},$$

$$\zeta_i = \frac{\sigma \log p_i}{p_i - 1},$$

$$\sigma = \left( \frac{1}{2} \sum_p \left( \frac{\log p}{p-1} \right)^2 \right)^{-1/2}.$$

Moreover, $\sup_x |F_j(x) - \psi(x)| \leq e^{-x/3 + 4}$ for any $x > 0$.

Let $\varrho_N = X_1 * \ldots * X_N$. The sequence of splines $\{\varrho_N\}_N$ converges to $\varrho$ and is given explicitly by

Theorem 2. For $N > 1$,

$$\varrho_N = A_N \sum_{j=1}^{2N-1} s_N(j) \left( \langle \vartheta_N(j) + x \rangle^{N-1} + \langle \vartheta_N(j) - x \rangle^{N-1} \right),$$

where
where

\[ A_N = \frac{1}{4(N-1)} \prod_{j=1}^{N} \zeta_j^{-1}, \]
\[ s_N(j) = 2^{1-N}(-1)^{0<j<N} \sum_{0<i<N} \xi^{(j)}i, \]
\[ \theta_N(j) = \zeta_j + \sum_{0<i<N} (-1)^{i} \zeta_{j+i}, \]
\[ \epsilon_i(j) = \left[ \frac{2}{j(j+1)} \right], \]
\[ \langle x \rangle_M = x^M \text{sgn}(x). \]

Since logarithms of primes are linearly independent over \( Q \), the nodes of the spline \( \xi_N \) consist of the \( 2^N \) distinct points \( \{ \pm \theta_N(j)^{-1} \}_{j=1}^{2^N-1} \). Moreover, \( \xi_N \) is supported on an interval symmetric about the origin of length \( < \log N \). Since the complexity of \( \xi_N \) makes it difficult to calculate, the representations

\[ \hat{\psi}(t) = \prod_{k>0} \frac{\sin(\xi_k t)}{\xi_k t} = \exp \left\{ - \sum_{n>0} \frac{4^n B_n}{n(2n)!} \sum_{k>0} (\xi_k t)^{2n} \right\} \]

(see [2]) for the characteristic function of \( \psi \) were used instead of calculating the following result which is stated as a conjecture (this computation has not been independently verified).

**Conjecture.** Let \( f(x) = \exp(-0.954 - 0.434 x^2 - 0.11 x^4) \). Then

\[ |f(x) - \varphi(x)| < 0.001 \quad \text{for} \quad |x| < 3. \]

The probability distribution \( \psi \) concentrates mass about the origin more so than does the Gaussian distribution. We have the following result as \( x \to \infty \):

**Theorem 3.** Both \( 1 - \psi(x) \) and \( \varphi(x) \) are \( o(\exp(-e^{(e-1) - \delta x})) \) for any \( \delta > 0 \).

**Demonstrations.**

**Proof of Theorem 1.** Define \( \mu_n(j) \) for positive integer \( n \) by

\[ \mu_n = (12^{-1}) \frac{(x+1)^n - 1}{(\log p)^n} \]

The convergence of \( F_j \) to \( \psi \) follows from Theorem 1 of [2] which in the present case reduces to:

**Theorem.** A necessary and sufficient condition for \( F_j \) to converge to a distribution \( \psi \) is that for each \( n \) the limits \( a_n = \lim_{j \to \infty} \frac{\mu_n}{\mu_2} \) exist. In this case \( \psi \)

\[ \hat{\psi}(z) = \exp \left\{ - \sum_{n>0} \frac{6B_n}{n(2n)!}(a_n z)^{2n} \right\} \]

are the Bernoulli numbers.

The limits \( a_n \) are computed from the prime factorization

\[ j = p_1^a \cdots p_k^a, \]

where \( a_i = \sum_{l>0} [j p_l^{-1}]. \)

It follows that

\[ a_l = \frac{j}{h_l} + O \left( \frac{\log f}{\log p_l} \right), \]

and, for any \( \epsilon > 0 \),

\[ (\mu_2(j))^2 = \frac{\log p}{2 \epsilon} \sum_{n>0} \frac{(\log p)^n}{p-1} \epsilon^n (1 + O(\epsilon^{-1}))^2. \]

Hence the limit distribution \( \psi \) exists, and \( \hat{\psi} \) is represented near the origin by

\[ \hat{\psi}(z) = \exp \left\{ - \sum_{n>0} \frac{4^n B_n}{n(2n)!} \sum_{k>0} (\xi_k t)^{2n} \right\} \]

\[ = \prod_{k>0} \frac{\sin(\xi_k t)}{\xi_k t} \]

Therefore, \( \varphi = X_1 \ast X_2 \ast \ldots \)

To estimate the rate of convergence of \( F_j \) to \( \psi \), we use the Berry–Esseen inequality [1]:

For all \( T > 0 \),

\[ \sup_{x} |F_j(x) - \psi(x)| \leq \frac{1}{T} + \int_{-T}^{T} \left| F_j(t) - \psi(t) \right| dt. \]

We will use the following representations of \( \hat{F_j} \):

\[ \hat{F_j}(t) = \exp \left\{ - \sum_{n>0} \frac{6B_n}{n(2n)!}(\mu_n/\mu_2)^{2n} t^{2n} \right\}. \]
where
\[
R_i(t) = \frac{\sin\left(\frac{t}{2\mu^2} - (a_i + 1) \log p_i\right)}{(a_i + 1) \sin\left(\frac{t}{2\mu} \log p_i\right)},
\]
see [2]. From (1.1) and (1.2) follows the estimate
\[
R_i(t) = \frac{\sin(\xi_i t)}{\xi_i t} + O_s\left(\frac{t^2 \log^2 p_i}{1 - e^{-t^2}}\right).
\]
Let \(0 < \eta < 1/2\) and suppose \(1 \leq t \leq N\). Since \(\xi_i \approx i^{-1}\), Sterling's formula applied to the product representations of \(\hat{F}_j\) and \(\hat{\psi}\) gives
\[
|\hat{F}_j(t)| + |\hat{\psi}(t)| \leq \sqrt{\pi} e^{-i t}.
\]
Assuming the further condition that \(t^2 = O(N^2)\), and using
\[(a \sin x)^{-1} \sin ax = 1 + O((1 + a^2)x^2),\]
we have
\[
\prod_{j=1}^{N} R_i(t) = 1 + O_s(T^2N^{-\eta}).
\]
Since the tail of the product representing \(\hat{\psi}\) is similarly small, (1.4) applied to the product representation for \(\hat{F}_j\) allows us to conclude:
\[
|\hat{F}_j(t)| \leq \sqrt{\pi} e^{-i t} \quad \text{for} \quad 1 \leq t \leq N,
\]
\[
|\hat{F}_j(t)| \leq \frac{t^{1/2} e^{-i t}}{\sqrt{\pi}} + N^{-2} \quad \text{for} \quad 1 \leq t^2 = O(N^2).
\]
We base our estimate of \(|\hat{F}_j - \hat{\psi}|\) for small values of \(t\) on (1.4). Using (1.2) we have
\[
(\mu_2 \mu_3)^n = \frac{4^n}{12} \sum_{k \geq 0} (\xi_k (1 + O(\eta^{-1})))^n,
\]
therefore
\[
\hat{F}_j(t) = \hat{\psi}(t(1 + O(\eta^{-1}))).
\]
Since \(\hat{\psi}(t)\) is Lipschitz for small \(t\), and since \(\hat{\psi}(0) = 0\), we obtain
\[
|\hat{F}_j(t)| \leq t^{1/2} e^{-i t} \quad \text{for} \quad |t| < \frac{1}{2}.
\]
Using our estimates of \(|\hat{F}_j - \hat{\psi}|\) in (1.3) with \(T = N^2\) and \(\eta = \frac{1}{2} + \epsilon\) completes the proof. \(\blacksquare\)

Proof of Theorem 2. Let \(t_N(x) = \left(-\frac{d}{dx}\right)^N \cos x\). Since
\[
\prod_{j=1}^{N} \sin \xi_j = \sum_{j=1}^{2N-1} s_N(j) t_N(\theta_N(j)),
\]
Fourier inversion gives
\[
\phi_N(x) = \lim_{\eta \to 0} \frac{4(N - 1)!}{\pi} A_N \sum_{j=1}^{2N-1} s_N(j) \int_0^\infty \left(\frac{\sin(u)}{2u^{2N} + 1}\right) t_N(\theta_N(j)) u^{\eta} du.
\]
Assume \(N > 1\) is odd, say \(N = 2n + 1\). Let \(\eta^+ = |\theta_N(j) + x|\) and \(\eta^- = |\theta_N(j) - x|\). Then the integral in (2.1) is
\[
\int_0^\infty \left(\frac{\sin(u)}{2u^{2N} + 1}\right) t_N(\theta_N(j)) u^{\eta} du = (-1)^n.
\]
Since
\[
\int_0^\infty \left(\frac{\sin(u)}{2u^{2N} + 1}\right) u^{\eta} du = \left(H_n(u) + \frac{1}{2n+1} \int_0^\infty \sin(u) \right) (-1)^n,
\]
where
\[
H_n(u) = \frac{1}{u(n+1)!} \sum_{k=0}^{n} \frac{(-1)^k (2k)!}{u^{2k}} \left(\cos u + \frac{2k + 1}{u} \sin u\right).
\]
(2.2) becomes
\[
\left(\theta_N(j) + x\right)^{N-1} \left(\frac{\pi}{4(2n)!} + o(1) - H_n(\eta^+)\right)
\]
\[
+ \left(\theta_N(j) - x\right)^{N-1} \left(\frac{\pi}{4(2n)!} + o(1) - H_n(\eta^-)\right).
\]
Therefore \(\phi_N\) is represented by
\[
A_N \sum_{j=1}^{2N-1} s_N(j) \left(\left(\theta_N(j) + x\right)^{N-1} + \left(\theta_N(j) - x\right)^{N-1}\right)
\]
\[
- \lim_{\eta \to 0} \frac{4(N - 1)!}{\pi} A_N \sum_{j=1}^{2N-1} s_N(j) \left(\left(\theta_N(j) + x\right)^{N-1} H_n(\eta^+)\right)
\]
\[
- \left(\theta_N(j) - x\right)^{N-1} H_n(\eta^-)\right).
\]
Now view the \(\xi_i\) as indeterminates so that \(\phi_N\) is a function of the \(\xi_i\) and of \(x\). Notice that if \(\xi_i\) and \(x\) are algebraic, then both \(\phi_N(x)\) and the first term of
(2.3) are algebraic. Since π is transcendental, it follows that the limit as ε → 0 of the sum in the second term of (2.3) is either transcendental or zero. Since $H_{x}(u)$ is meromorphic, with rational coefficients in its Laurer expansion about zero, this limit must therefore be zero.

Before proving Theorem 3, we establish the following Lemma. For any $ε > 0$, $\hat{\varrho}(iy) \ll \exp\left((\sigma + ε) y \log y\right)$ as $y \to \infty$.

Proof of the Lemma. We have

$$\hat{\varrho}(iy) \ll \prod_{k=1}^{\infty} (\xi_{k}y)^{-1} \sinh(\xi_{k}y) = \exp\left(\sum_{k=1}^{\infty} \log((\xi_{k}y)^{-1} \sinh(\xi_{k}y))\right).$$

Since $x^{-1} \sinh x < e^{x^{2}}$ for $x \geqslant \frac{1}{2}$, and since $\xi_{k} \sim \sigma k^{-1}$, we have

$$\sum_{k \leqslant 3\varepsilon y/2} \log((\xi_{k}y)^{-1} \sinh(\xi_{k}y)) \ll (\sigma + ε) y \log y,$$

for any $ε > 0$ provided $y$ is sufficiently large (depending on $ε$). If $|x| < 1$, the

$$\log\left(\frac{\sinh x}{x}\right) \ll x^{2},$$

so that

$$\sum_{k \geqslant 3\varepsilon y/2} ((\xi_{k}y)^{-1} \sinh(\xi_{k}y)) \ll y.$$

Combining these estimates completes the proof of the lemma.

Proof of Theorem 3. Let $f(x) = \chi_{[0, \infty)}(x) e^{-\alpha(x)}$, $g(x) = e^{\alpha(x)} \varrho(x)$, $h_{k}(x) = \frac{1}{2\lambda} \delta_{x-\lambda k}t(x)$, where $\alpha(y)$ is a function to be chosen later, and $\lambda$ is positive parameter. It follows that

$$1 - \psi(y) = \int f \ast h_{0}(x) g(x) dx + \int g(x) (f(x) - f \ast h_{0}(x)) dx,$$

which by the Parseval identity applied to the first integral is

$$\int \left(\frac{1}{\sqrt{2\pi}} e^{-i\alpha(y)t} \right)\left(\frac{1}{\sqrt{2\pi}} \frac{\sin(t\lambda)}{i\lambda} \right) \hat{\varrho}(t+i\lambda y) dt + \int e^{i\alpha(x)} g(x) H(x) dx$$

where

$$H(x) = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} (f(x) - f(x-t)) dt.$$

Note that

$$H(x) = \begin{cases} 0 & \text{if } x < y - \lambda, \\ e^{-\alpha(y)x} \left(1 - \frac{\sinh(\lambda x(y))}{\lambda x(y)}\right) & \text{if } x > y + \lambda, \\ e^{-\alpha(y)(y-x)} & \text{if } y - \lambda \leqslant x \leqslant y + \lambda. \end{cases}$$

Assuming that $\alpha(y) \to \infty$ as $y \to \infty$, $\lambda = o(\alpha(y)^{-1})$, and estimating $\hat{\varrho}(i\alpha(y))$ by the lemma produces

$$1 - \psi(y) \ll \lambda^{-1} \exp\left(-\alpha(y)(y-(\sigma + ε) \log \alpha(y))\right) + \lambda \varrho(y - \lambda).$$

Choosing $\alpha(y) = \exp((\sigma + ε^{-1} y - 1)$, and noting that $\varrho(y + 1) < 1 - \psi(y)$, we obtain

$$\varrho(y + 1) \ll \lambda^{-1} \exp\left(-\exp\left((\sigma^{-1} - \varepsilon_{0})y\right)\right) + \lambda \varrho(y - \lambda),$$

for any $\varepsilon_{0} > 0$. Choosing

$$\lambda = \exp\left(-\frac{1}{2} \exp\left((\sigma^{-1} - \varepsilon_{0})y\right)\right)$$

completes the proof since $\varepsilon_{0} > 0$ was arbitrary.

References


Received on 9.5.1986