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Rational approximations to the Rogers–Ramanujan continued fraction

by

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1. Introduction. Let $F(\alpha)$ be defined by

$$F(\alpha) = F(\alpha, x) = \sum_{n=0}^{\infty} \frac{\alpha^n x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)} \quad (|x| < 1).$$

Then $F(\alpha)$ satisfies

$$F(\alpha) = F(\alpha x) + \alpha x F(\alpha x^2),$$

so that $F(\alpha)/F(\alpha x)$ can be developed in the Rogers–Ramanujan continued fraction

(1) $$\frac{F(\alpha)}{F(\alpha x)} = 1 + \frac{\alpha x}{1} + \frac{\alpha x^2}{1} + \frac{\alpha x^3}{1} + \dots$$

In particular, by virtue of the Rogers–Ramanujan identities, we have

$$1 + \frac{x}{1} + \frac{x^2}{1} + \dots = \frac{\sum_{n=0}^{\infty} \frac{x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)}}{\sum_{n=0}^{\infty} \frac{x^{n^2+n}}{(1-x)(1-x^2)\dots(1-x^n)}} = \prod_{n=0}^{\infty} \frac{(1-x^{5n+2})(1-x^{5n+3})}{(1-x^{5n+1})(1-x^{5n+4})}.$$

(For details see for example [1], [5].) We put for brevity

$$f(\alpha, x) = F(\alpha)/F(\alpha x).$$

In 1971 Osgood [8], [9] proved that, if a, b, and d are non-zero integers with $|d| \geqslant 2$, then, for any $\varepsilon > 0$, there is a positive constant $q_0 = q_0(a, b, d, \varepsilon)$ such that

$$\left| f\left(\frac{a}{b}, \frac{1}{d}\right) - \frac{p}{q} \right| > q^{-2-\varepsilon}$$

for all integers p, q $(\geqslant q_0)$.

For the values of the exponential function at rational points more precise results have been obtained (cf. Bundschuh [2], Durand [4], Mahler [7], Shiokawa [10]): If a/b is a non-zero rational number, then there are explicit positive constants $B = B(a/b)$ and $C = C(a/b)$ such that

$$\left| e^{a/b} - \frac{p}{q} \right| > Cq^{-2-B/\log\log q}$$

for all integers p, q $(\geqslant 3)$. Especially, Davis [3] proved that, if b is a non-zero integer and

$$C = \begin{cases} 1/|b| & \text{if } b \text{ is even,} \\ 1/|4b| & \text{otherwise,} \end{cases}$$

then, for any $\varepsilon > 0$,

$$\left| e^{2/b} - \frac{p}{q} \right| < (C+\varepsilon)\, q^{-2} \frac{\log\log q}{\log q}$$

for infinitely many integers p, q, while there is a positive constant $q_0 = q_0(b, \varepsilon)$ such that

$$\left| e^{2/b} - \frac{p}{q} \right| > (C-\varepsilon)\, q^{-2} \frac{\log\log q}{\log q}$$

for all integers p, q $(\geqslant q_0)$.

Comparing these results, we see that it would be interesting to replace, if possible, the ε in Osgood's theorem stated above by a function of q. In this connection, we prove in this paper the following theorems.

THEOREM 1. *Let a, b, c, and d be non-zero integers with*

$$(2) \qquad |d| > |c|^2.$$

Then $f(a/b, c/d)$ is an irrational number, and furthermore, there is a positive constant $C = C(a, b, c, d)$ such that

$$\left| f\left(\frac{a}{b}, \frac{c}{d}\right) - \frac{p}{q} \right| > Cq^{-2-2A-B/\sqrt{\log q}}$$

for all integers p, q $(\geqslant q_0)$, where

$$A = \frac{\log|c|}{\log|d/c^2|}$$

and

$$B = \frac{\log|a^2 d| - A\log|b/a^2|}{\sqrt{\log|d/c^2|}}.$$

COROLLARY. *Let a, b, and d be non-zero integers with $|d| \geqslant 2$. Then there is a positive constant $C = C(a, b, d)$ such that*

$$\left| f\left(\frac{a}{b}, \frac{1}{d}\right) - \frac{p}{q} \right| > Cq^{-2-B/\sqrt{\log q}}$$

for all integers p, q $(\geqslant 2)$, where

$$B = \frac{\log|a^2 d|}{\sqrt{\log|d|}}.$$

Theorem 1 is in a sense best possible since we have the following theorem:

THEOREM 2. *Let a, b, and d be positive integers such that $(a, b) = 1$, $d \geqslant 2$, and a divides d, and let*

$$C = \begin{cases} \sqrt{\dfrac{b}{a}} & \text{if} \quad \left(\dfrac{a}{b}\right)^2 > d, \\ \sqrt{\dfrac{a}{bd}} & \text{otherwise.} \end{cases}$$

Then, for any $\varepsilon > 0$,

$$\left| f\left(\frac{a}{b}, \frac{1}{d}\right) - \frac{p}{q} \right| < (C+\varepsilon)\, q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

for infinitely many integers p, q $(\geqslant 0)$, while there is a positive constant $q_0 = q_0(a, b, d, \varepsilon)$ such that

$$\left| f\left(\frac{a}{b}, \frac{1}{d}\right) - \frac{p}{q} \right| > (C-\varepsilon)\, q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

for all integers p, q $(\geqslant q_0)$.

2. A lemma. We shall make use of the following lemma.

LEMMA. *Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that*

$$|a_n a_{n+1}| > 4 \quad (n \geqslant 1)$$

and

$$\sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1} = \sigma < \infty.$$

Define as usual $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geqslant 1)$ with $p_0 = q_{-1} = 0$, $p_{-1} = q_0 = 1$. Then $p_n/(a_2 a_3 \ldots a_n)$ and $q_n/(a_1 a_2 \ldots a_n)$ converge to finite non-zero limits, and they satisfy

$$e^{-4\sigma} < |p_n/(a_2 a_3 \ldots a_n)| < e^{2\sigma},$$

$$e^{-4\sigma} < |q_n/(a_1 a_2 \ldots a_n)| < e^{2\sigma},$$

so that the continued fraction

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots = \lim_{n\to\infty}\frac{p_n}{q_n}$$

is convergent.

For the proof see [6], § 4.4; [10].

To apply the lemma, we transform the continued fraction (1) by using the formula

$$\frac{b_1}{1}+\frac{b_2}{1}+\frac{b_3}{1}+\ldots = \frac{1}{\frac{1}{b_1}}+\frac{1}{\frac{b_1}{b_2}}+\frac{1}{\frac{b_2}{b_1 b_3}}+\frac{1}{\frac{b_1 b_3}{b_2 b_4}}+\ldots$$

(cf. [6], (2, 3, 24)) and obtain the regular continued fraction

(1′) $$f(\alpha, x) = 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots,$$

where

(3) $$a_{2k-1} = \alpha^{-1}x^{-1}, \quad a_{2k} = x^{-k} \quad (k \geqslant 1).$$

We note here that

(4) $$a_1 a_2 \ldots a_{2k-1} = \alpha^{-k}x^{-k^2}, \quad a_1 a_2 \ldots a_{2k} = \alpha^{-k}x^{-k^2-k} \quad (k \geqslant (1),$$

and hence

(5) $$\log|a_1 a_2 \ldots a_n| = -\tfrac{1}{4}n^2\log|x| - \tfrac{1}{2}n\log|\alpha x| + O(1).$$

3. Proof of Theorem 1. Let $\alpha = a/b$ and $x = c/d$ be as in Theorem 1. Then a_n, and hence, p_n, q_n are rational numbers for which $d_n p_n$, $d_n q_n$ are integers for all $n \geqslant 1$, where

$$d_{2k-1} = |a^k c^{k^2}|, \quad d_{2k} = |a^k c^{k^2+k}|,$$

so that

(6) $$\log d_n = \tfrac{1}{4}n\log|c| + \tfrac{1}{2}n\log|ac| + O(1).$$

Here and in what follows constants implied in O-symbols as well as positive constants m, n_0, c_0, c_1, … depend possibly on a, b, c, d (and ε in Section 4).

Since $a_n a_{n+1} = \alpha^{-1}x^{-n}$ $(n \geqslant 1)$ with $|x| < 1$, the series $\sum_{n=1}^{\infty}(a_n a_{n+1})^{-1}$ is absolutely convergent and there exists an integer $m \geqslant 1$ such that $|a_n a_{n+1}| > 4$ $(n \geqslant m)$. We may thus apply the lemma and find that the continued fraction

(7) $$\frac{1}{a_{n+1}}+\frac{1}{a_{n+2}}+\frac{1}{a_{n+3}}+\ldots = \theta_n, \quad \text{say,}$$

is convergent for each $n \geqslant m$ and

(8) $$e^{-6\sigma} < |a_{n+k+1}\theta_{n+k}| < e^{6\sigma}, \qquad e^{-6\sigma} < |a_{n+k+1}q_{n,k}/q_{n,k+1}| < e^{6\sigma} \quad (n \geqslant m,\ k \geqslant 1),$$

where $p_{n,k}/q_{n,k}$ is the kth convergent of the continued fraction (7) and $\sigma = \sum_{n=1}^{\infty}|a_n a_{n+1}|^{-1}$. Hence

$$\left|\theta_n - \frac{p_{n,k}}{q_{n,k}}\right| = \frac{1}{|q_{n,k}(q_{n,k+1}+\theta_{n+k+1}q_{n,k})|} < \frac{2}{|q_{n,k}^2 a_{n+k+1}|}$$

for all sufficiently large k. But using again the lemma with (5) and (6), we get

$$\frac{\log|q_{n,k}^2 a_{n+k+1}|}{\log|d_{n+k+1}q_{n,k}|} > 2 - \frac{2\log|c|}{\log|d|} - \frac{C_0}{k},$$

so that, for any $\varepsilon > 0$,

$$\left|\theta_n - \frac{d_{n+k}p_{n,k}}{d_{n+k}q_{n,k}}\right| < |d_{n+k}q_{n,k}|^{-2+2(\log|c|)/\log|d|+\varepsilon}$$

for all sufficiently large k. This establishes the irrationality of θ_n $(n \geqslant m)$, since $d_{n+k}p_{n,k}$, $d_{n+k}q_{n,k}$ are integers and $2(\log|c|)/\log|d| < 1$ by (2).

Now we may assume $p_m q_m \neq 0$, since at least one of $p_{n-1}q_{n-1}$, $p_n q_n$ is different from zero, because $a_n \neq 0$ $(n \geqslant 1)$. It follows from the formula $p_n = p_m q_{m,n-m} + p_{m-1}p_{m,n-m}$, $q_n = q_m q_{m,n-m} + q_{m-1}p_{m,n-m}$ that

$$\frac{p_n}{a_2 a_3 \ldots a_n} = \frac{p_m}{a_2 a_3 \ldots a_n}\,\frac{q_{m,n-m}}{a_{m+1}\ldots a_n}\left(1+\frac{p_{m-1}}{p_m}\,\frac{p_{m,n-m}}{q_{m,n-m}}\right),$$

$$\frac{q_n}{a_1 a_2 \ldots a_n} = \frac{q_m}{a_1 a_2 \ldots a_n}\,\frac{q_{m,n-m}}{a_{m+1}\ldots a_n}\left(1+\frac{q_{m-1}}{q_m}\,\frac{p_{m,n-m}}{q_{m,n-m}}\right).$$

By the lemma, quantities on the right-hand side above converge as $n \to \infty$ to finite limits which are different from zero, because of the fact that θ_m is irrational and $p_m q_m \neq 0$. Hence the continued fraction (1′) converges to $f(a/b, c/d)$, which, as is easily seen, is also irrational. Thus we have, using (5),

(9) $$\log|q_n| = \frac{n^2}{4}\log\left|\frac{d}{c}\right| + \frac{n}{2}\log\left|\frac{bd}{ac}\right| + O(1),$$

and so, using (6),

(10) $$\log\left|\frac{q_{n+1}}{d_{n+1}}\right| - \log\left|\frac{q_n}{d_n}\right| = \frac{n}{2}\log\left|\frac{d}{c^2}\right| + O(1).$$

Hence, noticing (2) and (8), we can choose $n_0 \geqslant m$ such that

(11) $$|\theta_n| < 1/2, \quad |q_{n-1}| < |q_n|, \quad |q_{n-1}/d_{n-1}| < |q_n/d_n| \quad (n \geqslant n_0).$$

Now let p, q be given non-zero integers. We may assume that $|q_{n_0}/d_{n_0}| < 4q$. Then by (10) and (11), there is an integer $n = n(q) \geqslant n_0$ such that

$$|q_{n-1}/d_{n-1}| \leqslant 4q < |q_n/d_n|. \tag{12}$$

By virtue of the formula $p_n q_{n-1} - p_{n-1} q_n = \pm 1$, at least one of $p_{n-1} q - q_{n-1} p$, $p_n q - q_n p$ is different from zero. Assume first that $p_n q - q_n p \neq 0$. Then we have

$$d_n q_n \left(f\left(\frac{a}{b}, \frac{c}{d}\right) - \frac{p}{q} \right) = \frac{d_n (p_n q - q_n p)}{q} + d_n \left(q_n f\left(\frac{a}{b}, \frac{c}{d}\right) - p_n \right),$$

where $|d_n (p_n q - q_n p)| \geqslant 1$ and

$$\left| d_n \left(q_n f\left(\frac{a}{b}, \frac{c}{d}\right) - p_n \right) \right| = \frac{d_n}{|q_{n+1} + \theta_{n+1} q_n|} \leqslant \frac{2d_n}{|q_n|} < \frac{1}{2q},$$

so that

$$\left| f\left(\frac{a}{b}, \frac{c}{d}\right) - \frac{p}{q} \right| > \tfrac{1}{2} q^{-1-(\log|d_n q_n|)/\log q}. \tag{13}$$

The same inequality will be obtained also in the case of $p_{n-1} q - q_{n-1} p \neq 0$.

It remains to estimate $|d_n q_n|$ from above in terms of q. Combining (3), (6), (9), and (12), we get

$$\log|d_n q_n| \leqslant \log q + \log(d_{n-1} d_n) + \log|a_n| + C_1$$
$$\leqslant \log q + \tfrac{1}{2} n^2 \log|c| + \tfrac{1}{2} n \log|a^2 d| + C_2.$$

Here it follows from (12) with (6) and (9) that

$$\frac{n^2}{4} \log\left|\frac{d}{c^2}\right| + \frac{n}{2} \log\left|\frac{b}{a^2}\right| - C_3 < \log q < \frac{n^2}{4} \log\left|\frac{d}{c^2}\right| + \frac{n}{2} \log\left|\frac{bd}{a^2 c^2}\right| + C_4,$$

so that

$$n = 2\sqrt{\log q}/\sqrt{\log|d/c^2|} + o(1),$$

and hence

$$n^2 \leqslant \frac{4 \log q}{\log|c/d^2|} - \frac{4\sqrt{\log q} \log|b/a^2|}{\sqrt{\log|c/d^2|} \log|c/d^2|} + C_5.$$

Therefore, we obtain

$$\frac{\log|d_n q_n|}{\log q} < 1 + A + \frac{B}{\sqrt{\log q}},$$

which together with (13) leads to Theorem 1.

4. Proof of Theorem 2. Let $\alpha = a/b$ and $x = 1/d$ as in Theorem 2. Then $f(a/b, 1/d)$ can be developed in the regular continued fraction

$$f\left(\frac{a}{b}, \frac{1}{d}\right) = 1 + \frac{1}{\frac{b}{a} d +} \frac{1}{d +} \frac{1}{\frac{b}{a} d^2 +} \frac{1}{d^2 +} \cdots \frac{1}{+\frac{b}{a} d^n +} \frac{1}{d^n +} \cdots,$$

whose partial denominators are positive integers, so that its convergents p_n/q_n $(n \geqslant 1)$ are just all the best approximations to $f(a/b, 1/d)$. Thus we have only to estimate

$$\left| f\left(\frac{a}{b}, \frac{1}{d}\right) - \frac{p_n}{q_n} \right| = \frac{1}{\left| 1 + \frac{\theta_{n+1}}{a_{n+1}} + \frac{q_{n-1}}{a_{n+1} q_n} \right|} \frac{1}{|q_n^2 a_{n+1}|}. \tag{14}$$

We note first that

$$\lim_{n\to\infty} \theta_{n+1}/a_{n+1} = \lim_{n\to\infty} q_{n-1}/(q_n a_{n+1}) = 0.$$

If $n = 2k$, then by (3)

$$\log a_{2k+1} = k \log d + \log(db/a).$$

But by (4)

$$k = \frac{\sqrt{\log q_{2k}}}{\sqrt{\log d}} - \frac{\log(db/a)}{2 \log d} + o(1),$$

and hence

$$\frac{\log a_{2k+1}}{\log q_{2k}} \geqslant \frac{\sqrt{\log d}}{\sqrt{\log q_{2k}}} + \tfrac{1}{2} \log(db/a) + o(1).$$

Similarly, we get

$$\frac{\log a_{2k}}{\log q_{2k-1}} \geqslant \frac{\sqrt{\log d}}{\sqrt{\log q_{2k-1}}} + \tfrac{1}{2} \log(a/b) + o(1).$$

(14) together with these estimates yields Theorem 2.

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Poincaré series and Kloosterman sums for SL(3, Z)

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Appendix: Estimation of SL(3, Z) Kloosterman sums by Michael Larsen

1. Introduction. For $z = x+iy$, $y > 0$, let

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1-e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}$$

by the Ramanujan cusp form of weight 12. Ramanujan conjectured that

$$\tau(n) = O(n^{11/2+\varepsilon})$$

for any $\varepsilon > 0$. This conjecture was proved by Deligne [2] in 1974. Actually, Deligne proved the more general result (Petersson conjecture) that

$$a(n) = O(n^{(k-1)/2+\varepsilon}) \tag{1.1}$$

where $a(n)$ is the nth Fourier coefficient of a holomorphic cusp form of weight k associated to a congruence subgroup of SL(2, Z).

Ramanujan's conjecture can be generalized to non-holomorphic cusp forms (Maass wave forms) associated to arithmetic discrete subgroups of GL(r, R), $r \geqslant 2$, and in this form, the conjecture is still open even for $r = 2$. We now briefly describe the generalized Ramanujan conjecture.

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