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Weighted restricted partitions*

by

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In this paper we investigate the powers of the function

$$\varphi = \prod \frac{1-x^{nq}}{1-x^n} \tag{1}$$

which enumerates partitions into parts not divisible by q, where q is a positive integer. We set

$$\varphi^s = \sum q_s(n) x^n$$

where s is an integer. Here as in what follows all products are extended from 1 to ∞ and all sums from 0 to ∞, unless otherwise indicated. The positive powers of φ then enumerate certain weighted partitions in which the parts occurring in a partition are counted in a composition like way. Thus if $a_1, a_2, \ldots$ are the integers not divisible by q, then the number of solutions of

$$a_1x_1 + a_2x_2 + \ldots = n$$

in non-negative integers x_i is just $q_1(n)$; while if $s > 0$ then the number of solutions of

$$a_1 \sum_{j=1}^{s} x_{1j} + a_2 \sum_{j=1}^{s} x_{2j} + \ldots = n$$

in non-negative integers x_{ij} is just $q_s(n)$. In the latter expression the order in which the variables x_{ij} occur for a fixed $i \geqslant 1$ and $1 \leqslant j \leqslant s$ is relevant, which makes clear the combined partition-composition nature of $q_s(n)$. For another interpretation of weighted partitions see Petersson [5].

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It is quite simple to show by logarithmic differentiation that if $\sigma(k)$ denotes the sum of the divisors of k, then $q_s(n)$ for all integral n satisfies the recurrence

$$nq_s(n) = s\sum_{k=1}^{n}\left\{\sigma(k)-q\sigma\left(\frac{k}{q}\right)\right\}q_s(n-k),$$

which is useful for numerical calculation. Further, if $s = s_1+s_2$ then for all integral n

$$q_s(n) = \sum_{k=0}^{n} q_{s_1}(k)\,q_{s_2}(n-k).$$

In this article however we will be interested in determining linear recurrence relations of length *independent of* n for the coefficients $q_s(n)$, of which the following is typical:

$$pq_1(np+\nu)+(-1)^{\nu}q_1\left(\frac{n}{p}\right) = 2^{(p-1)/2}q_p(n),$$

where $q = 2$, p is a prime such that $5 \leqslant p \leqslant 23$, and $\nu = \frac{1}{24}(p^2-1)$.

We shall permit s to be negative, and indeed for $q = 2$ the coefficients so defined possess a simple number-theoretic meaning since

$$\prod\left\{\frac{1-x^{2n}}{1-x^n}\right\}^{-1} = \prod(1-x^{2n-1}),$$

enumerates (apart from sign) partitions into distinct odd parts.

The method we use is the "subgroup" method, which deals with functions invariant with respect to the substitutions of a subgroup of the modular group Γ, regarded as linear fractional transformations. Γ is the group of rational integral matrices

$$A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

of determinant 1. We shall be concerned with the subgroups $\Gamma_0(m)$, m a positive integer, defined as follows: $A\in\Gamma_0(m)$ if and only if $m|c$.

If τ is a complex number, then by $A\tau$ we shall mean

$$\frac{a\tau+b}{c\tau+d}.$$

The basic modular function we will be interested in is best described in terms of the Dedekind modular form

$$\eta(\tau) = \exp\frac{i\pi\tau}{12}\prod(1-x^n), \qquad x = \exp 2\pi i\tau$$

whose properties are well-known and have been studied extensively (see [7] for a good discussion of this function). In particular, for the generators

$$S = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$$

of Γ we have that

(2) $$\eta(S\tau) = \exp\frac{i\pi}{12}\,\eta(\tau),$$

(3) $$\eta(T\tau) = (-i\tau)^{1/2}\,\eta(\tau).$$

We set

$$B(\tau) = \frac{\eta(q\tau)}{\eta(\tau)} = x^t\varphi$$

where

$$t = \tfrac{1}{24}(q-1)$$

and $\varphi = \varphi(\tau)$ is defined by (1). Then $B(\tau)$ is an entire modular function of level q. We also set

$$g(\tau) = g_{r,s}(\tau) = B^r(\tau)B^s(p\tau)$$

and assume from now on that q is a prime, r and s are integers, and p is a prime > 3 which is different from q.

Then

$$g(\tau) = x^{\delta}\varphi^r(\tau)\varphi^s(p\tau), \qquad \text{where} \qquad \delta = t(r+sp).$$

We shall require the transformation formulas for $B(\tau)$, which may be derived from those of $\eta(\tau)$. These have been treated by Rademacher and Whiteman in their paper [6], in terms of the associated Dedekind sums $s(a, c)$. In addition the author has shown in [1] that if

$$A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}, \quad a > 0, \quad c > 0, \quad (a, 6) = 1$$

then

(4) $$s(a, c)-\frac{a+d}{12c} \equiv \frac{1}{12}a(c-b-3)-\frac{1}{2}\left(1-\left(\frac{c}{a}\right)\right) \pmod 2,$$

where (c/a) is the generalized Legendre-Jacobi symbol. From (4) we prove

LEMMA 1. *Suppose that*

$$A = \begin{pmatrix} a & b\\ qc & d\end{pmatrix}\in\Gamma_0(q), \quad c > 0, \quad (a, 6) = 1.$$

Then

(5) $$B(A\tau) = \left(\frac{q}{|a|}\right)\exp 2\pi ita(b+c)\cdot B(\tau).$$

Proof. Set

$$A_0 = \begin{pmatrix} a & qb \\ c & d \end{pmatrix} \epsilon \Gamma.$$

Then by the transformation formula for $\eta(\tau)$, we have that

$$B(A\tau) = \frac{\eta(A_0 q\tau)}{\eta(A\tau)} = \exp(-i\pi\lambda)B(\tau),$$

where

$$\lambda = \left\{s(a, c) - \frac{a+d}{12c}\right\} - \left\{s(a, qc) - \frac{a+d}{12qc}\right\}.$$

Suppose first that $a > 0$. Then (4) implies that

$$\lambda \equiv \frac{1}{12}a(c-qb-3) - \frac{1}{2}\left(1-\left(\frac{c}{a}\right)\right) - \frac{1}{12}a(qc-b-3) + \frac{1}{2}\left(1-\left(\frac{qc}{a}\right)\right) \pmod 2,$$

so that

$$\lambda \equiv -2act - 2abt - \frac{1}{2}\left(1-\left(\frac{q}{a}\right)\right) \pmod 2,$$

which implies (5). If $a < 0$ we use instead the matrix

$$\tilde{A} = \begin{pmatrix} -a & b \\ qc & -d \end{pmatrix} \epsilon \Gamma_0(q)$$

and notice that for the associated $\tilde{\lambda}$,

$$\tilde{\lambda} = -\lambda$$

since $s(-a, c) = -s(a, c)$. The lemma is thus proved. Furthermore, (2) implies that

(6) $$B(S\tau) = \exp 2\pi it\cdot B(\tau);$$

and defining the transformation T_q by

$$T_q f(\tau) = f(-1/q\tau),$$

(3) implies that

(7) $$T_q B(\tau) = q^{-1/2}B^{-1}(\tau).$$

We may rewrite $g(\tau)$ as

$$g(\tau) = \left\{\frac{\eta(p\tau)}{\eta(\tau)}\right\}^{-s}\left\{\frac{\eta(q\tau)}{\eta(\tau)}\right\}^{r}\left\{\frac{\eta(pq\tau)}{\eta(\tau)}\right\}^{s}.$$

In this form we at once obtain from Theorem 1 of [1]

LEMMA 2. *$g(\tau)$ is an entire modular function on $\Gamma_0(pq)$, provided that δ and*

$$\varepsilon = \tfrac{1}{2}(r+s)$$

are integers.

We assume from now on that this is indeed so.

If $f = f(r, s)$ is a function of the variables r, s then we will denote $f(-r, -s)$ by f^-. Thus $\delta^- = -\delta$, $\varepsilon^- = -\varepsilon$.

Since p and q are relatively prime, integers p_0 and q_0 may be determined so that

$$p_0 p - q_0 q = 1.$$

Then

$$R = \begin{pmatrix} 1 & -1 \\ -q_0 q & p_0 p \end{pmatrix} \epsilon \Gamma_0(q).$$

It is known (see [2]) that $\Gamma_0(pq)$ is of index $p+1$ in $\Gamma_0(q)$, and that

$$R_k = \begin{pmatrix} 1 & 0 \\ -kq & 1 \end{pmatrix}, \quad 0 \leqslant k \leqslant p-1, \quad R_p = R$$

forms a complete set of right coset representatives for $\Gamma_0(q)$ modulo $\Gamma_0(pq)$. Thus by Theorem (2.2) of [4] and Lemma 2 we have

LEMMA 3. *The function*

$$G(\tau) = G_{r,s}(\tau) = \sum_{k=0}^{p} g(R_k\tau)$$

is an entire modular function on $\Gamma_0(q)$.

Since q is prime the fundamental region of $\Gamma_0(q)$ has only the parabolic points $\tau = i\infty$, $\tau = 0$; and since $G(\tau)$ is an entire modular function its singularities (if any) occur only at these points and become polar when measured in the proper uniformizing variables.

At $\tau = i\infty$ the uniformizing variable is $x = \exp 2\pi i\tau$. To study the singularity at $\tau = 0$ however we find it convenient to form the function $T_q G(\tau)$, which is also an entire modular function on $\Gamma_0(q)$ (see [3]), and study its singularity at $\tau = i\infty$. We make the remark that the transformation T_q permutes the parabolic vertices $\tau = i\infty$, $\tau = 0$.

We consider $\tau = i\infty$ first. Suppose that $1 \leqslant k \leqslant p-1$. Then

$$pR_k\tau = \begin{pmatrix} p & 0 \\ -kq & 1 \end{pmatrix}\tau.$$

Since $(p, 6kq) = 1$ integers k_0, k_1 may be determined so that

$$\begin{pmatrix} p & 0 \\ -kq & 1 \end{pmatrix} = \begin{pmatrix} -p & 24k_0 \\ kq & -k_1 \end{pmatrix}\begin{pmatrix} -1 & -24k_0 \\ 0 & -p \end{pmatrix},$$

where

$$pk_1 - 24qkk_0 = 1.$$

Thus k and k_0 simultaneously run over a reduced set of residues modulo p. Setting

$$A_k = \begin{pmatrix} -p & 24k_0 \\ kq & -k_1 \end{pmatrix}$$

we have that

$$g(R_k\tau) = B^r(R_k\tau)B^s\left(A_k\frac{\tau+24k_0}{p}\right);$$

and Lemma 1 implies easily that

$$g(R_k\tau) = \left(\frac{q}{p}\right)^s B^r(\tau)B^s\left(\frac{\tau+24k_0}{p}\right). \tag{8}$$

Writing $k:p$ in a summation to indicate that k runs over a reduced set of residues modulo p, (8) implies that

$$\sum_{k:p} g(R_k\tau) = \left(\frac{q}{p}\right)^s B^r(\tau)\sum_{k:p} B^s\left(\frac{\tau+24k}{p}\right). \tag{9}$$

In addition,

$$pR_p\tau = \begin{pmatrix} p & -p \\ -q_0q & p_0p \end{pmatrix}\tau = \begin{pmatrix} p & -1 \\ -q_0q & p_0 \end{pmatrix}\frac{\tau}{p} = A_p\frac{\tau}{p},$$

and $A_p \in \Gamma_0(q)$. Then

$$g(R_p\tau) = B^r(R_p\tau)B^s\left(A_p\frac{\tau}{p}\right)$$

and once again Lemma 1 may be used to give

$$g(R_p\tau) = \left(\frac{q}{p}\right)^s B^r(\tau)B^s\left(\frac{\tau}{p}\right). \tag{10}$$

Combining (9) and (10) we obtain

LEMMA 4. *We have*

$$G(\tau) = B^r(\tau)\left\{B^s(p\tau) + \left(\frac{q}{p}\right)^s\sum_{k=0}^{p-1} B^s\left(\frac{\tau+24k}{p}\right)\right\}. \tag{11}$$

We now go on to $\tau = 0$. We have that

$$R_kT = TS^{qk}, \quad 0 \leqslant k \leqslant p-1.$$

Then

$$g(R_kT\tau) = g(TS^{qk}\tau) = B^r\big(T(\tau+qk)\big)B^s\left(T\frac{\tau+qk}{p}\right)$$

and since k and $24k$ simultaneously run over a complete set of residues modulo p, we find from (7) that

$$T_q\sum_{k=0}^{p-1} g(R_k\tau) = q^{-s}B^{-r}(\tau)\sum_{k=0}^{p-1} B^{-s}\left(\frac{\tau+24k}{p}\right). \tag{12}$$

Further, we have that

$$pR_pT\tau = A_pTp\tau$$

from which it is easy to find by Lemma 1 and (7) that

$$T_qg(R_p\tau) = \left(\frac{q}{p}\right)^s q^{-s}B^{-r}(\tau)B^{-s}(p\tau). \tag{13}$$

Combining (12) and (13) and taking (11) into account we have

LEMMA 5. *The function $G(\tau)$ satisfies*

$$T_qG(\tau) = \left(\frac{q}{p}\right)^s q^{-s}G^{-}(\tau). \tag{14}$$

Thus we need only know the Fourier expansion of $G(\tau)$ at $\tau = i\infty$ in order to have complete information about its singularities. We make the special choice

$$r = -sp.$$

Then $\delta = 0$ and $\varepsilon = -\frac{1}{2}s(p-1)$ is an integer, since p is odd. The conditions of Lemma 2 therefore are satisfied for all integral s. $G(\tau)$ is now independent of r, so that we can write

$$G(\tau) = G_s(\tau);$$

and (14) may be rewritten as

$$T_qG_s(\tau) = q^{s(p-1)/2}\left(\frac{q}{p}\right)^s G_{-s}(\tau). \tag{15}$$

From (11) we find easily that

$$G_s(\tau) = \varphi^{-sp}(\tau)\left\{\sum q_s\left(\frac{n}{p}\right)x^n + p\left(\frac{q}{p}\right)^s x^{-\Delta_1}\sum q_s(np+\Delta_0)x^n\right\}, \tag{16}$$

where

$$\Delta = \tfrac{1}{24}s(p^2-1)(q-1),$$

Δ_0 is the least non-negative residue of Δ modulo p, and Δ_1 is defined by

$$\Delta_0 = \Delta - p\Delta_1.$$

We remark that if $s \geqslant 0$ then $\Delta_1 = \left[\frac{\Delta}{p}\right]$ and $\Delta_0 = \Delta - p\left[\frac{\Delta}{p}\right]$. The expansion (16) together with (15) now implies

THEOREM 1. *If $s \geqslant 0$ then $G_s(\tau)$ has a pole of order $\left[\frac{\Delta}{p}\right]$ at $i\infty$ and is pole-free at 0. If $s < 0$ then $G_s(\tau)$ is pole-free at $i\infty$ and has a pole of order $\left[\frac{-\Delta}{p}\right]$ at 0.*

Suppose now that q is limited to the values

$$q = 2, 3, 5, 7, 13.$$

For these values $\Gamma_0(q)$ is of genus 0, while

$$\frac{24}{q-1} = \frac{1}{t}$$

is an even integer and

$$B(\tau)^{24/(q-1)}$$

is a hauptmodul for $\Gamma_0(q)$ having a zero of order 1 at $i\infty$ and a pole of order 1 at 0 (see [2]). We know therefore that $G_s(\tau)$ is a polynomial in $B(\tau)^{-24/(q-1)}$ of degree $\left[\frac{\Delta}{p}\right]$ for $s \geqslant 0$, and a polynomial in $B(\tau)^{24/(q-1)}$ of degree $\left[\frac{-\Delta}{p}\right]$ for $s < 0$. Therefore we have on comparing coefficients

THEOREM 2. *Suppose that $s \geqslant 0$. Then there are constants c_k, d_k such that for all integral n*

$$q_s\left(\frac{n}{p}\right) + p\left(\frac{q}{p}\right)^s q_s(np+\Delta) = \sum_{k=0}^{[\Delta/p]} c_k q_{sp-24k/(q-1)}(n+k), \tag{17}$$

$$q_{-s}\left(\frac{n}{p}\right) + p\left(\frac{q}{p}\right)^s q_{-s}(np-\Delta) = \sum_{k=0}^{[\Delta/p]} d_k q_{-sp+24k/(q-1)}(n-k). \tag{18}$$

These are the principal identities we wished to derive.

Of particular interest are those identities for which $\left[\frac{\Delta}{p}\right] = 0$; that is, $\Delta < p$. This necessitates

$$\tfrac{1}{24}s(p^2-1)(q-1) < p \tag{19}$$

and all instances of (19) are easily enumerated. We find as a matter of fact the following:

THEOREM 3. *The identities*

$$q_s\left(\frac{n}{p}\right) + p\left(\frac{q}{p}\right)^s q_s(np+\Delta) = c_0 q_{sp}(n), \tag{20}$$

$$q_{-s}\left(\frac{n}{p}\right) + p\left(\frac{q}{p}\right)^s q_{-s}(np-\Delta) = q_{-sp}(n), \tag{21}$$

where

$$c_0 = 1 + p\left(\frac{q}{p}\right)^s q_s(\Delta), \tag{22}$$

are valid for the values of q, s, p given below:

q	s	p
2	1	5, 7, 11, 13, 17, 19, 23
	2	5, 7, 11
	3	5, 7
	4	5
3	1	5, 7, 11
	2	5

There are in addition some identities not covered by the previous discussion for $p = 2, 3$. We list these without proof.

$$3q_3(3n+1) - q_3\left(\frac{n}{3}\right) = 8q_9(n), \qquad q = 2,$$

$$q_{-3}\left(\frac{n}{3}\right) - 3q_{-3}(3n-1) = q_{-9}(n), \qquad q = 2.$$

$$q_4\left(\frac{n}{2}\right) + 2q_4(2n+1) = 9q_8(n), \qquad q = 3,$$

$$q_{-4}\left(\frac{n}{2}\right) + 2q_{-4}(2n-1) = q_{-8}(n), \qquad q = 3.$$

$$q_2\left(\frac{n}{2}\right) + 2q_2(2n+1) = 5q_4(n), \qquad q = 5,$$

$$q_{-2}\left(\frac{n}{2}\right) + 2q_{-2}(2n+1) = q_{-4}(n), \qquad q = 5.$$

$$q_1\left(\frac{n}{3}\right) + 3q_1(3n+2) = 7q_3(n), \qquad q = 7,$$

$$q_{-1}\left(\frac{n}{3}\right) + 3q_{-1}(3n-2) = q_{-3}(n), \qquad q = 7.$$

On final observation: The quantity c_0 defined in (22) may be shown to be

$$\left(\frac{q}{p}\right)^s q^{s(p-1)/2}$$

for all values given in Theorem 3.

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Verwendung der Zeta-Funktion beim Sieb von Selberg*

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Einleitung. Ziel dieser Arbeit ist der Beweis des folgenden Satzes (siehe Seite 402): *In jeder arithmetischen Reihe* $km+l$, $0<l<k$, $(l,k)=1$ *gibt es eine Primzahl oder eine aus zwei verschiedenen Primzahlen bestehende Zahl, die* $\leqslant k^{15+\varepsilon}$ *ist* ($k\geqslant k_0(\varepsilon)$ *und* $\varepsilon>0$ *beliebig klein, fest)* (vgl. die Fußnote auf Seite 402). Wir führen die allgemeinen Formeln gleich für s-tupel von Primzahlen aus, spezialisieren dann aber auf $s=1$ (bei variablem k, Seite 388).

§ 1. Allgemeines. Die in [6] skizzierte und z. B. in [3], Kp. II, § 3 bzw. in [11] näher ausgeführte Siebmethode wollen wir hier (in § 2) soweit wiedergeben, als wir sie benötigen, da sie die Grundlage für alles folgende bildet. Dabei beschränken wir uns gleich auf Zahlensysteme der Form $a_i m+b_i$, $1\leqslant i\leqslant s$, $1\leqslant m\leqslant N$, s und N natürliche Zahlen, a_i, b_i ganz. Weiters möge für diese gelten: $(a_i,b_i)=1$, $a_ib_k-a_kb_i\neq 0$ für $i\neq k$; damit ist auch

$$E=\prod_{1\leqslant i\leqslant s}a_i\prod_{1\leqslant i<k\leqslant s}(a_ib_k-a_kb_i)\neq 0. \tag{1.1}$$

Sodann bilden wir die Zahlen

$$n_m=\prod_{i=1}^{s}|a_im+b_i|, \tag{1.2}$$

die im weiteren an Stelle der s-tupel untersucht werden. Vorerst einige Bezeichnungen und Bemerkungen: mit p,q bezeichnen wir durchwegs Primzahlen; mit $\omega(d)$ die Anzahl der mod d verschiedenen Restklassen, die die Kongruenz

$$n_m\equiv 0 \pmod d \tag{1.3}$$

* Die vorliegende Arbeit stellt einen Auszug aus meiner Dissertation dar. Herrn Prof. K. Prachar möchte ich für die Anregung zu dieser Arbeit, wertvolle Hinweise und die Durchsicht des Manuskriptes meinen aufrichtigen Dank aussprechen.