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On the minimal overlap problem of Erdös

by

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Let $\mathcal{E}$ be a separation of the integers $1, 2, \ldots, 2n$ into two disjoint classes $\{a_i\}$ and $\{b_j\}$ with n elements in each class, say

$$(1)\qquad \begin{aligned} \{a_i\}:\ a_1 < a_2 < \ldots < a_n, \quad \sum_{i=1}^{n} a_i = A, \\ \{b_j\}:\ b_1 < b_2 < \ldots < b_n, \quad \sum_{j=1}^{n} b_j = B. \end{aligned}$$

Let M_k denote the number of solutions of $a_i - b_j = k$, i. e.

$$(2)\qquad M_k = \sum_{a_i - b_j = k} 1 \quad (-2n \leqslant k \leqslant 2n)$$

and let

$$(3)\qquad M = M(n) = \min_{\mathcal{E}} \max_{k} M_k.$$

Erdös [1] conjectured that for n even, $M = \frac{1}{2}n$ but later (written communication) disproved this and showed by a probability method that for large n, $M \leqslant \frac{4}{9}n$. Independently, Selfridge, Motzkin and Ralston [3] used the electronic computer SWAC to find an $\mathcal{E}$ with $n = 15$ and $\max_k M_k = 6$. A slight modification of their example shows that for infinitely many n,

$$(4)\qquad M \leqslant 0{,}4n.$$

On the other hand, Erdös [2] proved that

$$(5)\qquad M \geqslant 0{,}25n$$

and Scherk (written communication) improved this to

$$(6)\qquad M > \left(1 - \frac{1}{\sqrt{2}}\right) n > 0{,}2929n.$$

The object of this note is to prove

$$M > \frac{\sqrt{2}}{4}(n-1) > 0{,}3525(n-1). \tag{7}$$

We can combine our method with that of Scherk to obtain

$$M > \sqrt{4-\sqrt{15}}(n-1) > 0{,}3570(n-1). \tag{8}$$

However, the proof of (8) is somewhat involved and the improvement over (7) is small so we will not prove (8) here.

We proceed to the proof of (7). First consider

$$P = \sum_k kM_k = \sum_{i,j}(a_i-b_j) = n(A-B). \tag{9}$$

Next note that

$$Q = \sum_k k^2 M_k = \sum_{i,j}(a_i-b_j)^2 = n\big(1^2+2^2+\ldots+(2n)^2\big)-2AB. \tag{10}$$

Finally compute

$$R = \sum_k (k-P/n^2)^2 M_k = \sum_{i,j}(a_i-b_j-P/n^2)^2 = Q-(A-B)^2. \tag{11}$$

Since $(A-B)^2+2AB = \frac{1}{2}(A+B)^2+\frac{1}{2}(A-B)^2$ and $A+B = 1+2+\ldots+2n$ we easily find that

$$R = \tfrac{1}{6}n^2(4n^2-1)-\tfrac{1}{2}(A-B)^2 < \tfrac{2}{3}n^4. \tag{12}$$

We now obtain a lower bound for R which involves M. Note that in the sum $R = \sum_{i,j}(a_i-b_j-P/n^2)^2$, the distinct elements a_i-b_j-P/n^2 differ by integers. Further, the total number of terms in this sum is n^2, and no term can be repeated more than M times. Hence R is decreased by replacing it by $2M$ terms of 0^2 each, $2M$ terms of 1^2 each, $\ldots, 2M$ terms of r^2 each, where r is determined by

$$2Mr \leqslant n^2 < 2M(r+1). \tag{13}$$

With this value of r we therefore have

$$R \geqslant 2M(0^2+1^2+\ldots+r^2) = \tfrac{1}{3}Mr(r+1)(2r+1); \tag{14}$$

using the right hand side of (13) gives

$$R > \frac{M}{3}\left(\frac{n^2}{2M}-1\right)\left(\frac{n^2}{2M}\right)\left(\frac{n^2}{M}-1\right). \tag{15}$$

Combining (12) and (15) yields the required result (7).

References

[1] P. Erdös, *Some results on number theory*, Riveon Lematematika 9 (1955), p. 48.

[2] — *Problems and results in additive number theory*, Colloque sur la théorie des nombres, 1955, p. 135-137.

[3] T. S. Motzkin, K. E. Ralston and J. L. Selfridge, *Minimal overlap under translation*, Abstract, Bull. Amer. Math. Soc. 62 (1956), p. 558.

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