On sign-changes in the remainder-term of the prime-number formula, IV

by

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1. The subject of this paper is to pursue some further questions concerning the oscillatory nature of the “Abel mean” of the remainder-term of the prime-number formula:

\[ A_5(x) = \sum_{n=2}^{\infty} (A(n)-1) e^{-n/x}, \quad x > 0. \]

(1.1)

In contrast with part III of this cycle [3], our main results are unconditional.

Let \( V_5(T) \) denote, as usual, the number of sign-changes of \( A_5 \) in the interval \((0, T]\).

**Theorem 1.** For \( T \) tending to infinity we have the estimate

\[ V_5(T) = o(1/\log^2 T). \]

(1.2)

Let us remark that (1.2) cannot be much improved without an additional information about the distribution of zeros of the Riemann zeta-function near the line

\[ \sigma = \Theta := \sup \Re \zeta. \]

(1.3)

For example, if \( \Theta > 1/2 \) and there exist a positive-valued function \( g \), monotonically decreasing to zero, and a sequence of zeta-zeros \( \zeta_m = \beta_m + i\gamma_m \), \( m = 1, 2, \ldots \), such that

\[ \beta_m \to \Theta, \quad \gamma_m \to \infty \quad \text{as} \quad m \to \infty, \]

(1.4)

and if region \( s = \sigma + it \),

\[ \beta_m - g(\gamma_m) \leq \sigma \leq \Theta, \]

(1.5)

\[ 0 < t < \gamma_m / g(\gamma_m) \]

(1.6)

contains only one zero \( \zeta_m \), then

\[ V_5(T) = \Omega (g(\log T) \log^2 T). \]

(1.7)
This follows easily from the well-known explicit formula

\[ A_2(x) = \sum \Gamma(\theta) x^\theta + O(1), \]

where \( \Gamma \) denotes the Euler gamma-function.

We shall deduce Theorem 1 from Theorem 5.1 of [3] and the following

**Theorem 2.** Suppose \( \Theta > 1/2 \). Then for every \( \epsilon > 0 \) there exist two positive constants \( c_0 \) and \( T_0 \) depending on \( \epsilon \) such that, for every \( T > T_0 \),

\[
\max_{T \leq x \leq (1+\Theta)T} |A_2(x)| \geq c_0 T^{\Theta - \epsilon}.
\]

Our main tool in the proof is the power sum method due to P. Turán. According to the considerations in Sections 11–14 of [3], we have, under the Riemann hypothesis, the inequality

\[
\max_{T \leq x \leq (1+\Theta)T} |A_2(x)| \geq c_1 T \sqrt{T},
\]

satisfied for every positive \( \epsilon \) and sufficiently large \( T \). Thus we can formulate a completely unconditional theorem concerning “large values” of \( A_2 \):

**Theorem 3.** For every positive \( \epsilon \) there exist two positive constants \( c_2 \) and \( T_1 \) such that

\[
\max_{T \leq x \leq (1+\Theta)T} |A_2(x)| \geq c_2 T^{1/2} \quad \text{for} \quad T \geq T_1.
\]

This theorem is for large \( T \) much stronger than the result of S. Knapowski and W. Stas [5], with regard both to the localization and to the lower estimate. However, our estimate (1.11) is ineffective.

2. For the reader’s convenience we now state all necessary lemmas.

**Lemma 1.** Let \( m \) be a non-negative number and \( z_1, z_2, \ldots, z_N \) complex numbers such that

\[
1 = |z_1| \geq |z_2| \geq \cdots \geq |z_{m-N}|,
\]

\[
|z_m| \geq 2N/(m+N).
\]

Then there exists an integer \( \nu \) with

\[
m \leq \nu \leq m+N
\]

such that

\[
|\sum_{r=1}^N b_r z_r| \geq \left( \frac{2N}{3(2N+m)} \right)^\nu \left( \frac{1}{24e^{2N+m}} \right)^\nu \min_{h \leq s \leq N} |b_1 + \cdots + b_s|.
\]

This is Turán’s second main theorem in the form given by S. Knapowski [4]. Let us notice that Knapowski has formulated a somewhat less precise

inequality, with the factor \( (|z_m|/2)^\nu \) in place of \( \left( |z_m| - \frac{2N}{3(2N+m)} \right)^\nu \), but in fact his proof leads to (2.4).

**Lemma 2.** Let \( \lambda_j = \sigma_j + it_j, \ j = 1, \ldots, N, \) denote complex numbers such that

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N.
\]

Let further

\[
f(t) = \sum_{j=1}^N b_j e^{\lambda_j t}, \quad t \in \mathbb{R},
\]

where \( b_j \) are arbitrary complex numbers.

Suppose \( \xi \) and \( \eta \) are real numbers satisfying

\[
|\sigma_j - \sigma_N| \leq \xi,
\]

and

\[
a \geq 2e^\eta.
\]

Then for every \( d, 0 < d \leq 1 \), and every \( h, 1 \leq h \leq N, \) we have

\[
\max_{0 \leq k \leq h} |f(t)| \geq e^{\eta|\sigma_j - \sigma_N|} \left( 24e^{1+\eta\varepsilon} \left( 2 + \frac{a}{d} \right) \right)^{-N} \min_{0 \leq k \leq h} \sum_{j=1}^N |b_j|.
\]

**Proof.** To prove (2.9) it suffices to apply Lemma 1 with

\[
z_j := \exp \left( \frac{\lambda_j - \sigma_N}{N} \right) \quad \text{and} \quad m := \frac{aN}{d}.
\]

Let us only remark that, by virtue of (2.5), condition (2.1) is satisfied. Further, (2.7) and (2.8) imply (2.2) for all \( h \).

Moreover,

\[
\left( \frac{|z_m| - \frac{2N}{3(2N+m)}}{|z_m|} \right)^\nu \left( \frac{1}{24e^{2N+m}} \right)^\nu \geq \left( 1 - \frac{2e^\eta d}{3a} \right)^\nu \geq \exp \left( -\frac{4\varepsilon^d}{3a} \right) \geq \exp(-3\varepsilon N),
\]

so that

\[
\left( \frac{1}{24e^{2N+m}} \right)^\nu \left( \frac{|z_m| - \frac{2N}{3(2N+m)}}{|z_m|} \right)^\nu \geq \left( 24e^{1+\eta\varepsilon} \left( 2 + \frac{a}{d} \right) \right)^{-N} \left| \sum_{r=1}^N b_r z_r \right|.
\]

and the assertion follows.
Lemma 3. Let $\sigma \geq 1/2$, $T > 0$ and let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + iy$ of the Riemann zeta-function in the rectangle $\sigma < \beta < 1$, $0 < |y| < T$. Then

$$N(\sigma, T) \ll T^{4\epsilon \theta(1-\sigma)^{-\epsilon}}$$

for every $\epsilon > 0$.

This is a well-known density estimate, first proved by F. Carlson [1]. Much better estimates are also known, but (2.10) is sufficient for our purposes.

3. Proof of Theorem 2. It suffices to prove the theorem for $\epsilon$ small enough. Let us assume that

$$0 < \epsilon < \Theta - 1/2$$

and let us fix a real number $\Theta_1$ with

$$1/2 < \Theta - \epsilon < \Theta_1 < \Theta.$$ 

There exists $\lambda_0 \in (0, \epsilon/2)$ such that

$$|\sum_{\rho : \beta > \lambda_1} \Gamma(\rho) e^{\epsilon \rho}| = 3\lambda > 0,$$

$\lambda$ depending on $\epsilon$.

Let $\rho_0 = \beta_0 + iy_0$ denote a zeta zero with

$$\Theta_1 < \beta_0 < \Theta$$

satisfying the condition

$$\sum_{\rho : \beta > \lambda_1} |\Gamma(\rho)| e^{\epsilon \rho} < \lambda.$$ 

Using (3.1) we can write

$$A_5(e^{\Theta_1 + \lambda_0}) = \sum_{\rho : \beta > \lambda_1} \Gamma(\rho) e^{\rho(t + \lambda_0)} + O(1) = F(t) + O(e^{\Theta_1 t}),$$

where

$$F(t) = \sum_{\rho : \beta > \lambda_1, |\rho| < T} a_{\rho} e^{\rho t},$$

$$a_{\rho} = \Gamma(\rho) e^{\rho t}.$$ 

In order to find “large values” of $F$ we apply Lemma 2 with the numbers $\rho = \beta + iy$, $\beta > \Theta_1$, $|\rho| < T$, taken for $\lambda_1$'s (arranged according to decreasing real parts), with the numbers $a_{\rho}$ as $h$'s, and with $N = N(\Theta_1, T)$, $a = T$, $d = \epsilon/2$, $\kappa = 1/2$. Moreover, let $h, 1 < h \ll N$, be the integer for which $h = \lambda_0$. Then, owing to Lemma 2, we get

$$\max_{T \leq t \leq T + \epsilon} |F(t)| \gg e^{\eta T} T^{-2N} B_{\epsilon_0},$$

where

$$B_{\epsilon_0} = \min_{A} \sum_{t \in A} a_{\rho},$$

and the minimum is taken over all sets of zeros contained in the rectangle $\sigma > \Theta_1$, $|\rho| < T$ and containing all zeros $\rho = \beta + iy$ with $\beta > \beta_0$, $|\rho| < T$.

Lemma 3 implies that, for sufficiently large $T$,

$$N = N(\Theta_1, T) \ll T^\delta \log T \quad \text{with} \quad 0 < \delta_0 < 1.$$ 

Thus

$$T^{-2N} \gg e^{-2T^0} T^{-2N} \gg e^{-\frac{\beta_0 - \Theta_1}{2} T}$$

for $T$ large enough.

Further, (3.3), (3.5) and (3.10) imply

$$B_{\epsilon_0} \gg \sum_{\beta > \Theta_1} \sum_{|\rho| < T} \sum_{\beta > \beta_0} \sum_{|\rho| < T} a_{\rho} e^{\epsilon \rho} \gg 3\lambda - O(e^{-\Theta_1 \epsilon}) \gg \alpha$$

for large $T$.

Estimates (3.9)-(3.13) yield

$$\max_{T \leq t \leq T + \epsilon} |F(t)| \gg e^{\eta T} T^{-2N} B_{\epsilon_0},$$

Hence, using (3.6) and (3.14) we get

$$\max_{T \leq t \leq T + \epsilon} |A_5(e^{\Theta_1 + \lambda_0})| \gg e^{\eta T} T^{-2N} - O(e^{\Theta_1 T}) \gg e^{\Theta_1 T},$$

and the result follows.

4. Proof of Theorem 1. We may assume that $\Theta > 1/2$, since otherwise our assertion (and even more) follows from Theorem 5.1 of [3].

Let us fix $\epsilon > 0$. We shall prove that the function $A_5(e^z)$ changes sign at most $\epsilon T$ times in every interval of the form $[T, T+1]$, for sufficiently large $T$. The theorem hence easily follows.

We shall use the same method as in the proof of Theorem 3.1 of [3].

As in [3], we consider the function

$$G(z) = A_5(e^z),$$

which is regular in the strip $|\text{Im} z| < \pi/2$.

Then for every $z = x + iy$, $|y| < \pi/4$, we have

$$|G(z)| < e^{\pi x}.$$
From Theorem 2 we know that there exists $x_0$ in the interval $[T, T+1]$, $T \geq T_0(\varepsilon)$, such that

$$|G(x_0)| \geq c_0(\varepsilon)e^{\theta - nT}.$$  

Let $\tau$ denote the conformal mapping defined for $|w| < 1$ by the formula

$$\tau(w) = x_0 + \frac{1}{2} \log \frac{1 + w}{1 - w}.$$  

Then there exists a real number $r$, $0 < r < 1$, independent of $x_0$ and $\varepsilon$, such that

$$[T, T+1] = \tau(K(0, r)).$$  

The number of sign-changes of $A_\varepsilon(\varepsilon)$ in the interval $[T, T+1]$ is less than or equal to the number $n_0(r)$ of zeros of the function

$$G_1(w) = G(\tau(w))$$  

in the disc $|w| < r$. Using the Jensen inequality and (4.2), (4.3), we get

$$n_0(r) \leq \log \left| \max_{|w| < (r+1)/2} |G_1(w)|/|G_1(0)| \right|$$

$$\leq \log \left| c_3(r)e^{\theta r}/c_0(\varepsilon)e^{\theta - nT} \right| \leq \varepsilon T + O(1).$$

The proof of Theorem 1 is thus complete.

Remark. The author wishes to avail himself of the opportunity to correct some misprints in part II of this cycle of papers (see [2]): in formula (3.10) $\max$ has to be replaced by $\max$; in formula (3.20) replace $\lim_{1 \leq p < k_{\eta}}$ by $\lim$, and on page 73 (lines 2 and 7 from below) replace $R^+$ by $R^*$.

References


