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On the error function in the asymptotic formula for the counting function of k-full numbers

by

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To the memory of Professor D. Suryanarayana

1. Introduction. Let k be an integer $\geqslant 2$. We say that a positive integer n is *k-full* if either $n = 1$ or for every prime p dividing n, p^k also divides n. We are interested in the Ω results for the error function mentioned in the title. It is nice to recall that the asymptotic formula was first considered by Erdős and Szekeres in 1935 ([8]).

2. Statement of the theorem. Let $a_n = a_n(k) = \{1$ if n is k-full; 0 otherwise$\}$. We write as usual $s = \sigma + it$. Then $F(s) = F_k(s) = \sum_{n=1}^{\infty} a_n/n^s$ has an analytic continuation in $\sigma > 1/(4k+4)$ and further, in the region $\sigma > 1/(2k)$, it has only simple poles at $1/k, 1/(k+1), \ldots, 1/(2k-1)$ (see § 5). For $k \leqslant j \leqslant 2k-1$, let $b_j x^{1/j}$ be the residue of $F(s)x^s/s$ at the pole $s = 1/j$. Let

$$M(x) = \sum_{k \leqslant j \leqslant 2k-1} b_j x^{1/j} \quad \text{and} \quad E(x) = \sum_{n<x} a_n - M(x).$$

Let ϱ_k be the greatest lower bound for $\{\varrho\colon E(x) = O(x^{\varrho})\}$. Then we prove roughly that $\varrho_k \geqslant \frac{r-1}{r(2k+r-1)}$ for any integer $r \geqslant 1$. Since the maximum of $(r-1)/r(2k+r-1)$ is obtained, for integer $r \geqslant 1$, at $[\sqrt{2k}]$ or $[\sqrt{2k}]+1$, we can assume that $r = [\sqrt{2k}]$ or $[\sqrt{2k}]+1$. More precisely we prove

THEOREM 1. *For suitable constants $B > 0$ and $C > 0$, we have*

$$\int_T^{\infty} \frac{|E(u)|^2}{u^{2\alpha+1}} \exp(-u/T^B)\, du \geqslant C \log T$$

where $\alpha = (r-1)/r(2k+r-1)$ for $k \geqslant 3$ and $\alpha = 1/10$ for $k = 2$.

In case, $2\alpha(k+r)$ is an integer, one can improve Theorem 1 to

THEOREM 2. *For suitable constants $B>0$ and $C>0$, we have*

$$\int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} \exp(-u/T^B)\,du \geqslant C\log^2 T$$

where $\alpha=(r-1)/r(2k+r-1)$ and $k\geqslant 3$.

COROLLARY. *If $k\geqslant 3$, then $E(x)=\Omega(x^{1/(2(k+r))}\sqrt{\log x})$ where r is the least integer such that $r(r-1)\geqslant 2k$.*

3. Remarks. We now record a few O and Ω results known in this direction. The earlier O-results are due to Erdős and Szekeres [8], Bateman and Grosswald [7], Krätzel [12] and Ivić [9], [10]. The best known results are due to Ivić and Shiu [11], who proved, among other things that $\varrho_k\leqslant 1/(2k)$ for $k\leqslant 13$. As is to be expected, these results could be improved if one assumes some unproved hypothesis. For example, it easily follows from the Lindelöf hypothesis for the Riemann zeta-function that $\varrho_k\leqslant 1/(2k)$ for all $k\geqslant 2$. Further, on the Riemann Hypothesis, we know that $\varrho_2\leqslant 13/81$ ([16]) which can be improved to $\varrho_2\leqslant 11/72$ ([2]). About Ω results, it was noted in [7] that $\varrho_2\geqslant(\operatorname{Re}\varrho)/6$ if $\zeta(\varrho)=0$, $\zeta(\varrho/2)\neq 0$ and $\zeta(\varrho/3)\neq 0$.

Now about the theorem. The method of proof extends to other cases also. What we prove amounts roughly to the following. If $F(s)$ is given by a Dirichlet's series, convergent in some half plane, and admits an analytic continuation and if $M(x)$ and $E(x)$ are defined suitably, then

$$\limsup \frac{|E(x)|}{x^\alpha} \geqslant 0$$

where α is such that $\int_T^{2T} |F(\alpha+it)|^2\,dt \gg T^2$ for all $T\geqslant 0$.

Thus this method can be used for

$$\zeta^2(s),\quad \frac{\zeta(as)\zeta(bs)}{\zeta(2bs)},\quad \frac{\zeta(as)\zeta(bs)}{\zeta(abs)},\quad \frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(bs)}$$

and other similar functions under suitable conditions on a and b. In this connection, we refer the reader to [3], [4] and [5] also.

4. Notations. Let $\{a_n\}$ be the special sequence

$$a_n=a_n(k)=1 \text{ if } n \text{ is } k\text{-full};\quad 0 \text{ otherwise}.$$

$$F(s)=F_k(s)=\sum_{n=1}^\infty a_n/n^s.$$

$\zeta(s)$ is the Riemann zeta function, p denotes a (general) prime. $r=[\sqrt{2k}]$ or $[\sqrt{2k}]+1$; $\alpha=(r-1)/r(2k+r-1)$; A is a big constant, not necessarily same at each occurrence. T is a real number sufficiently large. For any complex

number z, we denote the real part by $\operatorname{Re}z$ and the imaginary part by $\operatorname{Im}z$;

$y=T^B$ where B is a sufficiently large constant;

$J=\{T^{1-\alpha}\leqslant t\leqslant 2T$: for any complex number z with $\operatorname{Re}z\geqslant 1/(4k+3)$, and $|\operatorname{Im}z-t|\leqslant(\log T)^{20}$, $\prod_{2k+2\leqslant j\leqslant 4k+3}\zeta(jz)\neq 0\}$;

$J_1=\{T^{1-\alpha}\leqslant t\leqslant 2T$: for any complex number z with $\operatorname{Re}z\geqslant 1/(4k+3)$ and $|\operatorname{Im}z-t|\leqslant(\log T)^{15}$, $\prod_{2k+2\leqslant j\leqslant 4k+3}\zeta(jz)\neq 0\}$.

(An interpretation about J is given below Lemma 14.)

Note that, if $t\in J$, then $t+\omega\in J_1$ if $|\omega|\leqslant(\log T)^4$. Throughout the proof, we assume that $k\geqslant 3$. The case $k=2$ is similar and easy.

5. Analytic continuation. In this section, we deal with the analytic continuation of $F_k(s)$.

PROPOSITION 1. (a) *We have*

$$F_2(s)=\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}.$$

(b) *For $k\geqslant 3$,*

$$F_k(s)=\prod_{k\leqslant j\leqslant 2k-1}\zeta(js)\prod_{2k+2\leqslant j\leqslant 4k+3}(\zeta(js))^{-e_j}\prod_{2k+2\leqslant j\leqslant 4k+3}(\zeta(2js))^{e_j}G(s)$$

where e_j, $2k+2\leqslant j\leqslant 4k+3$, are suitable nonnegative integers and $G(s)$ is absolutely convergent in $\sigma>1/(4k+4)$.

Proof. Proposition 1(a) is standard. We now prove 1(b). One checks by direct verification that

$$(1+x^k+x^{k+1}+\ldots)\prod_{k\leqslant j\leqslant 2k-1}(1-x^j)=1+\sum_{n\geqslant 2k+2}c_n x^n$$

for suitable integers c_n. Here $c_n\leqslant 0$ for $n\leqslant 4k+3$ and since

$$(1+x^k+x^{k+1}+\ldots)(1-x^k)$$

is a polynomial, $c_n=0$ for all large n. Now

$$\Big(1+\sum_{n\geqslant 2k+2}c_n x^n\Big)\prod_{2k+2\leqslant n\leqslant 4k+3}(1+x^n)^{-c_n}=1+\sum_{n\geqslant 4k+4}d_n x^n$$

with $d_n=0$ for all large n. Hence

$$\begin{aligned}(1+x^k+x^{k+1}+\ldots)&=\prod_{k\leqslant j\leqslant 2k-1}(1-x^j)^{-1}\prod_{2k+2\leqslant j\leqslant 4k+3}(1+x^j)^{c_j}\Big(1+\sum_{n\geqslant 4k+4}d_n x^n\Big)\\&=\prod_{k\leqslant j\leqslant 2k-1}(1-x^j)^{-1}\prod_{2k+2\leqslant j\leqslant 4k+3}(1-x^j)^{-c_j}\\&\quad\times\prod_{2k+2\leqslant j\leqslant 4k+3}(1-x^{2j})^{c_j}\Big(1+\sum_{n\geqslant 4k+4}d_n x^n\Big).\end{aligned}$$

We now put $x = p^{-s}$ and take the product over all primes. This yields the proposition. Since d_n is bounded by a function of k, the absolute convergence of $G(s) = \prod_p (1 + \sum_{n \geq 4k+4} d_n p^{-ns})$ is guaranteed.

6. A trivial upper bound for $F(s)$. We recall the definition of J_1 (see § 4). We now prove

PROPOSITION 2. *If* $\operatorname{Re} s \geq 1/(4k+3)$ *and* $t \in J_1$ *then*

$$F(s) = O\big((|t|+2)^A\big) \quad \textit{for a suitable } A > 0.$$

We need

LEMMA 1. *There holds the following inequality: For any* $\varepsilon > 0$,

$$\frac{1}{\zeta(s)} = O_\varepsilon\big((|t|+2)^\varepsilon\big) \quad \textit{in } \operatorname{Re} s \geq \tfrac{1}{2}+\varepsilon,$$

provided $\zeta(z) \neq 0$ *for* $\operatorname{Re} z > \frac{1}{2}+\varepsilon$ *and* $|\operatorname{Im} z - \operatorname{Im} s| \leq (\log T)^5$ *and* $|\operatorname{Im} s| \geq 1$.

Proof. This is a standard result. One can refer to Theorem 14.2 of [17] where a similar result has been proved.

LEMMA 2. *The following inequalities are true*:

(a) *If* $\operatorname{Re} s > 0$ *and* $|\operatorname{Im} s| \geq 1$ *then* $\zeta(ls) = O\big((|t|+2)^A\big)$ *for a suitable* $A > 0$.

(b) *If* $\operatorname{Re} s \geq 1/(4k+3)$ *and* $l \geq 2k+2$ *and* $t \in J_1$, $1/\zeta(ls) = O\big((|t|+2)\big)$.

(c) *For* $\operatorname{Re} s \geq 1/(4k+3)$, $G(s)$ *is bounded by a function of* k.

Proof. (a) follows from Chapter 5 of [17]. (b) is a consequence of Lemma 1. (c) is true because of the absolute convergence of $G(s)$.

Now Proposition 2 follows from Proposition 1 and Lemma 2.

7. A mean value upper bound for $F(s)$. Since our aim is to prove the result that

$$\int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} \exp\left(-\frac{u}{y}\right) du \text{ is large,}$$

we can assume that

$$(*) \qquad \int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} \exp\left(-\frac{u}{y}\right) du \leq \log^2 T.$$

Our aim in this section is to prove, under the assumption $(*)$,

PROPOSITION 3. *We have*

$$\int_{\substack{J \\ \operatorname{Re} s = \alpha}} \frac{|F(s)|^2}{|s|^2} dt \ll 1 + \int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} e^{-2u/y} du.$$

LEMMA 3. *If* $0 \leq \operatorname{Re} z \leq 1$ *and* $|\operatorname{Im} z| \geq (\log T)^3$, *then*

$$\int_T^\infty \frac{e^{-u/y}}{u^z} du = -\frac{T^{1-z}}{1-z} + O(T^{-10}).$$

Proof. We have

$$\int_T^\infty \frac{e^{-u/y}}{u^z} du = \int_0^\infty \frac{e^{-u/y}}{u^z} du - \int_0^T \frac{du}{u^z} - \int_0^T \frac{(e^{-u/y}-1)}{u^z} du$$

$$= y^{1-z}\Gamma(1-z) - \frac{T^{1-z}}{1-z} + O\left(\int_0^T \frac{u/y}{u^{\operatorname{Re} z}} du\right)$$

and hence the result.

LEMMA 4. *Under the assumption* $(*)$, *there exists* T_0, $T \leq T_0 \leq 2T$ *such that*

(a) $\dfrac{E(T_0)\, e^{-T_0/y}}{T_0^\alpha} = O(\log T)$.

(b) $\dfrac{1}{y} \displaystyle\int_{T_0}^\infty \frac{|E(u)|\, e^{-u/y}}{u^\alpha} du = O(\log T)$.

Proof. Because of $(*)$,

$$\log^2 T > \int_T^{2T} \frac{|E(u)|^2}{u^{2\alpha+1}} \exp\left(-\frac{u}{y}\right) du$$

$$\geq \min_{T \leq u \leq 2T} \left(\frac{|E(u)|}{u^\alpha} \exp\left(-\frac{u}{y}\right)\right)^2 \int_T^{2T} \frac{du}{u}$$

and hence (a).

To prove (b), observe that

$$\left(\frac{1}{y}\int_{T_0}^{\infty}\frac{|E(u)|\,e^{-u/y}}{u^{\alpha}}du\right)^2 \leqslant \left(\int_{T_0}^{\infty}\frac{|E(u)|^2\,e^{-u/y}}{u^{2\alpha+1}}du\right)\left(\frac{1}{y^2}\int_{T_0}^{\infty}e^{-u/y}u\,du\right)$$

$$= O(\log^2 T).$$

Lemma 5. *If $t\in J$ and $\operatorname{Re}s=\alpha$, then*

$$F(s) = \sum_{n\leqslant T_0}\frac{a_n}{n^s}e^{-n/y} + s\int_0^1 \sum_{n>T_0}\frac{E(n+u)}{(n+u)^{s+1}}e^{-(n+u)/y} + O\left((\log T)^{20}\right).$$

Proof. We start with

$$\sum_{n=1}^{\infty}\frac{a_n e^{-n/y}}{n^s} = \frac{1}{2\pi i}\int_{\operatorname{Re}w=2} F(s+w)\,y^w\,\Gamma(w)\,dw.$$

Now we break off the portion of the integral $|\operatorname{Im}w|\geqslant(\log T)^4$ with a small error and move the line of integration to $\operatorname{Re}w=\dfrac{1}{4k+3}-\alpha$; now, using the estimate of $F(s)$ given in Proposition 2 (and assuming that B is large enough) and using the fact that, since $t\in J$, $t+\operatorname{Im}w\in J$, we see that the value of the integral on the horizontals $t+\sin w\in J_1$, as well as the vertical $\operatorname{Re}w=\dfrac{1}{4k+3}-\alpha$ is small. This proves that $\sum_{n=1}^{\infty}\dfrac{a_n}{n^s}e^{-n/y}$ equals nearly the sum of the residues inside the contour. Thus we have

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-n/y} + O(T^{-10}) = \sum_{n\leqslant T_0} + \sum_{n>T_0} + O(T^{-10}).$$

Now

$$\sum_{n>T_0}\frac{a_n}{n^s}e^{-n/y} = \int_{T_0}^{\infty}\frac{1}{u^s}e^{-u/y}\,d\Big(\sum_{n\leqslant u}a_n\Big) = \int_{T_0}^{\infty}\frac{1}{u^s}e^{-u/y}\,d\big(M(u)+E(u)\big)$$

$$= \int_{T_0}^{\infty}\frac{1}{u^s}e^{-u/y}M'(u)\,du + \int_{T_0}^{\infty}\frac{1}{u^s}e^{-u/y}\,d\big(E(u)\big)$$

$$= S_1 + S_2,\ \text{say}.$$

Since $M(u)$ is of the form $\sum c_j u^{1/j}$, the first integral S_1 is of the form considered in Lemma 3 and hence small. Now S_2 is (on integration by parts)

$$\left[\frac{E(u)\,e^{-u/y}}{u^s}\right]_{T_0}^{\infty} + s\int_{T_0}^{\infty}\frac{E(u)\,e^{-u/y}}{u^{s+1}}du + \frac{1}{y}\int_{T_0}^{\infty}\frac{E(u)\,e^{-u/y}}{u^s}du.$$

Hence, using Lemma 4,

$$S_2 = s\int_{T_0}^{\infty}\frac{E(u)}{u^{s+1}}e^{-u/y}du + O(\log T)$$

$$= s\sum_{n\geqslant T_0}\int_0^1\frac{E(n+u)}{(n+u)^{s+1}}e^{-(n+u)/y}du + O(\log T).$$

This proves the lemma.

Lemma 6. *We have, for any sequence of complex numbers b_n,*

$$\int_0^T |\sum_n b_n n^{it}|^2\,dt = \sum_n (T+O(n))(|b_n|^2),$$

provided the right side is convergent.

Proof. For the proof of this lemma, we refer the reader to [13] or [14].

Lemma 7. *We have*

$$\int_{\substack{J\\ \operatorname{Re}s=\alpha}} \left|\sum_{n\leqslant T_0}\frac{a_n}{n^s}e^{-n/y}\right|^2\frac{dt}{t^2} \ll 1.$$

Proof. It is sufficient to prove that

$$\int_{T^{1-\alpha}}^{2T} \left|\sum_{n\leqslant T_0}\frac{a_n}{n^s}e^{-n/y}\right|^2\frac{dt}{t^2} \ll 1.$$

By Lemma 6, we have

$$\int_{2^m}^{2^{m+1}} \left|\sum_{n\leqslant T_0}\frac{a_n}{n^{\alpha}}e^{-n/y}\right|^2 dt = \sum_{n\leqslant T_0}\left(2^m+O(n)\right)\frac{|a_n|^2}{n^{2\alpha}}$$

$$\ll \sum_{n<T_0}\frac{(2^m+n)}{n^{2\alpha}} \ll 2^m\,T_0^{1-2\alpha} + T_0^{2-2\alpha}.$$

Hence

$$\int_{2^m}^{2^{m+1}} \Big|\sum_{n\leqslant T_0} \frac{a_n}{n^s} e^{-n/y}\Big|^2 \frac{dt}{t^2} \ll \frac{T_0^{1-2\alpha}}{2^m} + \frac{T_0^{2-2\alpha}}{2^{2m}}.$$

Now summing over m, $T^{1-\alpha}/2 < 2^m \leqslant 4T$, it follows that

$$\int_{T^{1-\alpha}}^{2T} \Big|\sum_{n\leqslant T_0} \frac{a_n}{n^s} e^{-n/y}\Big|^2 \frac{dt}{t^2} \ll 1.$$

Lemma 8. *We have*

$$\int_{\substack{J\\ \mathrm{Re}\, s=\alpha}} \Big|\int_0^1 \sum_{n\geqslant T_0} \frac{E(n+u)e^{-(n+u)/y}}{(n+u)^{s+1}}\Big|^2 dt \ll \int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} e^{-2u/y}\, du.$$

Proof. By Hölder's inequality,

$$\Big|\int_0^1 \sum_{n\geqslant T_0} \frac{E(n+u)e^{-(n+u)/y}}{(n+u)^{s+1}} du\Big|^2 \leqslant \int_0^1 \Big|\sum_{n\geqslant T_0} \frac{E(n+u)e^{-(n+u)/y}}{(n+u)^{s+1}}\Big|^2 du.$$

Hence the integral on the left side of the lemma is bounded by

$$\int_0^{2T} dt \int_0^1 \Big|\sum_{n\geqslant T_0} \frac{E(n+u)}{(n+u)^{s+1}} e^{-(n+u)/y}\Big|^2 du.$$

We now interchange the order of integration and use Lemma 5. Hence the integral in question is bounded by

$$\int_0^1 \sum_{n\geqslant T_0} \frac{(2T+O(n))|E(n+u)|^2}{(n+u)^{2\alpha+2}} e^{-2(n+u)/y}\, du$$

$$\ll \int_0^1 \sum_{n\geqslant T_0} \frac{|E(n+u)|^2}{(n+u)^{2\alpha+1}} e^{-2(n+u)/y}\, du$$

$$= \sum_{n\geqslant T_0} \int_n^{n+1} \frac{|E(u)|^2}{u^{2\alpha+1}} e^{-2u/y}\, du \leqslant \int_T^\infty \frac{|E(u)|^2}{u^{2\alpha+1}} e^{-2u/y}\, du.$$

Now Proposition 3 follows from Lemmas 5, 7 and 8.

8. A mean value lower bound for $F(s)$. In this section, we prove

Proposition 4. (a) *If* $(k+r)\alpha$ *is not an integer then*

$$\int_{\substack{J\\ \mathrm{Re}\, s=\alpha}} \frac{|F(s)|^2}{|s|^2} dt \gg \log T.$$

(b) *If* $(k+r)\alpha$ *is an integer, then*

$$\int_{\substack{J\\ \mathrm{Re}\, s=\alpha}} \frac{|F(s)|^2}{|s|^2} dt \gg (\log T)^2.$$

First we give the proof of Proposition 4(b).

Lemma 9. *Let* $f(s) = \sum_{n=1}^{\infty} b_n/n^s$ *be convergent in some half plane and analytically continuable in* $\sigma > 1/(4k+4)$, $A \leqslant t \leqslant A+H$. *Then*

$$\int_A^{A+H} |f(\alpha+it)|^2\, dt \gg H \sum_{n\leqslant H^{1/2}} |b_n|^2/n^{2\alpha},$$

provided $H \geqslant (\log A)$, $b_1 = 1$ *and maximum of* $|f(s)|$ *in* $A \leqslant t \leqslant A+H$ *and* $\mathrm{Re}\, s \geqslant \alpha$ *is* $\leqslant e^A$.

Proof. This result, in slightly stronger forms, can be found in [6] and [15] and in a weaker form in [1] (Theorem 4). We define

$$c_j(n) = \frac{1}{n^{1-2\alpha j}}, \quad G_j(s) = \sum_{n=1}^{\infty} \frac{c_j(n)}{n^s},$$

$$f(s) = F(s) \prod_{k\leqslant j\leqslant k+r-1} (\zeta(js))^{-1} \prod_{k\leqslant j\leqslant k+r-1} G_j(js) = \sum b_n/n^s.$$

We are interested in the value of b_n, when n is a $(k+r)$-th power.

Lemma 10. *If* $n = \lambda^{k+r}$ *and* $\mu(\lambda) \neq 0$, *then* $b_n = 1$.

Proof. We have

$$f(s) = \prod_p \Big\{(1+p^{-ks}+p^{-(k+1)s}+\ldots) \times \prod_{\substack{j\\ k\leqslant j\leqslant k+r-1}} \Big(1-\frac{1}{p^{js}}\Big) \prod_{\substack{j\\ k\leqslant j\leqslant k+r-1}} \Big(1+\frac{c_j(p)}{p^{js}}+\ldots\Big)\Big\}$$

$$= \prod_p \Big(1+\frac{g(p)}{p^s}+\ldots\Big).$$

Clearly $g(p^{k+r}) = 1$ and this proves the result.

LEMMA 11. *There holds*

$$\int_{A}^{A+H} |f(\alpha+it)|^2\,dt \gg H\log A \quad \text{if} \quad H \geqslant A^{\varepsilon}.$$

Proof. By Lemma 9,

$$\int_{A}^{A+H} |f(\alpha+it)|^2\,dt \gg H \sum_{n\leqslant H^{1/2}} \frac{|b_n|^2}{n^{2\alpha}} \gg H \sum_{\lambda\leqslant H^{1/2(k+r)}} \frac{1}{\lambda} \gg H\log H \gg H\log A.$$

We now assume that $\prod_{k\leqslant j\leqslant k+r-1} \zeta(js) \neq 0$ for $\mathrm{Re}\, s = \alpha$ and $A \leqslant \mathrm{Im}\, s \leqslant A+H$. Then we have

LEMMA 12. *We have*

$$\int_{A}^{A+H} |F(\alpha+it)|^2\,dt \gg AH\log A \quad \text{if} \quad H \geqslant A^{\varepsilon}.$$

Proof. On $\mathrm{Re}\, s = \alpha$, we note that $|\zeta(1-js)| = |G_j(js)|$. Using the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, we have

$$|F(s)| = |f(s)|\Big|\prod_{\substack{j\\ k\leqslant j\leqslant k+r-1}} \zeta(js)\Big|\Big|\prod_{\substack{j\\ k\leqslant j\leqslant k+r-1}} (G_j(js))^{-1}\Big|$$

$$= |f(s)|\Big|\prod_j \chi(js)\Big|\Big|\prod_j \zeta(1-js)\Big|\Big|\prod_j G_j(js)^{-1}\Big|$$

$$= |f(s)|\prod_j |\chi(js)| \gg |f(s)|\prod_j |t|^{1/2-j\alpha} \gg |f(s)|\,|t|^{1/2}$$

and hence the lemma.

Now we need a few results about the set J. Define

$$J(x) = J\cap[x, 2x] \quad \text{for any } x,\ T^{1-\alpha} \leqslant x \leqslant 2T.$$

Let $N(x)$ be the number of zeros of $\prod_{k\leqslant j\leqslant 2k-1} \zeta(js)$ with $t\in[x, 2x]$ and $\mathrm{Re}\, s = \alpha$.

LEMMA 13. *The number of zeros of $\zeta(s)$ in $\sigma \geqslant 1/2+\varepsilon$ and $|t| \leqslant x$ is $O(x^{1-(11\varepsilon/10)})$.*

Proof. By Theorem 9.19 (B) of [17], the number of zeros of $\zeta(s)$ in $\sigma \geqslant \alpha$, $|t| \leqslant T$ is

$$O\big(T^{\frac{3}{2-\alpha}(1-\alpha)}(\log T)^5\big)$$

and hence the result.

LEMMA 14. *We have*

$$N(x) \ll x^{1-(11\varepsilon/10)}.$$

Proof. This follows from Lemma 13 and the definition of $N(x)$.

We now give an interpretation for $J(x)$. Consider the interval $[x, 2x]$. Corresponding to every zero $\varrho = \beta+i\gamma$, $\beta \geqslant 1/2+\varepsilon$, $x-(\log T)^{20} \leqslant \gamma \leqslant 2x+(\log T)^{20}$, delete the portion $[\gamma-(\log T)^{20}, \gamma+(\log T)^{20}]$ from the interval $[x, 2x]$. The rest gives us J. Hence $J(x)$ consists of at most $N(x)$ disjoint intervals and the total length of $J(x)$ is $\gg x$. Now from $J(x)$ delete the connected components whose length is $\leqslant x^{\varepsilon}$. The total length of the deleted portion is $O(N(x)x^{\varepsilon}) = O(x^{1-(\varepsilon/10)})$. Hence if $J_2(x)$ is the remaining portion, the total length of $J_2(x)$ is $\gg x$. Now applying Lemma 12 to each connected component of $J_2(x)$ and adding, we get

LEMMA 15. *We have*

$$\int_{J_2(x)} |F(\alpha+it)|^2\,dt \gg x^2\log T.$$

Hence there follows

$$\int_{\substack{J_2(x)\\ \mathrm{Re}\, s=\alpha}} \frac{|F(s)|^2}{|s|^2} \gg \log T$$

and consequently Proposition 4(b). Now Theorem 2 follows from Propositions 3 and 4(b).

The proof of Proposition 4(a) is similar and easy. We give the main steps only.

LEMMA 11 (a). *There holds*

$$\int_{A}^{A+H} |f(\alpha+it)|^2\,dt \gg H \quad \text{if} \quad H \geqslant A^{\varepsilon}.$$

Proof. The result follows from Lemma 9 since $b_1 = 1$.

LEMMA 12 (a). *We have*

$$\int_{A}^{A+H} |F(\alpha+it)|^2\,dt \gg AH.$$

LEMMA 15 (a). *We have*

$$\int_{J_1(x)} |F(\alpha+it)|^2\,dt \gg x^2.$$

This proves Proposition 4(a), which together with Proposition 3, proves Theorem 1.

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Note to a paper of Bambah, Rogers and Zassenhaus

by

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It is known [7] that the density of a packing of translates of a convex domain C cannot exceed the density of the densest lattice packing of C. It is conjectured ([5], p. 205) that an analogous statement holds for coverings: The density of a covering of the plane with translates of a convex domain C cannot be less than the density of the thinnest lattice covering with C.

For a closed convex domain C let $a(C)$ denote the area of C, $\vartheta(C)$ the infimum of the lower densities of all coverings of the plane by translates of C and $h(C)$ the maximum area of a hexagon inscribed in C. According to a general result of L. Fejes Tóth [4] (see also [1]) we have

$$\vartheta(C) \geqslant \frac{a(C)}{h(C)}.$$

This proves the truth of the above conjecture for centrally symmetric domains. For, if C is centrally symmetric then, by a theorem of Dowker [3], there is a centrally symmetric hexagon of area $h(C)$ inscribed in C. There is a lattice tiling of the plane by translates of this hexagon, and the corresponding translates of C provide a lattice covering with C with density $\vartheta(C) = a(C)/h(C)$.

The proof of the inequality $\vartheta(C) \geqslant a(C)/h(C)$ is based on a construction which associates with each domain from the covering a convex polygon inscribed in the respective domain such that these polygons form a tiling. Carrying out this construction for a lattice covering with C we obtain congruent centro-symmetric hexagons providing a lattice tiling of the plane. It immediately follows that the density of the thinnest lattice covering with C is equal to $a(C)/h^*(C)$, where $h^*(C)$ denotes the supremum of the areas of all centrally symmetric hexagons contained in C. Thus the conjecture above can be reformulated as follows:

Conjecture. *For any convex domain C in the plane we have*

$$\vartheta(C) = \frac{a(C)}{h^*(C)}.$$