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On sign-changes in the remainder-term of the prime-number formula, IV

by

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1. The subject of this paper is to pursue some further questions concerning the oscillatory nature of the "Abel mean" of the remainder-term of the prime-number formula:

$$\Delta_5(x) = \sum_{n=2}^{\infty} (\Lambda(n)-1)e^{-n/x}, \qquad x > 0. \tag{1.1}$$

In contrast with part III of this cycle [3], our main results are unconditional.

Let $V_5(T)$ denote, as usual, the number of sign-changes of Δ_5 in the interval $(0, T]$.

THEOREM 1. *For T tending to infinity we have the estimate*

$$V_5(T) = o(\log^2 T). \tag{1.2}$$

Let us remark that (1.2) cannot be much improved without an additional information about the distribution of zeros of the Riemann zeta-function near the line

$$\sigma = \Theta := \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho. \tag{1.3}$$

For example, if $\Theta > 1/2$ and there exist a positive-valued function g, monotonically decreasing to zero, and a sequence of zeta-zeros $\varrho_m = \beta_m + i\gamma_m$, $m = 1, 2, \ldots$, such that

$$\beta_m \to \Theta, \quad \gamma_m \to \infty \quad \text{as } m \to \infty, \tag{1.4}$$

and if region $s = \sigma + it$,

$$\beta_m - g(\gamma_m) \leqslant \sigma \leqslant \Theta, \tag{1.5}$$

$$0 < t < \gamma_m/g(\gamma_m) \tag{1.6}$$

contains only one zero ϱ_m, then

$$V_5(T) = \Omega(g(\log T)\log^2 T). \tag{1.7}$$

This follows easily from the well-known explicit formula

$$\Delta_5(x) = \sum_{\varrho} \Gamma(\varrho) x^{\varrho} + O(1), \tag{1.8}$$

where Γ denotes the Euler gamma-function.

We shall deduce Theorem 1 from Theorem 5.1 of [3] and the following

THEOREM 2. *Suppose $\Theta > 1/2$. Then for every $\varepsilon > 0$ there exist two positive constants c_0 and T_0 depending on ε such that, for every $T > T_0$,*

$$\max_{T \leqslant x \leqslant (1+\varepsilon)T} |\Delta_5(x)| \geqslant c_0 T^{\Theta-\varepsilon}. \tag{1.9}$$

Our main tool in the proof is the power sum method due to P. Turán.

According to the considerations in Sections 11–14 of [3] we have, under the Riemann hypothesis, the inequality

$$\max_{T \leqslant x \leqslant (1+\varepsilon)T} |\Delta_5(x)| \geqslant c_1(\varepsilon)\sqrt{T}, \tag{1.10}$$

satisfied for every positive ε and sufficiently large T. Thus we can formulate a completely unconditional theorem concerning "large values" of Δ_5:

THEOREM 3. *For every positive ε there exist two positive constants c_2 and T_1 such that*

$$\max_{T \leqslant x \leqslant (1+\varepsilon)T} |\Delta_5(x)| \geqslant c_2\sqrt{T} \quad \textit{for} \quad T \geqslant T_1. \tag{1.11}$$

This theorem is for large T much stronger than the result of S. Knapowski and W. Staś [5], with regard both to the localization and to the lower estimate. However, our estimate (1.11) is ineffective.

2. For the reader's convenience we now state all necessary lemmas.

LEMMA 1. *Let m be a non-negative number and $z_1, z_2, \ldots, z_N$ complex numbers such that*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_N|, \tag{2.1}$$

$$|z_h| \geqslant 2N/(m+N). \tag{2.2}$$

Then there exists an integer ν with

$$m \leqslant \nu \leqslant m+N \tag{2.3}$$

such that

$$\Big|\sum_{i=1}^{N} b_i z_i^{\nu}\Big| \geqslant \left(|z_h| - \frac{2N}{3(2N+m)}\right)^{\nu} \left(\frac{1}{24e}\frac{N}{2N+m}\right)^{\nu} \min_{h \leqslant j \leqslant N} |b_1 + \ldots + b_j|. \tag{2.4}$$

This is Turán's second main theorem in the form given by S. Knapowski [4]. Let us notice that Knapowski has formulated a somewhat less precise

inequality, with the factor $(|z_h|/2)^{\nu}$ in place of $\left(|z_h| - \frac{2N}{3(2N+m)}\right)^{\nu}$, but in fact his proof leads to (2.4).

LEMMA 2. *Let $\lambda_j = \sigma_j + it_j$, $j = 1, \ldots, N$, denote complex numbers such that*

$$\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_N. \tag{2.5}$$

Let further

$$f(t) = \sum_{j=1}^{N} b_j e^{\lambda_j t}, \quad t \in R, \tag{2.6}$$

where b_j are arbitrary complex numbers.

Suppose $\varkappa$ and a are real numbers satisfying

$$\sigma_1 - \sigma_N \leqslant \varkappa, \tag{2.7}$$

and

$$a \geqslant 2e^{\varkappa}. \tag{2.8}$$

Then for every d, $0 < d \leqslant 1$, and every h, $1 \leqslant h \leqslant N$, we have

$$\max_{a \leqslant t \leqslant a+d} |f(t)| \geqslant e^{\sigma_h a - \varkappa d}\left(24e^{1+3e^{\varkappa}}\left(2+\frac{a}{d}\right)\right)^{-N} \min_{h \leqslant k \leqslant N} \Big|\sum_{j=1}^{k} b_j\Big|. \tag{2.9}$$

Proof. To prove (2.9) it suffices to apply Lemma 1 with

$$z_j := \exp\left(\frac{\lambda_j - \sigma_1}{N} d\right) \quad \text{and} \quad m := \frac{aN}{d}.$$

Let us only remark that, by virtue of (2.5), condition (2.1) is satisfied. Further, (2.7) and (2.8) imply (2.2) for all h.

Moreover,

$$\left(\frac{|z_h| - \frac{2N}{3(2N+m)}}{|z_h|}\right)^{\nu} \geqslant \left(1 - \frac{2e^{\varkappa} d}{3a}\right)^{\nu} \geqslant \exp\left(-\frac{4\nu e^{\varkappa} d}{3a}\right) \geqslant \exp(-3e^{\varkappa} N),$$

so that

$$\left(\frac{1}{24e}\frac{N}{2N+m}\right)^{\nu}\left(|z_h| - \frac{2N}{3(2N+m)}\right)^{\nu} \geqslant \left(24e^{1+3e^{\varkappa}}\left(2+\frac{a}{d}\right)\right)^{-N} |z_h|^{\nu}$$

$$\geqslant \left(24e^{1+3e^{\varkappa}}\left(2+\frac{a}{d}\right)\right)^{-N} e^{-(\sigma_1 - \sigma_h)(a+d)}$$

$$\geqslant \left(24e^{1+3e^{\varkappa}}\left(2+\frac{a}{d}\right)\right)^{-N} e^{\sigma_h a - \varkappa d} e^{-\sigma_1 a}$$

and the assertion follows.

LEMMA 3. *Let $\sigma \geqslant 1/2$, $T > 0$ and let $N(\sigma, T)$ denote the number of zeros $\varrho = \beta + i\gamma$ of the Riemann zeta-function in the rectangle $\sigma < \beta < 1$, $0 < |\gamma| < T$. Then*

(2.10) $$N(\sigma, T) \ll_\varepsilon T^{4\sigma(1-\sigma)+\varepsilon} \quad \textit{for every } \varepsilon > 0.$$

This is a well-known density estimate, first proved by F. Carlson [1]. Much better estimates are also known, but (2.10) is sufficient for our purposes.

3. Proof of Theorem 2. It suffices to prove the theorem for ε small enough. Let us assume that

(3.1) $$0 < \varepsilon < \Theta - 1/2$$

and let us fix a real number Θ_1 with

(3.2) $$1/2 < \Theta - \varepsilon < \Theta_1 < \Theta.$$

There exists $\lambda_0 \in (0, \varepsilon/2)$ such that

(3.3) $$\Big|\sum_{\beta > \Theta_1} \Gamma(\varrho) e^{\varrho\lambda_0}\Big| = 3\alpha > 0,$$

α depending on ε.

Let $\varrho_0 = \beta_0 + i\gamma_0$ denote a zeta zero with

(3.4) $$\Theta_1 < \beta_0 < \Theta$$

satisfying the condition

(3.5) $$\sum_{\Theta_1 < \beta < \beta_0} |\Gamma(\varrho)| e^{\beta\lambda_0} < \alpha.$$

Using (1.8) we can write

(3.6) $$\Delta_5(e^{t+\lambda_0}) = \sum_{\varrho} \Gamma(\varrho) e^{\varrho(t+\lambda_0)} + O(1) = F(t) + O(e^{\Theta_1 t}),$$

where

(3.7) $$F(t) = \sum_{\substack{\varrho = \beta + i\gamma \\ \beta > \Theta_1, |\gamma| < T}} a_\varrho e^{\varrho t},$$

(3.8) $$a_\varrho = \Gamma(\varrho) e^{\varrho\lambda_0}.$$

In order to find "large values" of F we apply Lemma 2 with the numbers $\varrho = \beta + i\gamma$, $\beta > \Theta_1$, $|\gamma| < T$, taken for λ_j's (arranged according to decreasing real parts), with the numbers a_ϱ as b_j's, and with $N = N(\Theta_1, T)$, $a = T$, $d = \varepsilon/2$, $\varkappa = 1/2$. Moreover, let h, $1 \leqslant h \leqslant N$, be the integer for which $\lambda_h = \varrho_0$. Then, owing to Lemma 2, we get

(3.9) $$\max_{T \leqslant t \leqslant T+\varepsilon/2} |F(t)| \gg e^{\beta_0 T} T^{-2N} B_{\varrho_0},$$

where

(3.10) $$B_{\varrho_0} = \min_{A} \Big|\sum_{\varrho \in A} a_\varrho\Big|,$$

and the minimum is taken over all sets of zeros contained in the rectangle $\sigma > \Theta_1$, $|\gamma| < T$ and containing all zeros $\varrho = \beta + i\gamma$ with $\beta > \beta_0$, $|\gamma| < T$.

Lemma 3 implies that, for sufficiently large T,

(3.11) $$N = N(\Theta_1, T) \leqslant T^{\delta_0}/\log T \quad \text{with} \quad 0 < \delta_0 < 1.$$

Thus

(3.12) $$T^{-2N} \geqslant e^{-2T^{\delta_0}} \geqslant e^{-\frac{\beta_0 - \Theta_1}{2} T}$$

for T large enough.

Further, (3.3), (3.5) and (3.10) imply

(3.13) $$B_{\varrho_0} \geqslant \Big|\sum_{\beta > \Theta_1} a_\varrho\Big| - \sum_{\substack{\beta > \Theta_1 \\ |\gamma| \geqslant T}} |a_\varrho| - \sum_{\substack{\Theta_1 < \beta < \beta_0 \\ |\gamma| < T}} |a_\varrho| \geqslant 3\alpha - O(e^{-T/2}) - \alpha \geqslant \alpha$$

for large T.

Estimates (3.9)–(3.13) yield

(3.14) $$\max_{T \leqslant t \leqslant T+\varepsilon/2} |F(t)| \gg e^{\frac{\Theta_1 + \beta_0}{2} T}.$$

Hence, using (3.6) and (3.14) we get

(3.15) $$\max_{T \leqslant t \leqslant T+\varepsilon} |\Delta_5(e^t)| \geqslant \max_{T \leqslant t \leqslant T+\varepsilon/2} |\Delta_5(e^{t+\lambda_0})| \gg_\varepsilon e^{\frac{\Theta_1 + \beta_0}{2} T} - O(e^{\Theta_1 T}) \gg e^{(\Theta - \varepsilon)T},$$

and the result follows.

4. Proof of Theorem 1. We may assume that $\Theta > 1/2$, since otherwise our assertion (and even more) follows from Theorem 5.1 of [3].

Let us fix $\varepsilon > 0$. We shall prove that the function $\Delta_5(e^t)$ changes sign at most εT times in every interval of the form $[T, T+1]$, for sufficiently large T. The theorem hence easily follows.

We shall use the same method as in the proof of Theorem 3.1 of [3].

As in [3], we consider the function

(4.1) $$G(z) = \Delta_5(e^z),$$

which is regular in the strip $|\text{Im } z| < \pi/2$.

Then for every $z = x + iy$, $|y| < \pi/4$, we have

(4.2) $$|G(z)| \ll e^{\Theta x}.$$

From Theorem 2 we know that there exists x_0 in the interval $[T, T+1]$, $T \geqslant T_0(\varepsilon)$, such that

$$|G(x_0)| \geqslant c_0(\varepsilon)\, e^{(\Theta-\varepsilon)T}. \tag{4.3}$$

Let τ denote the conformal mapping defined for $|w| < 1$ by the formula

$$\tau(w) = x_0 + \frac{1}{2}\log\frac{1+w}{1-w}. \tag{4.4}$$

Then there exists a real number r, $0 < r < 1$, independent of x_0 and ε, such that

$$[T, T+1] \subset \tau\big(K(0, r)\big). \tag{4.5}$$

The number of sign-changes of $\Delta_5(e^t)$ in the interval $[T, T+1]$ is less than or equal to the number $n_0(r)$ of zeros of the function

$$G_1(w) = G\big(\tau(w)\big) \tag{4.6}$$

in the disc $|w| < r$. Using the Jensen inequality and (4.2), (4.3), we get

$$\begin{aligned} n_0(r) &\ll_r \log\{\max_{|w|<(r+1)/2} |G_1(w)|/|G_1(0)|\} \\ &\leqslant \log\{c_3(r)\, e^{\Theta T}/c_0(\varepsilon)\, e^{(\Theta-\varepsilon)T}\} \ll \varepsilon T + O(1). \end{aligned} \tag{4.7}$$

The proof of Theorem 1 is thus complete.

Remark. The author wishes to avail himself of the opportunity to correct some misprints in part II of this cycle of papers (see [2]): in formula (3.10) $\max\limits_{1\leqslant\mu\leqslant k_m}$ has to be replaced by $\max\limits_{\substack{1\leqslant\mu\leqslant k_m\\ \mu\neq\nu}}$; in formula (3.20) replace lim by $\underline{\lim}$, and on page 73 (lines 2 and 7 from below) replace R^+ by R^*.

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