Consequently if we put

\[(6.5) \quad F_r = \begin{cases} B_k(n^r) & \text{for } r = 0, \\ \frac{k^{-r-1}(n^{r+1} \eta^r)}{\phi_k} & \text{for } r = 1, \ldots, n-1, \end{cases} \]

it follows that the matrix

\[(6.6) \quad (F_{rs}) \quad (r, s = 0, 1, \ldots, n-1)\]

has the characteristic roots

\[(6.7) \quad n^k B_k \left( x - \frac{r}{n} \right) \quad (r = 0, 1, \ldots, n-1).\]

In particular we can evaluate the determinants of (6.3) and (6.5).

References


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A note on the real zeros of Dirichlet's $L$-functions

by

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1. For $s = \sigma + it$, the $L$-functions of Dirichlet belonging to a modulus $k$ are defined for $\sigma > 1$ by

\[(1.1) \quad L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \]

where $\chi(n)$ are the characters of the group of the reduced residue-classes mod $k$. It is well known that the study of zeros of these functions give the key to the distribution of primes in the arithmetical progressions mod $k$ and the essentially new difficulties, compared to those connected with the zeros of the Riemann zeta-function, are due to the appearance of real zeros. Concerning them we know (1) that for a suitable positive (2) $c_1$, at most one of the $L(s, \chi)$-functions mod $k$ can vanish in the interval

\[(1.2) \quad 1 - \frac{c_1}{\log k} \leq \sigma \leq 1\]

and, if such an exceptional $L(s, \chi)$ exists, it has here a single simple zero (called exceptional zero and denoted by $\beta$). The possibility of an exceptional zero gives a lot of trouble in the number-theory. A typical example is furnished by the formula (Page, [2]), valid for $\chi \neq \chi_0$

\[(1.3) \quad \left| \sum_{x \leq n} A(n) \chi(n) \right| \leq \left( c_2 \left( x - \sqrt{\frac{x}{\varphi(k)}} + x^\delta \right) \right)\]

$\chi$ is an exceptional character or not, respectively; here $A(n)$ stands for the known Dirichlet symbol and $\varphi(k)$ is the usual Euler function.

(1) This is essentially due to E. Landau [1].
(2) In what follows, $c_1$, $c_2$, ..., stand for explicitly calculable positive numerical constants; as an exception $c_1 = c_1(k)$ is not and depends on $k$. 
For \( \beta \) we know at present only the estimation\(^{(4)}\)
\[
\beta < 1 - \varepsilon_0(\varepsilon) / \varepsilon
\]
for \( 0 < \varepsilon \leq 1 \), where — curiously enough — no explicit form of \( \varepsilon_0(\varepsilon) \) is known and also that the exceptional \( \varepsilon \)-values (i.e., those with an exceptional \( L(s, \chi) \)) if they exist at all lie very dispersed (see Landau [1]). Taking into account all these it is of some interest to note that for the greatest real zero \( \gamma = \gamma(\varepsilon) \geq \frac{1}{2} \) of any \( L(s, \chi) \) function belonging to the modulus \( k \) (if there exists such a zero) only the "small" primes are responsible.

More exactly we shall prove the following

**Theorem.** With \( P = \varepsilon d \log k \log \log \log k \) we have for \( \varepsilon > e_0 \) the inequality
\[
(\frac{1}{2} < \frac{1}{2}) \log \frac{\log k}{\log \log k} + \frac{1}{\log P} \max_{\chi \neq \chi_0} \log \left| \sum_{n \leq x} A(n) \chi(n) \right|
\]
for each \( \chi \neq \chi_0 \) (if there are any real zeros of \( L(s, \chi) \)).

By more careful treatment of the details one could have the constants in the theorem replaced by smaller numerical values and the estimation refined so that it could be used also for numerical calculations with prime tables. We shall not do this. The proof of the theorem will be based, as in many former applications, on the following theorem\(^{(5)}\).

For any arbitrary non-negative integer \( m \) and complex \( b, a \)-numbers with
\[
1 = |a_1| \geq |a_2| \geq \ldots \geq |a_m|
\]
there is an integer \( v \) with
\[
m - 1 \leq v \leq m + n
\]
and
\[
\sum_{f \leq x} \frac{b_f}{2} \geq \frac{n}{3 \log (m + n)} \min \left\{ b_1 + \ldots + b_{\frac{n}{3}} \right\}
\]

This theorem will be applied here in the case \( b_1 = b_2 = \ldots = b_n = 1 \) in the following form (which can easily be derived from (1.4)-(1.5)-(1.6)).

If \( m > 0 \) and max \( |a_i| \geq 1 \) and \( 2 \leq n \leq N \), then
\[
\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \frac{b_1 + \ldots + b_n}{2} \right| \geq \frac{N}{22 (m + N)}^{\frac{1}{2}}
\]

The proof of our present theorem is in principle similar to a previous one\(^{(4)}\) on the remainder-term of the prime-number formula but the appearance of the parameter \( k \) necessitates unexpected changes in the choice of the parameters of the proof.

2. We turn to the proof of the theorem. We fix an arbitrary \( \chi(n) \) mod \( k \) and suppose that \( L(s, \chi) \) has real zeros and \( \gamma = \frac{1}{2} \) is the maximal one. We shall make repeated use of the fact that for \( k > e_0 \) and any real \( r \) the number of zeros of \( L(s, \chi) \) in the parallelogram
\[
\sigma \geq 1, \quad t \leq r \leq t + 1
\]
does not exceed
\[
o \log k \sigma \log k \leq \omega \sigma \leq \log k \log k \log \log k
\]

The integer \( \omega \) will be exactly determined later; at this moment we require only

Further let \( M \) be such that
\[
\max_{\sigma > 1} \left| L(1, \chi) \right| \leq M \leq \epsilon \log \log \log k
\]
and then fixed; further let
\[
\xi = M^{e_1 + 1}
\]
then owing to (2.4) and (2.3) we have
\[
\xi \leq \varepsilon d \log \log k \log \log k
\]

Finally let
\[
J(x) = \frac{1}{2\pi i} \int_{x+it}^{x-\frac{1}{2}} \frac{L'}{L} (s, \chi) ds
\]

\(^{(4)}\) See § 9 of the German edition of my book [5].
Owing to the well-known coefficient formula we have

$$J(\chi) = \sum_{\chi \sim \chi} A(n) \chi(n) \log^2 \frac{\varepsilon}{\beta}.$$  

Putting

$$\int_{\mathbb{R}} \left( \sum_{\chi \sim \chi} A(n) \chi(n) U(\varepsilon, \chi) \right) \, d\varepsilon = \int_{\mathbb{R}} U(\varepsilon, \chi) \, d\varepsilon,$$

we obtain

$$J(\chi) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{U(\varepsilon, \chi)}{\varepsilon} \, d\varepsilon,$$

i.e. (2.6) and (2.3)

$$J(\chi) \leq \frac{\log^2 \varepsilon}{\alpha} \max_{1 < \varepsilon < \infty} |U(\varepsilon, \chi)|,$$

$$\leq \left( \frac{2 \log \log \log(k)}{\alpha} \right)^{s+1} \max_{1 < \varepsilon < \infty} |U(\varepsilon, \chi)|,$$

$$\leq \left( \frac{2 \log \log \log(k)}{\alpha} \right)^{s+1} \max_{1 < \varepsilon < \infty} |U(\varepsilon, \chi)|.$$

3. Using (2.2) and the known fact (see [4]) that if \( \varepsilon = \alpha + it \), stand for the zeros of an \( L(s, \chi) \) with any fixed \( \chi \), then for \( \sigma \geq \frac{1}{2} \) and any real \( t \) the inequality

$$\frac{L'}{L}(s, \chi) - \sum_{\chi \neq \chi} \frac{1}{\varepsilon - \xi} \leq \frac{1}{\log \log(k) \log(k)|\xi|+1}$$

holds, we easily get the existence of a connected broken line \( F \), consisting of segments alternately parallel to the axes, all lying in the infinite vertical strip

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log \log(k) \log(k)|\xi|+1},$$

on which the inequality

$$\frac{L'}{L}(s, \chi) \leq \frac{1}{\log \log(k) \log(k)|\xi|+1}$$

holds. For later reasons let us observe that \( F \) does not depend upon the choice of \( \omega \). Using this and the other known fact that for any real \( r \) with \( |r| \geq 2 \) we have between \( r \) and \( r+1 \) a \( \varepsilon = t \), so that on the segment

$$0 \leq \sigma \leq \frac{1}{2}, \quad t = t,$$

the inequality

$$\frac{L'}{L}(s, \chi) \leq \frac{1}{\log \log(k) |\xi|}$$

holds, we infer by usual contour integration that

$$J(\chi) \leq \frac{c_{\alpha}}{\log \log(k) \log(k)|\xi|+1}.$$
On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two

by

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Summary

§ 1. Introduction.

§ 2. Construction of a complete \((2q + 4)_{2q}\) for \(q = 4\).

§ 3. Construction of a complete \((2q + 3)_{2q}\) for any \(q > 2\).

§ 4. Two additional lemmas.

§ 5. The polarity defined by an ovaloid.

§ 6. On the plane sections of an ovaloid.

§ 7. On ovaloids of \(S_{4,4}\) which are not quadrics.

§ 1. Introduction

The study of the geometry of a Galois space \(S_{r,q}\), i.e., of a projective \(r\)-dimensional space over a Galois field of order

\[ q = p^h, \]

where \(p, h\) are positive integers and \(p\) is a prime (the characteristic of the field), has recently been pursued and developed along new lines (1). In it, both algebraic-geometric and arithmetical methods have been applied, including the use of electronic calculating machines; moreover, some of the problems dealt with are deeply connected with information theory, especially with the construction of \(g\)-ary error-correcting codes.

It is actually a chapter of arithmetical geometry, which reduces to the investigation of certain questions on congruences mod\(p\) in the particular case when \(h = 1\).

A set of \(k\) distinct points of \(S_{r,q}\), no three of which lie on a line, is denoted by \(k_s\), and called a \(k\)-span if \(r = 2\) and a \(k\)-span if \(r > 2\). Any such \(k_s\) is said to be complete when it is not a subset of a \((k-1)_{k_q}\). For given \(r\) and \(q\), a \(k_s\) having maximum \(k\) is called an oval if \(r = 2\) and an ovaloid if \(r > 2\), and then it is consequently always complete.

\(^{(1)}\) See especially (8); further historical and bibliographical informations are contained in (7).