On the error function in the asymptotic formula for the counting function of \( k \)-full numbers

by

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To the memory of Professor D. Suryanarayana

1. Introduction. Let \( k \) be an integer \( \geq 2 \). We say that a positive integer \( n \) is \( k \)-full if either \( n = 1 \) or for every prime \( p \) dividing \( n \), \( p^k \) also divides \( n \). We are interested in the \( \Omega \) results for the error function mentioned in the title. It is nice to recall that the asymptotic formula was first considered by Erdős and Szekeres in 1933 (\cite{8}).

2. Statement of the theorem. Let \( a_n = a_n(k) = [1 \text{ if } n = k\text{-full}; 0 \text{ otherwise}] \). We write as usual \( s = \sigma + it \). Then \( F(s) = F_k(s) = \sum_{n=1}^{\infty} a_n/n^s \) has an analytic continuation in \( \sigma > 1/(4k+4) \) and further, in the region \( \sigma > 1/(2k) \), it has only simple poles at \( 1/k, 1/(k+1), \ldots, 1/(2k-1) \) (see \S 5). For \( k \leq j \leq 2k-1 \), let \( b_j x^{1/j} \) be the residue of \( F(s)x^s/s \) at the pole \( s = j/k \). Let

\[
M(x) = \sum_{k \leq j \leq 2k-1} b_j x^{1/j} \quad \text{and} \quad E(x) = \sum_{n < x} a_n - M(x).
\]

Let \( \varrho_k \) be the greatest lower bound for \( \varrho: E(x) = O(x^{\varrho}) \). Then we prove roughly that

\[
\varrho_k \geq \frac{r-1}{r(2k+r-1)}
\]

for any integer \( r \geq 1 \). Since the maximum of

\[
(r-1)/(2k+r-1)
\]

is obtained, for integer \( r \geq 1 \), at \( \lceil \sqrt{2k} \rceil \) or \( \lceil \sqrt{2k} \rceil + 1 \), we can assume that \( r = \lceil \sqrt{2k} \rceil \) or \( \lceil \sqrt{2k} \rceil + 1 \). More precisely we prove

Theorem 1. For suitable constants \( B > 0 \) and \( C > 0 \), we have

\[
\int_{T}^{\infty} \frac{|E(u)|^2}{u^{a+1}} \exp(-u/T^B) du \geq C \log T
\]

where \( a = (r-1)/(2k+r-1) \) for \( k \geq 3 \) and \( a = 1/10 \) for \( k = 2 \).
In case, 2s(k + r) is an integer, one can improve Theorem 1 to Theorem 2. For suitable constants B > 0 and C > 0, we have
\[ \int_0^\infty \frac{|E_\alpha|^2}{u^{s+1}} \exp(-u/T^\delta) \, du \geq C \log^2 T \]
where \( x = (r-1)/(2k+r-1) \) and \( k \geq 3 \).

**Corollary.** If \( k \geq 3 \), then \( E(x) = O(x^{1/2(2k+n)}/\sqrt{\log x}) \) where \( r \) is the least integer such that \( r(r-1) \geq 2k \).

**3. Remarks.** We now record a few \( O \) and \( \Omega \) results known in this direction. The earlier \( O \)-results are due to Erdős and Szekeres [8], Bateman and Grosswald [7], Krätzel [12] and Ivic [9, 10]. The best known results are due to Ivic and Shiu [11], who proved, among other things that \( \theta_k \leq 1/2(k) \) for \( k \leq 13 \). As is to be expected, these results could be improved if one assumes some unproved hypothesis. For example, it easily follows from the Lindelöf hypothesis for the Riemann zeta-function that \( \theta_k \leq 1/2(k) \) for all \( k \geq 2 \). Further, on the Riemann Hypothesis, we know that \( \theta_2 \leq 13/81 \), which can be improved to \( \theta_2 \leq 11/72 \) ([22]). About \( \Omega \) results, it was noted in [7] that \( \theta_2 \geq (\text{Re} \theta_0)/6 \) if \( \zeta(\theta_0) = 0 \), \( \zeta(2\theta) \neq 0 \) and \( \zeta(3\theta) \neq 0 \).

Now, about the theorem. The method extends to other cases also. What we prove amounts roughly to the following. If \( F(s) \) is given by a Dirichlet's series, convergent in some half plane, and admits an analytic continuation and if \( M(x) \) and \( E(x) \) are defined, then
\[ \limsup_{x \to \infty} \frac{|E(x)|}{x^\delta} \geq 0 \]
where \( x \) is such that \( \int_T^\infty |F(x+i\varepsilon)|^2 \, dx \geq T^2 \) for all \( T > 0 \).

Thus this method can be used for
\[ \zeta^2(s), \quad \zeta(s) \zeta(b), \quad \frac{\zeta(s) \zeta(2s)}{\zeta(b)s} \]
and other similar functions under suitable conditions on \( a \) and \( b \). In this connection, we refer the reader to [3], [4] and [5].

**4. Notations.** Let \( \{a_n\} \) be the special sequence
\[ a_n = a_n(k) = 1 \text{ if } n \text{ is } k\text{-full}; \quad 0 \text{ otherwise.} \]
\[ F(s) = F_k(s) = \sum_{n=1}^{\infty} a_n/n^s. \]
\( \zeta(s) \) is the Riemann zeta function, \( p \) denotes a (general) prime. \( r = \lfloor \sqrt{2k} \rfloor + 1; \quad \alpha = (r-1)/(2k+r-1) \); \( A \) is a big constant, not necessarily same at each occurrence. \( T \) is a real number sufficiently large. For any complex number \( z \), we denote the real part by \( \text{Re} z \) and the imaginary part by \( \text{Im} z \); \( y = T^\delta \) where \( B \) is a sufficiently large constant;
\[ J = \{ T^{1-x} \leq t \leq 2T \}; \quad \text{for any complex number } z \text{ with } \text{Re} z \geq 1/(4k+3) \]
and \( |\text{Im} z - t| \leq (\log T)^{2n}, \quad 1 \leq 2k+n \leq 3j-1 \}
\[ J_1 = \{ T^{1-x} \leq t \leq 2T \}; \quad \text{for any complex number } z \text{ with } \text{Re} z \geq 1/(4k+3) \]
and \( |\text{Im} z - t| \leq (\log T)^{2n}, \quad 1 \leq 2k+n \leq 3j-1 \} \]
(An interpretation about \( J \) is given below Lemma 14.)
Note that, if \( t \in J \), then \( t+\omega \in J_1 \) if \( |\omega| \leq (\log T)^2 \). Throughout the proof, we assume that \( k \geq 3 \). The case \( k = 2 \) is similar and easy.

**5. Analytic continuation.** In this section, we deal with the analytic continuation of \( F_k(s) \).

**Proposition 1.** (a) We have
\[ F_2(s) = \frac{\zeta(2s) \zeta(3s)}{\zeta(6s)}. \]
(b) For \( k \geq 3 \),
\[ F_k(s) = \prod_{\kappa \in \{2k-1\}} \zeta(\kappa s) \prod_{2k+2 \leq j \leq 4k+3} (\zeta(j))^{e_j} \prod_{2k+2 \leq j \leq 4k+3} (\zeta(2j))^{e_j} G(s) \]
where \( e_j \), \( 2k+2 \leq j \leq 4k+3 \), are suitable nonnegative integers and \( G(s) \) is absolutely convergent in \( \sigma > 1/(4k+4) \).

**Proof.** Proposition 1(a) is standard. We now prove 1(b). One checks by direct verification that
\[ (1+x^a+x^{a+1}+\ldots) \prod_{\kappa \leq 2k-1} (1-x^\kappa) = 1 + \sum_{n \geq 2k+2} c_n x^n \]
for suitable integers \( c_n \). Here \( c_n \leq 0 \) for \( n \leq 4k+3 \) and since
\[ (1+x^a+x^{a+1}+\ldots) (1-x^a) \]
is a polynomial, \( c_n \leq 0 \) for all large \( n \). Now
\[ (1 + \sum_{n \geq 2k+2} c_n x^n) \prod_{2k+2 \leq j \leq 4k+3} (1+x^j)^{e_j} (1+x^{j+1})^{e_j} \]
with \( d_n = 0 \) for all large \( n \). Hence
\[ (1+x^a+x^{a+1}+\ldots) \prod_{k \leq 2k-1} (1-x^\kappa)^{e_\kappa} \prod_{2k+2 \leq j \leq 4k+3} (1-x^{j+1})^{e_j} \]

\[ \times \prod_{2k+2 \leq j \leq 4k+3} (1-x^{j+1})^{e_j} (1 + \sum_{n > 4k+4} d_n x^n). \]
We now put \( x = p^{-z} \) and take the product over all primes. This yields the proposition. Since \( d_n \) is bounded by a function of \( k \), the absolute convergence of 
\[ G(s) = \prod_p \left( 1 + \sum_{n \gg 4k+4} d_n p^{-sn} \right) \]
is guaranteed.

6. A trivial upper bound for \( F(s) \). We recall the definition of \( J_1 \) (see § 4).

We now prove

**Proposition 2.** If \( \text{Res} \geq 1/(4k+3) \) and \( t \in J_1 \) then

\[ F(s) = O((|t|+2)^A) \quad \text{for a suitable } A > 0. \]

We need

**Lemma 1.** There holds the following inequality: For any \( \varepsilon > 0 \),

\[ \frac{1}{\zeta(s)} = O((|t|+2)^A) \quad \text{in } \text{Res} \geq \frac{1}{2} + \varepsilon, \]

provided \( \zeta(z) \neq 0 \) for \( \text{Re} z > \frac{1}{2} + \varepsilon \) and \( |\text{Im} z - \text{Im} s| \leq (\log T)^\varepsilon \) and \( |\text{Im} s| > 1 \).

**Proof.** This is a standard result. One can refer to Theorem 14.2 of [17], where a similar result has been proved.

**Lemma 2.** The following inequalities are true:

(a) If \( \text{Res} > 1 \) and \( |\text{Im} s| \geq 1 \) then \( \zeta'(z) = O((|t|+2)^A) \) for a suitable \( A > 0 \).

(b) If \( \text{Res} \geq 1/(4k+3) \) and \( l \geq 2k+2 \) and \( t \in J_1 \), \( 1/\zeta'(z) = O((|t|+2)) \).

(c) For \( \text{Res} \geq 1/(4k+3) \), \( G(s) \) is bounded by a function of \( k \).

**Proof.** (a) follows from Chapter 5 of [17]. (b) is a consequence of Lemma 1. (c) is true because of the absolute convergence of \( G(s) \).

Now Proposition 2 follows from Proposition 1 and Lemma 2.

7. A mean value upper bound for \( F(s) \). Since our aim is to prove the result that

\[ \int_T^\infty \frac{|E(u)|^2}{u^{2k+1}} \exp\left(-\frac{u}{y}\right) du \text{ is large}, \]

we can assume that

\[ \int_T^\infty \frac{|E(u)|^2}{u^{2k+1}} \exp\left(-\frac{u}{y}\right) du \leq \log^2 T. \]

Our aim in this section is to prove, under the assumption (\( \ast \)),

**Proposition 3.** We have

\[ \int_T^\infty \frac{|F(s)|^2}{|y|^2} dt \ll 1 + \int_T^\infty \frac{|E(u)|^2}{u^{2k+1}} e^{-2uy} du. \]

**Lemma 3.** If \( 0 \leq \text{Re} z \leq 1 \) and \( |\text{Im} z| \geq (\log T)^\varepsilon \), then

\[ \int_T^\infty \frac{e^{-uy}}{u^t} du = -\frac{T^{1-z}}{1-z} + O(T^{-10}). \]

**Proof.** We have

\[ \int_T^\infty \frac{e^{-uy}}{u^t} du = \int_T^\infty \frac{e^{-uy}}{u^t} du - \int_T^\infty \frac{e^{-uy}}{u^t} du \]

\[ = y^{1-t} \Gamma(1-z) - \frac{T^{1-z}}{1-z} + O\left( \int_T^\infty \frac{u^y}{u^{k+r}} du \right) \]

and hence the result.

**Lemma 4.** Under the assumption (\( \ast \)), there exists \( T_0, T \leq T_0 \leq 2T \) such that

(a) \( \frac{E(T_0)e^{-T_0y}}{T_0} = O(\log T) \).

(b) \( \frac{1}{T_0} \int_T^\infty \frac{|E(u)| e^{-uy}}{u^t} du = O(\log T) \).

**Proof.** Because of (\( \ast \)),

\[ \log^2 T > \int_T^\infty \frac{|E(u)|^2}{u^{2k+1}} \exp\left(-\frac{u}{y}\right) du \]

\[ \geq \min_{T < u < 2T} \left| \int_T^u \frac{|E(u)|^2}{u^{2k+1}} \exp\left(-\frac{u}{y}\right) du \right|^2 \int_T^u \frac{du}{u} \]

and hence (a).
To prove (b), observe that
\[
\left(1 - \frac{1}{y} \right) \int_{\tau_0}^{\infty} \frac{E(u) e^{-\eta y}}{u^\theta} du \leq \left( \int_{\tau_0}^{\infty} |E(u)|^2 e^{-\eta y} du \right)^{1/2} \int_{\tau_0}^{\infty} e^{-\eta y} u^{\theta - 1} du = 0 \log^2 T.
\]

**Lemma 5.** If \( t \in J \) and \( \text{Res} = \alpha \), then
\[
F(s) = \sum_{n=1}^{\infty} \frac{a_n e^{-\eta n \tau}}{n^s} + s \sum_{n>\tau_0} \frac{E(n+\tau)}{n^{s+1}} e^{-(n+\tau)\eta} + O(( \log T)^2). \]

**Proof.** We start with
\[
\sum_{n=1}^{\infty} \frac{a_n e^{-\eta n \tau}}{n^s} = \frac{1}{2\pi i} \int_{C \cup \tau_0} F(s+w) \Gamma(s) d\nu. \]

Now we break off the portion of the integral \( \text{Im} \nu \geq (\log T)^4 \) with a small error and move the line of integration to \( \text{Re} \nu = \frac{1}{4k+3} - \alpha \); now, using the estimate of \( F(s) \) given in Proposition 2 (and assuming that \( B \) is large enough) and using the fact that, since \( t \in J \), \( t + \text{Im} \nu \in J_1 \), we see that the value of the integral on the horizontals \( t + \text{Im} \nu \in J_1 \), as well as the vertical \( \text{Re} \nu = \frac{1}{4k+3} - \alpha \), is small. This proves that \( \sum_{n=1}^{\infty} \frac{a_n e^{-\eta n \tau}}{n^s} \) equals nearly the sum of the residues inside the contour. Thus we have
\[
F(s) = \sum_{n=1}^{\infty} \frac{a_n e^{-\eta n \tau}}{n^s} + O(T^{-10}) = \sum_{n=1}^{\infty} \frac{a_n e^{-\eta n \tau}}{n^s} + O(T^{-10}).
\]

Now
\[
\sum_{n>\tau_0} \frac{a_n e^{-\eta n \tau}}{n^s} = \int_{\tau_0}^{\infty} \frac{1}{u^\theta} e^{-\eta y} d\left( \sum_{n=1}^{\infty} a_n \right) = \int_{\tau_0}^{\infty} \frac{1}{u^\theta} e^{-\eta y} d \left( M(u) + E(u) \right)
\]
\[
= \int_{\tau_0}^{\infty} \frac{1}{u^\theta} e^{-\eta y} M'(u) du + \int_{\tau_0}^{\infty} \frac{1}{u^\theta} e^{-\eta y} d \left( E(u) \right)
= S_1 + S_2, \text{ say.}
\]

Since \( M(u) \) is of the form \( \sum \eta_j \mu^{1/2} \), the first integral \( S_1 \) is of the form considered in Lemma 3 and hence small. Now \( S_2 \) is (on integration by parts)
\[
\left[ \frac{E(u) e^{-\eta y}}{u^\theta} \right]_{\tau_0}^{\infty} + \int_{\tau_0}^{\infty} \frac{E(u) e^{-\eta y}}{u^{\theta+1}} du + \frac{1}{\theta - 2} \int_{\tau_0}^{\infty} \frac{E(u) e^{-\eta y}}{u^\theta} du.
\]

Hence, using Lemma 4,
\[
S_2 = \int_{\tau_0}^{\infty} \frac{E(u) e^{-\eta y}}{u^{\theta+1}} du + O(\log T)
= \sum_{n>\tau_0} \frac{E(n+\tau)}{(n+\tau)^{\theta+1}} e^{-\eta (n+\tau)} du + O(\log T).
\]

This proves the lemma.

**Lemma 6.** We have, for any sequence of complex numbers \( b_n \),
\[
\int \sum_{n=1}^{\infty} b_n n^\theta |t|^{1/2} dt = \sum_{n=1}^{\infty} (T + O(n))(b_n n^\theta),
\]

provided the right side is convergent.

**Proof.** For the proof of this lemma, we refer the reader to [13] or [14].

**Lemma 7.** We have
\[
\int_{\gamma-\frac{1}{2}} \left| \sum_{n=1}^{\infty} \frac{a_n e^{-\eta y}}{n^s} \right|^2 dt \ll 1.
\]

**Proof.** It is sufficient to prove that
\[
\int_{\gamma-\frac{1}{2}} \left| \sum_{n=1}^{\infty} \frac{a_n e^{-\eta y}}{n^s} \right|^2 dt \ll 1.
\]

By Lemma 6, we have
\[
\int_{\gamma-\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{a_n e^{-\eta y}}{n^s} \right)^2 dt = \sum_{n=1}^{\infty} (2^m + O(n))(a_n n^\theta)
\]
\[
\ll \sum_{n>\tau_0} \frac{(2^m n^{1-\alpha} + 2 \tau_0^{-\alpha})}{n^\theta} \ll 2^m \tau_0^{-3\alpha} + \tau_0^{-2\alpha}.
\]
Hence
\[
\int_{2m}^{2m+1} \left| \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} e^{-n^{1/2}} \right|^2 \frac{dt}{t^2} \lesssim \frac{T_0^{1-2s}}{2^{m+\epsilon}} + \frac{T_0^{2-2s}}{2^{2m+\epsilon}}.
\]
Now summing over \( m \), \( T^{1-s}/2 < 2^m \leq 4T \), it follows that
\[
\int_{T^{-1-s}} T^{1-s} \left| \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} e^{-n^{1/2}} \right|^2 \frac{dt}{t^2} \lesssim 1.
\]

**Lemma 8.** We have
\[
\int_{R = \pi} \left| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right|^2 \frac{dt}{t^2} \lesssim \int_{0}^{2T} \left| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right|^2 \frac{du}{n^{1/2}}.
\]

**Proof.** By Hölder's inequality,
\[
\left| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right|^2 \lesssim \left\| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right\|_{L^2}^2 \lesssim \int_{0}^{2T} \left| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right|^2 \frac{du}{n^{1/2}}.
\]

Hence the integral on the left side of the lemma is bounded by
\[
\int_{R = \pi} \left| \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \right|^2 \frac{du}{n^{1/2}}.
\]

Hence the integral in question is bounded by
\[
\int_{0}^{1} \sum_{n \in \mathbb{N}} \frac{(2T - O(n)) |E(n+u)|^2}{(n+u)^{2\epsilon-1}} e^{-2(n+u)^{1/2}} \frac{du}{n^{1/2}} \lesssim \int_{0}^{1} \sum_{n \in \mathbb{N}} \frac{|E(n+u)|^2}{(n+u)^{2\epsilon-1}} e^{-2(n+u)^{1/2}} \frac{du}{n^{1/2}}.
\]

We now interchange the order of integration and use Lemma 5. Hence the integral in question is bounded by
\[
\int_{0}^{1} \sum_{n \in \mathbb{N}} \frac{E(n+u)}{(n+u)^{1/2}} e^{-(n+u)^{1/2}} \frac{du}{n^{1/2}}.
\]

8. **A mean value lower bound for \( F(s) \)**. In this section, we prove

**Proposition 4.** (a) If \( (k+r) \alpha \) is not an integer then
\[
\int_{R = \pi} \frac{|F(s)|^2}{|s|^2} \frac{dt}{t^2} \gg \log T.
\]

(b) If \( (k+r) \alpha \) is an integer, then
\[
\int_{R = \pi} \frac{|F(s)|^2}{|s|^2} \frac{dt}{t^2} \gg (\log T)^2.
\]

First we give the proof of Proposition 4(b).

**Lemma 9.** Let \( f(s) = \sum_{n=1}^{\infty} b_n n^s \) be convergent in some half plane and analytically continuable in \( \sigma > 1/(4k+4) \), \( A \leq t \leq A + H \). Then
\[
\int_{A}^{A+H} |f(x+it)|^2 \frac{dt}{t^2} \gg H \sum_{n < N/12} |b_n|^2 n^{-2s},
\]

provided \( H \gg (\log A) \), \( b_1 = 1 \) and maximum of \( |f(s)| \) in \( A \leq t \leq A + H \) and \( \mathrm{Re} \ s \gg \alpha \) is \( \approx v^A \).

**Proof.** This result, in slightly stronger forms, can be found in [6] and [15] and in a weaker form in [1] (Theorem 4). We define
\[
c_j(n) = \frac{1}{n^{-1/2}} \sum_{k \leq j \leq k+r-1} \frac{G_j(n)}{n},
\]

\[
f(s) = F(s) \prod_{k < j \leq k+r-1} \left( \zeta(j) \right)^{-1} \prod_{k < j \leq k+r-1} G_j(j) = \sum b_n n^s.
\]

We are interested in the value of \( b_n \), when \( n \) is a \((k+r)\)-th power.

**Lemma 10.** If \( n = 2^{k+r} \) and \( \mu(\lambda) \neq 0 \), then \( b_n = 1 \).

**Proof.** We have
\[
f(s) = \prod_{p} \left( 1 + p^{-2s} + p^{-(k+1)s} + \cdots \right)
\]
\[
\times \prod_{k < j \leq k+r-1} \left( 1 - \frac{1}{p^j} \right) \prod_{k < j \leq k+r-1} \left( 1 + \frac{c_j(p)}{p^{k+r}} + \cdots \right)
\]
\[
= \prod_{p} \left( 1 + \frac{g(p)}{p^{k+r}} + \cdots \right).
\]

Clearly \( g(p^{k+r}) = 1 \) and this proves the result.
Lemma 11. There holds
\[ \int_A^{A+H} |f(x + it)|^2 \, dt \gg H \log A \quad \text{if} \quad H \gg A^4. \]

Proof. By Lemma 9,
\[ \int_A^{A+H} |f(x + it)|^2 \, dt \gg H \sum_{\nu \leq H/2} \frac{|b_{\nu}|^2}{\nu^{2x}} \gg H \sum_{\nu \leq H/2 + \eta} \frac{1}{\nu^{2x}} \gg H \log H \gg H \log A. \]

We now assume that \( \prod_{k \leq k_{2k+\nu-1}} \zeta(jk) \neq 0 \) for \( \text{Re } s = \alpha \) and \( A \ll \text{Im } s \ll A + H \). Then we have

Lemma 12. We have
\[ \int_A^{A+H} |F(x + it)|^2 \, dt \gg AH \log A \quad \text{if} \quad H \gg A^4. \]

Proof. On \( \text{Re } s = \alpha \), we notice that \( |\zeta(1-jk)| = |G_j(jk)| \). Using the functional equation \( \zeta(s) = \chi(s) \zeta(1-s) \), we have
\[ |F(s)| = |f(s)| \prod_{k \leq k_{2k+\nu-1}} \zeta(jk) \prod_{k \leq k_\nu} (G_j(jk))^{-1} \]
\[ \quad = |f(s)| \prod_{j \neq \nu} z(jk) \prod_{j \neq \nu} \zeta(1-jk) \prod_{j \neq \nu} G_j(jk)^{-1} \]
\[ \quad = |f(s)| \prod_{j \neq \nu} \zeta(jk) \gg |f(s)| \prod_{j \neq \nu} |t|^{1/2-\delta} \gg |f(s)| |t|^{1/2} \]
and hence the lemma.

Now we need a few results about the set \( J \). Define
\[ J(x) = J \cap [x, 2x] \quad \text{for any } x, \quad T^{1-x} \lesssim x \lesssim 2T. \]
Let \( N(x) \) be the number of zeros of \( \zeta(s) \) with \( t \in [x, 2x] \) and \( \text{Re } s = \alpha \).

Lemma 13. The number of zeros of \( \zeta(s) \) in \( \sigma \gg 1/2 + \epsilon \) and \( |t| \leq x \) is \( O(x^{1-\epsilon(1+1/10)}) \).

Proof. By Theorem 9.19 (B) of [17], the number of zeros of \( \zeta(s) \) in \( \sigma \gg \alpha, |t| \leq T \) is
\[ O(T^{2-\delta(1-\sigma)}(\log T)^2) \]
and hence the result.

Lemma 14. We have
\[ N(x) \ll x^{1-(11/10)}. \]

Proof. This follows from Lemma 13 and the definition of \( N(x) \).

We now give an interpretation for \( J(x) \). Consider the interval \([x, 2x]\). Corresponding to every zero \( \xi = \beta + it \), \( \beta \gg 1/2 + \epsilon \), \( x - \log T \lesssim x + (\log T)^2 \), \( \gamma - (\log T)^2 \lesssim y + (\log T)^2 \) from the interval \([x, 2x]\). The rest gives us \( J \). Hence \( J(x) \) consists of at most \( N(x) \) disjoint intervals and the total length of \( J(x) \) is \( \gg x \). Now from \( J(x) \) delete the connected components whose length is \( \lesssim x^e \). The total length of the deleted portion is \( O(N(x)x^e) = O(x^{1-\epsilon(1+1/10)}) \). Hence if \( J_2(x) \) is the remaining portion, the total length of \( J_2(x) \) is \( \gg x \). Now applying Lemma 12 to each connected component of \( J_2(x) \) and adding, we get

Lemma 15. We have
\[ \int_{J_2(x)} |F(x + it)|^2 \, dt \gg x^2 \log T. \]

Hence there follows
\[ \int_{J_2(x)} \frac{|F(x + it)|^2}{|t|^2} \, dt \gg \log T \]
and consequently Proposition 4(b). Now Theorem 2 follows from Propositions 3 and 4(b).

The proof of Proposition 4(a) is similar and easy. We give the main steps only.

Lemma 11 (a). There holds
\[ \int_A^{A+H} |f(x + it)|^2 \, dt \gg H \quad \text{if} \quad H \gg A^4. \]

Proof. The result follows from Lemma 1 since \( b_1 = 1 \).

Lemma 12 (a). We have
\[ \int_A^{A+H} |F(x + it)|^2 \, dt \gg AH. \]

Lemma 15 (a). We have
\[ \int_{J_2(x)} |F(x + it)|^2 \, dt \gg x^2. \]

This proves Proposition 4(a), which together with Proposition 3, proves Theorem 1.
References


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Note to a paper of Bambah, Rogers and Zassenhaus

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It is known [7] that the density of a packing of translates of a convex domain \( C \) cannot exceed the density of the thinnest lattice packing of \( C \). It is conjectured ([5], p. 205) that an analogous statement holds for coverings: The density of a covering of the plane with translates of a convex domain \( C \) cannot be less than the density of the thinnest lattice covering with \( C \).

For a closed convex domain \( C \) let \( a(C) \) denote the area of \( C \), \( \delta(C) \) the infimum of the lower densities of all coverings of the plane by translates of \( C \) and \( h(C) \) the maximum area of a hexagon inscribed in \( C \). According to a general result of L. Fejes Tóth [4] (see also [1]) we have

\[
\delta(C) = \frac{a(C)}{h(C)}.
\]

This proves the truth of the above conjecture for centrally symmetric domains. For, if \( C \) is centrally symmetric then, by a theorem of Dowker [3], there is a centrally symmetric hexagon of area \( h(C) \) inscribed in \( C \). There is a lattice tiling of the plane by translates of this hexagon, and the corresponding translates of \( C \) provide a lattice covering with \( C \) with density \( \delta(C) = a(C)/h(C) \).

The proof of the inequality \( \delta(C) \geq a(C)/h(C) \) is based on a construction which associates with each domain from the covering a convex polygon inscribed in the respective domain such that these polygons form a tiling. Carrying out this construction for a lattice covering with \( C \) we obtain congruent centro-symmetric hexagons providing a lattice tiling of the plane. It immediately follows that the density of the thinnest lattice covering with \( C \) is equal to \( a(C)/h^*(C) \), where \( h^*(C) \) denotes the supremum of the areas of all centrally symmetric hexagons contained in \( C \). Thus the conjecture above can be reformulated as follows:

**Conjecture.** For any convex domain \( C \) in the plane we have

\[
\delta(C) = \frac{a(C)}{h^*(C)}.
\]