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A feature of Dirichlet's approximation theorem

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In [2] I have proved that the inequality $|ax-y| < c/x$ has for each irrational a infinitely many integer solutions with $x \equiv a \pmod{s}$, $y \equiv b \pmod{s}$, the numbers a, b, s being prescribed arbitrarily and the constant c depending only on s (compare also [1] and [3]). A related problem appears when we try to impose such supplementary conditions in the case of the classical theorem of Dirichlet stating that for any a and $t > 1$ there are integers x and y with

$$0 < x \leqslant t, \tag{1}$$

$$|ax-y| < \frac{1}{t}. \tag{2}$$

However, the result turns out to be different: the congruence conditions for x and y are in general inconsistent with this statement, even if (1) is replaced by

$$0 < x \leqslant ct \quad (c > 1). \tag{1'}$$

This is shown by the following

THEOREM 1. *For any prime p there is a real a such that, whatever be the constant $c > 0$, one has $|ax-y| \geqslant 1/t$ for some $t > 1$ and for each positive $x \leqslant ct$ which is not divisible by p.*

The inconsistency of the congruence conditions with the assertion of Dirichlet's theorem is not of "exceptional" character because of the following

THEOREM 2. *For almost every a, whatever be the constant $c > 0$, there is a $t > 1$ such that $|ax-y| \geqslant 1/t$ for each odd positive integer $x \leqslant ct$.*

First we prove two lemmas.

LEMMA 1. *If the inequalities* (1') *and* (2) *are fulfilled by two pairs of integers (x_1, y_1) and (x_2, y_2) $(x_1, x_2 > 0)$, and $x_1 y_2 - y_1 x_2 \neq 0$, then $(x_i, y_i) < 2c$ $(i = 1, 2)$.*

Proof. We have

$$\left|\frac{y_1}{x_1}-\frac{y_2}{x_2}\right| \leqslant \left|a-\frac{y_1}{x_1}\right|+\left|a-\frac{y_2}{x_2}\right| < \frac{1}{tx_1}+\frac{1}{tx_2} < \frac{2c}{x_1x_2}.$$

Hence $0 < |x_1y_2-y_1x_2| < 2c$, which yields the assertion.

For the next lemma we restrict ourselves to the case when c is an integer and $c \geqslant 4$.

LEMMA 2. *Denoting by p_i/q_i the convergents of the number a represented as a continued fraction and assuming, for a fixed i,*

$$\frac{q_{i+1}}{q_i} > c^2, \quad t = \tfrac{1}{2}cq_i$$

we have (1′) and (2) for some $x > 0$ and y with $(x, y) = 2c$.

Proof. Write $|aq_i-p_i| < 1/q_{i+1} < 1/c^2q_i$. Then $|2caq_i-2cp_i| < 2/cq_i = 1/t$.

Thus, $x = 2cq_i$ and $y = 2cp_i$ give the required solution.

From these lemmas it follows that

(*) If $q_{i+1}/q_i > c^2$ and $t = \frac{1}{2}cq_i$ ($c \geqslant 4$ integer), then all solutions (x, y) of the system (1′) and (2) are proportional to (q_i, p_i).

Proof of theorem 1. If the assumptions of Lemma 2 are fulfilled and if $p|q_i$, then (*) implies that there is no solution with x not divisible by p. Thus, the theorem will be proved if we construct a continued fraction $(0; a_1, a_2, \ldots)$ such as to have $p|q_{i_\nu}$ for infinitely many indices and $\lim_{\nu\to\infty}(q_{i_\nu+1}/q_{i_\nu}) = \infty$. If $p = 2$ put $a_1 = a_2 = a_3 = 1$ and $i_1 = 4$. If $p > 2$ put $a_1 = a_2 = 1$ and $i_1 = 3$. Then $p \nmid q_{i_1-1}$ and, since p is a prime, the congruence $rq_{i_1-1}+q_{i_1-2} \equiv 0 \pmod{p}$ can be solved in r. If r_1 is a positive solution, put $a_{i_1} = r_1$. Thus, we have $p|q_{i_1}$. Choose $a_{i_1+1} = i_1$. Then $q_{i_1+1}/q_{i_1} > i_1$. Since $p \nmid q_{i_1+1}$, we can solve the congruence $rq_{i_1+1}+q_{i_1} \equiv 0 \pmod{p}$ with a positive $r = r_2$ and put $a_{i_1+2} = r_2$. Write $i_2 = i_1+2$. Then we have $p|q_{i_2}$ and choosing $a_{i_2+1} = i_2$ we get $q_{i_2+1}/q_{i_2} > i_2$. Proceeding in this way we satisfy the required conditions with $i_3 = i_1+4$, $i_4 = i_1+6$ etc.

Proof of theorem 2. It suffices to show that given any constant $c > 0$ almost every continued fraction a has a convergent p_i/q_i with even q_i and $q_{i+1}/q_i > c^2$. Indeed, if the fixed $c \geqslant 4$ is an integer, then (*) implies that for $t = \frac{1}{2}cq_i$ there is no solution of (1′) and (2) with an odd x. Hence, taking the sets of a's with the above property for $c = 4, 5, \ldots$ and considering their common part, we obtain the assertion. Evidently we can restrict ourselves to *odd* integer values for c.

Assume to the contrary that for some odd c there is a set of positive measure of a's for which if a q_i is even then $q_{i+1}/q_i \leqslant c^2$. Denote by $\delta(a)$ the transformation sending the number $a = (0; a_1, a_2, \ldots)$ into $(0; a_2, a_3, \ldots)$. Then, according to the ergodic theorem for continued fractions proved by Ryll-Nardzewski [4], for any function $f \epsilon L(0,1)$ one has

$$(3) \qquad \frac{1}{n}\{f(a)+f(\delta(a))+\ldots+f(\delta^{n-1}(a))\} \to \frac{1}{\log 2}\int_0^1 \frac{f(\tau)d\tau}{1+\tau}$$

almost everywhere in $(0,1)$.

Put $f(a) = 1$ if $a_1 = a_2 = a_3 = c^2$ and $f(a) = 0$ otherwise. Evidently

$$\int_0^1 \frac{f(\tau)}{1+\tau}d\tau > 0$$

and hence (3) implies that for almost every a there is an index i_0 for which $a_{i_0-1} = a_{i_0} = a_{i_0+1} = c^2$ [1]. It follows from our assumption that there must be an a with this property for which $q_{i+1}/q_i \leqslant c^2$ holds whenever q_i is even. Then, since $q_i/q_{i-1} > c^2$ for $i = i_0-1, i_0, i_0+1$, the three numbers q_{i-1} are odd. But this contradicts the relation

$$q_{i_0} = c^2q_{i_0-1}+q_{i_0-2}$$

as c^2 is odd. Thus the proof is achieved.

References

[1] R. Descombes et G. Poitou, *Sur certains problèmes d'approximation*, Comptes Rendus de l'Académie des Sciences 234 (1952), p. 581-583 and 1522-1524.

[2] S. Hartman, *Sur une condition supplémentaire dans les approximations diophantiques*, Colloq. Math. 2 (1949), p. 48-51.

[3] J. F. Koksma, *Sur l'approximation des nombres irrationnels sous une condition supplémentaire*, Simon Stevin 28 (1951), p. 199-202.

[4] C. Ryll-Nardzewski, *On the ergodic theorems II (Ergodic theory of continued fractions)*, Studia Math. 12 (1951), p. 74-79.

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[1] This result is older than (3), as prof. J. F. Koksma pointed out to the author.