The construction of perfect and extreme forms I

by

E. S. Barnes (Sydney)

1. Introduction. Let

\[ f(x) = \sum_{i} a_i x_i^2 \quad (a_i = a_0) \]

be a positive definite quadratic form of determinant \( D \), and let \( M \) be the minimum of \( f(x) \) for integral \( x \neq 0 \). Then \( f(x) \) assumes the value \( M \) for a finite number of integral \( x = \pm m_1, \ldots, \pm m_s \), called its minimal vectors.

\( f(x) \) is said to be perfect if the \( s \) relations

\[ f(m_k) = \sum a_i m_i^2 = M \quad (k = 1, \ldots, s) \]

uniquely determine the \( \frac{1}{2}n(n+1) \) distinct coefficients \( a_{ij} \) of \( f \); i.e., if the equations

\[ \sum b_{ij} m_i^2 m_j^2 = 0 \quad (k = 1, \ldots, s) \quad (b_{ij} = b_{ji}) \]

have only the trivial solution \( b_{ij} = 0 \).

All classes of perfect forms are now known for \( n \leq 6 \), and a few classes are known for larger \( n \). Their interest lies mainly in the facts that (i) they lead to a method of reduction of quadratic forms under integral unimodular transformation; and (ii) they include all extreme forms, i.e., those for which \( M/D^{kn} \) is a local maximum, and hence all absolutely extreme forms, for which \( M/D^{kn} \) assumes its greatest value \( \gamma_n \).

Most known perfect and extreme forms are listed in Coxeter [5]; these include nearly all those previously published and several new types. All others, for \( n \geq 7 \), are: \( J_{11}, K_{18} \) given in [6]; \( T_n \) of [7] which is equivalent to \( \Phi_n \) of [4]; \( \Phi_n \) of [4]; the forms discussed in [1], which we shall denote here by \( M_n \); and the forms discussed in [8], which we shall denote here by \( F_n \).

The known methods of constructing perfect or extreme forms have shown themselves to be prohibitively laborious for large \( n \). In particular
we may mention Voronoi's algorithm [10] by which all perfect forms in a given number of variables may be found and which has been successfully applied for \( n \leq 6 \); and Minkowski's reduction [8], whereby all extreme forms appear as edges of a fundamental region in the coefficient-space.

With existing methods, therefore, one cannot hope to obtain a selection of the perfect forms for \( n \gg 6 \). It is for this reason that I present, in this and a succeeding paper, two new methods which yield large numbers of perfect forms with relatively little labour. Each method proceeds from a known perfect or extreme form and produces a new form, either by extending the range of values or by increasing the dimension of the known form.

In this article I treat only the first method, which will be described in detail in §2. In §§3, 4 and 5 I describe briefly, for completeness, the forms \( A_n, B_n, L_n, C_n, F_n \) (of which only \( L_n \) and \( Q_n \) do not appear in previous literature). Application of the method to some of these produces several new general classes of forms, of which \( A''_{n-q}, B''_{n-q}, A''_n, B''_n, M''_n, M''_d \) are discussed in §§6-9.

The applications here make no pretence of completeness, and I have preferred to obtain classes of forms describable in terms of parameters rather than make a complete analysis of any particular form. However, even these forms considerably extend the table of extreme forms given in [5]; thus, for the early values of \( n \), we find:

- for \( n = 7 \), 7 extreme and 1 perfect (non-extreme) form;
- for \( n = 8 \), 9 extreme and 2 perfect forms;
- for \( n = 9 \), 13 extreme and 2 perfect forms;
- for \( n = 10 \), 13 extreme and 3 perfect forms;
- for \( n = 11 \), 16 extreme and 5 perfect forms;
- for \( n = 12 \), 19 extreme and 6 perfect forms.

These numbers do not include \( D_n \) of [4], or \( E_{12} \) of [6]; these forms will be discussed in part II, where the above lists will be considerably extended.

2. Forms, lattices and refinements. If \( T \) is a regular \( n \times n \) matrix, the points

\[
\xi = T\alpha, \quad \alpha \text{ integral,}
\]

form a lattice \( \Lambda \) of determinant \( d(\Lambda) = |\det T| \). A positive quadratic form \( f(\alpha) \) is said to have lattice \( \Lambda = \Lambda(f) \), given by (2.1), if \( f(\alpha = \xi) = \xi^T T \xi \). Every positive form is representable as a sum of squares of linear forms, and may thus be associated (in infinitely many ways) with a lattice.

If \( U \) is an integral unimodular matrix, \( T \) and \( TU \) yield the same lattice (2.1) and correspond to the equivalent forms \( \alpha^T T \alpha \) and \( \alpha^T U^T T U \alpha \); thus a lattice determines a class of equivalent forms.

We shall sometimes find it convenient to extend this idea as follows. We say that \( f \) is the form \( g \) with lattice \( \Lambda \), (2.1), if \( f(\alpha) = g(T\alpha) \). Then the values of \( f \) (or of any form equivalent to \( f \)) for integral \( \alpha \) are precisely the values of \( g(\xi) \) for \( \xi \in \Lambda \). In this way, our forms will usually be representable by a simple form \( g \), with \( \Lambda \) a sublattice of the integral lattice.

A form \( f \) with lattice \( \Lambda \) has minimum \( M \) if and only if \( \Lambda \) is admissible for the sphere \( S = \xi^T \xi \leq M \) (i.e., if \( \Lambda \) has no point other than the origin in the interior of the sphere) while some point of \( \Lambda \) lies on the boundary of the sphere. The minimal vectors of \( f \), in \( \xi \)-coordinates, are simply the lattice points \( \pm \xi = \pm Tm \), lying on the boundary of \( S \).

With these notations, it follows easily that (i) \( f \) is perfect if there exists no quadratic cone containing all the minimal vectors of \( f \); (ii) \( f \) is extreme if any sufficiently near lattice \( \Lambda' \) which is admissible for \( S \) has \( d(\Lambda') \approx d(\Lambda) \).

If a lattice \( \Lambda \) is contained in a lattice \( \Lambda' \), we say that \( \Lambda \) is a sublattice of \( \Lambda' \) and \( \Lambda' \) a refinement of \( \Lambda \). Correspondingly, we say that a form \( f' \) is a refinement of \( f \) if \( \Lambda(f') \) is a refinement of \( \Lambda(f) \). We restrict ourselves here to the case when \( \Lambda \) and \( \Lambda' \) are both \( n \)-dimensional (and the case when \( \Lambda \) has lower dimension than \( \Lambda' \) will be taken up in Part II). The basic results on which the method of this paper rests are now easily proved:

**Theorem 2.1.** Let \( f' \) be a refinement of \( f \) with the same minimum \( M \). Then if \( f \) is perfect, so is \( f' \); and if \( f \) is extreme, so is \( f' \).

**Proof.** The lattice \( \Lambda' \) of \( f' \) is a refinement of \( \Lambda \), the lattice of \( f \), and both \( \Lambda' \) and \( \Lambda \) are admissible for the sphere \( S = \xi^T \xi \leq M \). Also, since \( M(f') = M(f) = M \), the minimal vectors of \( f' \) clearly include those of \( f \).

If now \( f \) is perfect, there is no quadratic cone containing the minimal vectors of \( f \), and a fortiori none containing those of \( f' \); hence \( f' \) is perfect.

Suppose next that \( f \) is extreme, and consider any neighbouring lattice \( \Lambda' \) of \( \Lambda \) which is admissible for \( S \). The points of \( \Lambda' \) which correspond to points of \( \Lambda \) form a lattice \( \Lambda' \); thus \( \Lambda \) is a neighbour of \( \Lambda \) and a sublattice of \( \Lambda' \). Also, if \( k \) is the density of \( \Lambda \) in \( \Lambda' \), then

\[
k = \frac{d(\Lambda)}{d(\Lambda')} = \frac{a(\Lambda)}{a(\Lambda')},
\]
Since \( f \) is extreme and \( \tilde{A} \) is admissible for \( S \) (since \( \tilde{A}' \) is), we have

\[
d(\tilde{A}) \geq d(A)
\]

if \( \tilde{A} \) is sufficiently close to \( A \). From (2.3), it follows that then \( d(A') \geq d(\tilde{A}') \), so that \( f' \) is extreme, as asserted.

We note that a refinement \( f' \) of a perfect non-extreme form \( f \) may well be extreme and not merely perfect. For this, it is easy to see that \( f' \) must have more minimal vectors than \( f \), but there is no simple sufficient condition. In such cases we can apply Voronoi’s direct criterion:

**Theorem 2.2 (Voronoi).** A perfect form \( f(x) \) is extreme if and only if it is eutactic, i.e., if its adjoint \( F(x) \) is expressible as

\[
F(x) = \sum_{i=1}^{n} \omega_i (m_i x_i)^2, \quad \omega_i > 0
\]

(where \( m_1, \ldots, m_n \) are the minimal vectors of \( f(x) \)).

For typographical convenience, we shall adopt throughout the convention

\[
m = n + 1.
\]

The \( n \)-dimensional integral lattice will be denoted by \( \Gamma_n \), and the unit vectors in \( n \)-space by \( e_1, \ldots, e_n \). We shall follow as far as possible the notation for forms used in [5]: a capital roman letter for a form, with a suffix denoting the number of variables and superscripts denoting various integral parameters. Following [5], we also set

\[
A = A(f) = \left( \frac{2^n D}{M} \right),
\]

(where \( f \) has minimum \( M \) and determinant \( D \)); and

\[
A_n = \min A(f),
\]

so that \( A(f) = A_n \) when \( f \) is any absolutely extreme form in \( n \) variables.

**3. The forms \( A_n, B_n \).** We may represent \( A_n \) and \( B_n \) (with \( m = n + 1 \)) by

\[
f(x) = \sum_{i=1}^{n} a_i x_i^2
\]

with lattices the sublattices of \( \Gamma_n \) given by:

\[
A(A_n): \sum_{i=1}^{n} a_i = 0
\]

\[
A(B_n): \sum_{i=1}^{n} a_i = 0 \mod 2.
\]

The construction of perfect and extreme forms I

Each form then has minimum \( M = 2 \) and

\[
A(A_n) = n + 1, \quad A(B_n) = 4.
\]

\( A_n \) has the \( \frac{1}{2} n (n + 1) \) minimal vectors \( e_i - e_j \) \((1 \leq i < j \leq m)\) and is perfect for all \( n \). \( B_n \) has the \( m(m-1) \) minimal vectors \( e_i + e_j \) \((1 \leq i < j \leq m)\) and is perfect for \( m \geq 4 \) (and \( B_n \) always perfect). These forms are all extreme, as is well known. In fact,

\[
\sum_{i=1}^{n} y_i^2 + \sum_{i<j} (y_i - y_j)^2 = n \sum_{i=1}^{n} y_i^2 - 2 \sum_{i<j} y_i y_j
\]

is the adjoint of \( A_n(x) = \sum_{i=1}^{n} a_i^2 + \left( \sum_{i=1}^{n} a_i \right)^2 \); and

\[
\sum_{i<j} (y_i - y_j)^2 + \sum_{i<j} (y_i + y_j)^2 = 2 (m - 1) \sum_{i=1}^{n} y_i^2
\]

is a multiple of the adjoint \( \sum_{i=1}^{n} y_i^2 \) of \( \sum_{i=1}^{n} a_i^2 \); so that both forms are eutactic.

**4. The forms \( L_n^a, M_n^a \).** We define \( L_n^a \) as the form

\[
f(x) = \sum_{i=1}^{n} (a_i^2 - a_i a_{2r+1} + a_{2r+1}^2) + \sum_{k=1}^{m} s_k a_k^2 \quad (m \geq 2r)
\]

with lattice the sublattice of \( \Gamma_m \) given by

\[
\sum_{i=1}^{r} a_i = 0 \mod 3.
\]

\( M_n^a \) (with \( m = n + 1 \)) has the same definition, save that (4.2) is replaced by

\[
\sum_{i=1}^{r} a_i = 0.
\]

Since \( f \) has determinant \( \frac{1}{2} f' \), we see that

\[
D(L_n^a) = \frac{3^{r+2}}{2^r}, \quad D(M_n^a) = \frac{3^r}{2^r} (n + 2r + 1).
\]

We shall also show that

\[
M(L_n^a) = 2, \quad \text{with} \quad D = \frac{3^{r+2}}{2^r}, \quad s = \frac{1}{4} (m-1) (r+7),
\]

\[
M(M_n^a) = 2, \quad \text{with} \quad D = \frac{3^r}{2^r} (n + 2r + 1), \quad s = \frac{1}{4} (m-1) (r+3).
\]
and that $\mathcal{L}_n$ is perfect if $r > 3$ or if $r = 2$ and $m > 5$, and that $M_n^r$ is perfect if $r \geq 3$.

These results for $M_n^3$, with $n \geq 2n$, are established in [1], where it is also shown that $M_n^r$ is eutactic only if $n \leq 4r - 2$.

The forms $\mathcal{L}_n^r$, $M_n^r$ present a clear generalization of some known perfect forms in five and six variables. The forms $A_n^r$ and $P_8$, $P_9$, $P_{10}$ of [2], [3] may in fact be represented as $\mathcal{L}_n^5$, $M_n^5$, $I_n^4$, and $\mathcal{L}_n^6$ respectively.

We set for convenience

$$v(x, y) = x^2 - xy + y^2.$$

Then $v \leq 2$ only when

$$v = 0, \quad x = y = 0,$$

or

$$v = 1, \quad \pm(x, y) = (1, 0), (0, 1) \text{ or } (-1, -1);$$

and each of these last three sets satisfies $x + y = 1 \pmod{3}$.

If now $\mathcal{L}_n^r(x) \leq 2$, $x \neq 0$, we must have either

(i) $v(x_i, x_{i+r}) = 0$ for $1 \leq i \leq r$, and then $\mathcal{L}_n^r = 2$ when some two of $x_{2r+1}, \ldots, x_m$ are $0$, and the rest are zero; or

(ii) $v(x_i, x_{i+r}) = 1$ for just one value of $i$, $1 \leq i \leq r$, and $i = 1$; then $x_{i+j} = (0, 1)$ or $(-1, -1)$; and so by (4.2), some $a_n = \frac{1}{2}(1)$ for $k > 2r$ and $L_n^r = 2$; or

(iii) $v(x_i, x_{i+r}) = 1$ for just two values of $i$, $1 \leq i \leq r$, say $i = 1, 2$; then $\mathcal{L}_n^r = 2$ and $x_1 = (1, 0)$, $(0, 1)$ or $(-1, -1)$; $x_{i+j} = (0, 1)$ or $(-1, -1)$, and all remaining $x_i$ are zero.

This shows that $M(\mathcal{L}_n^r) = 2$, as asserted, and the minimal vectors are:

(4.6) $e_i - e_j \quad (1 \leq i < j \leq m, \quad (i, j) \neq (1, r+1), \ldots, (r, 2r))$,

(4.7) $e_i + e_{i+r} - e_j - e_{j+r} \quad (1 \leq i < j \leq r)$,

(4.8) $e_i + e_{i+r} + e_j \quad (1 \leq i \leq r, \quad j \neq i, i+r)$.

Thus

$$s = \left[\frac{m}{2}\right] - r - \left[\frac{r}{2}\right] + m - 2,$$

which reduces to the result given in (4.4).

It now follows immediately, since the values assumed by $M_n^r$ form a subset of those assumed by $\mathcal{L}_n^r$, that also $M(\mathcal{M}_n^r) = 2$, and that the minimal vectors of $M_n^r$ are given by (4.6), (4.7). Thus the results (4.4), (4.5) are now established.

---

The construction of perfect and extreme forms I

It is clear that $\mathcal{L}_n^r$ is not perfect if $r < 2$ or if $r = 2, m = 4$, since then $s < \frac{1}{2}(m(m+1))$. Similarly $M_n^r$ is not perfect if $0 < r < 2$ (while $M_n^3 = \lambda_n$ and may be disregarded here).

To establish the perfection of these forms in all other cases, we begin by considering an arbitrary quadratic relation

$$\sum_{i=1}^{m} p_{ii} x_i y_i = 0 \quad (p_{ii} = p_{ii})$$

satisfied by all the vectors (4.6), (4.7), which are minimal vectors of both $\mathcal{L}_n^r$ and $M_n^r$. We set

$$q_i = q_0 = 2p_{ii} - p_{ii} - p_{ii} \quad (i \neq j).$$

Then, from the vectors (4.6), we have

$$q_0 = 0 \quad \text{for} \quad j \neq i \pm r, \quad 1 \leq i \leq r.$$

From (4.7), assuming as we may that $r > 2$, we obtain

$$q_{i+j} = 0 \quad (1 \leq i < j \leq r);$$

it follows that $g_{i+j} = 0$ for $1 \leq i < j \leq r$, provided that $r > 3$.

Thus if $r > 3$ the relation (4.9) must be of the form

$$\sum_{i=1}^{m} x_i \sum_{i=1}^{m} p_{ii} x_i = 0.$$

The perfection of $M_n^r$ for $r > 3$ follows at once. For $\mathcal{L}_n^r$, we have the further minimal vectors (4.8) satisfying (4.10). These yield

$$p_{ii} + p_{i+r} + p_{ii} = 0 \quad (1 \leq i \leq r, \quad j \neq i, i+r),$$

whence all $p_{ii} = 0$. Hence $\mathcal{L}_n^r$ is perfect for $r > 3$. For $r = 2, m > 5$, a similar analysis shows that $\mathcal{L}_n^3$ is again perfect.

We note in conclusion that $\mathcal{L}_n^r$ is eutactic (and so extreme if it is perfect) if and only if $m = 2r$ or $2r+1$. For the inverse of (4.1) is

$$f(y) = \frac{4}{3} \sum_{i=1}^{r} y_i^2 + y_{i+r} y_i + y_{i+r} + y_i^2,$$

If now $m \geq 2r+2$, $f(y)$ has zero coefficient of $y_{2r+1} y_{2r+2}$. Of the linear forms $y_i y_{i+1} + y_{i+r} + y_i y_{i+r} + y_i + y_{i+r}$ associated with the minimal vectors (4.6), (4.7) and (4.8), only the square of $y_{2r+1} y_{2r+2}$ involves a term in $y_{2r+2} y_{2r+3}$. Hence in any expression of $f(y)$ as

$$\sum_{i=1}^{m} q_i \lambda_i(y),$$

the coefficient $q_0$ of $(y_{2r+1} y_{2r+2})^2$ must be zero.
Thus $L_m^*$ is not cutastic for $m > 2r + 2$. A simple calculation shows that $L_m^*$ is cutastic for $m \leq 2r + 1$.

5. The forms $P_n, Q_n$. We define $Q_n$ to be

\[ f(x) = \sum_{i=1}^{n} x_i^2 \]

with lattice the sublattice of $P_m$ given by

\[ \sum_{i=1}^{n} x_i \equiv 0 \ (\text{mod} 4), \]

\[ \sum_{i=1}^{n} 2x_i \equiv 0 \ (\text{mod} m). \]

$P_n$ has the same definition, save that (5.2) is replaced by

\[ \sum_{i=1}^{n} x_i = 0. \]

Hence

\[ D(\mathbb{Q}_n) = 16m, \quad D(P_n) = m^n = (m+1)^n. \]

The form $P_n$ is discussed in [3], where it appears as a generalization of the new extreme ternary form found in [2]. It is stated there that $P_n$ has

\[ M = 4, \quad \lambda = \frac{(m+1)^n}{2^n}, \quad \kappa = \begin{cases} \frac{1}{2}(m-1)(m-2)(m-3) & \text{if } 2 \nmid m, \\ \frac{1}{3}(m-1)(m-2)(m-3) & \text{if } 2 \nmid m, \end{cases} \]

and is perfect and extreme for all $n \geq 6$. The proof that $P_n$ is perfect and extreme is carried through in [3] only for even $n$; a similar proof holds for odd $n$, though it is rather more intricate.

We shall show here that $Q_n$ has

\[ M = 4, \quad \lambda = \frac{m^n}{2}, \quad \kappa = \begin{cases} \frac{1}{3}(m-1)(m-2)(m-3) & \text{if } 2 \nmid m, \\ \frac{1}{2}(m-1)(m-2)(m-3) & \text{if } 2 \mid m, \end{cases} \]

\[ \text{and is perfect and extreme for all } n \geq 6. \]

The minimal vectors of $P_n$ are all of the form

\[ x = e_i + e_j - e_k - e_l, \]

where the suffixes $i, j, k, l$ are distinct and, by (5.3), satisfy

\[ a + b + c + d \equiv 0 \ (\text{mod } m). \]

A simple enumeration, as given in [3], now establishes (5.5).

For $Q_n$ we have, in addition to these, the minimal vectors

\[ x = e_i + e_j - e_k + e_l, \]

where the suffixes $i, j, k, l$ are distinct and satisfy

\[ a + b + c + d = 0 \ (\text{mod } m). \]

To count these, we need

**Lemma 5.1.** Let $N$ be the number of unordered sets $a, b, c, d$ of distinct integers chosen from $1, 2, \ldots, m$ satisfying (5.10). Then

\[ N = \begin{cases} \frac{1}{4}(m-1)(m-2)(m-3)(m-4) & \text{if } 2 \nmid m, \\ \frac{1}{2}(m-1)(m-2)(m^2-4m+6) & \text{if } 2 \mid m, \end{cases} \]

\[ \frac{1}{3}(m-1)(m-2)(m^2-3m+6) & \text{if } 4 \mid m. \]

**Proof.** Let

\[ p_a(t) = \prod_{x=1}^{m} (1 + at^x) = \sum_{t} q_a(t)a^t. \]

Then, if $q_a(t) = \sum q_{a'}t^s$, $q_a$ is the number of sets of $s$ distinct integers chosen from $1, 2, \ldots, m$ whose sum is $t$; thus

\[ N = \sum_{a} q_a. \]

If $a$ is a primitive $n$th root of unity, we therefore have

\[ mN = \sum_{a} q_a(a'c); \]

thus $mN$ is the coefficient of $a^n$ in the formal expansion of

\[ P_a(a) + P_a(a^2) + \ldots + P_a(a^n). \]

Now $a'$ is a primitive $n$th root of unity, where $s = m/(m, r)$, and

\[ p_a(a') = (1 + (-1)^{r-1} a')^{m/n}. \]

\[ \text{(*) I am indebted to Mr. W. B. Smith-White for the idea of this simple proof.} \]

*Acta Arithmetica* V.
the formal expansion of \( p_n(x) \) contains a term in \( x^s \) only when \( s = 1, 2 \) or 4. Since \( \varphi(1) = \varphi(2) = 1, \varphi(4) = 2 \), we therefore have

\[
\begin{aligned}
mN &= \begin{cases} 
\left(\frac{m}{4}\right) & \text{if } 2 \mid m, \\
\left(\frac{m}{2}\right) & \text{if } 4 \mid m, \\
\left(\frac{m}{4}\right) + \left(\frac{m}{2}\right) - 2\left(\frac{m}{4}\right) & \text{if } 4 \nmid m;
\end{cases}
\end{aligned}
\]

this gives (511).

Since \( N \) is the number of minimal vectors of \( Q_m \) of the type (5.8), we obtain the formulae (5.6) for \( x \) by adding (5.11) and (5.3) (with \( n = m - 1 \)).

The minimal vectors (5.7) are common to \( P_n \) and \( Q_m \), and any \( m \)-dimensional quadratic form which vanishes for all of them must be of the type

\[
(x_1 + x_2 + \ldots + x_n)(p_1x_1 + p_2x_2 + \ldots + p_nx_n),
\]

since \( P_n \) is perfect and its lattice lies in the plane \( x_1 + \ldots + x_n = 0 \). Since all the minimal vectors (5.9) satisfy \( \sum_{i=1}^n x_i = 0 \), it follows that the above quadratic form vanishes for all of these only if

\[
p_{n} + p_{0} + p_{c} + p_{d} = 0
\]

when (5.10) is satisfied.

Let now \( a, b, a', b' \) be any four distinct suffixes (mod \( m \)) with

\[
a + b = a' + b'.
\]

The number of distinct unordered pairs \( c, d \) (mod \( m \)) satisfying

\[
c + d = -(a + b)
\]

is \( \frac{1}{2}(m - 1) \) if \( m \) is odd, and \( \frac{m}{2} \) or \( \frac{1}{2}(m - 2) \) if \( m \) is even, and so is at least 5 if \( m \geq 11 \). Hence, for \( m \geq 11 \), there exists a pair \( c, d \) with both \( c \) and \( d \) distinct from \( a, b, a', b' \). Now (5.14) gives

\[
p_{n} + p_{0} + p_{c} + p_{d} = 0, \quad p_{n} + p_{0} + p_{c} + p_{d} = 0,
\]

whence

\[
p_{n} + p_{0} = p_{n} + p_{0}.
\]

(5.15)

It follows therefore that the linear form \( \sum p_i x_i \) vanishes for every vector (5.7) (where (5.8) holds), i. e. for all the minimal vectors of \( P_n \).

Hence, since \( P_n \) is perfect, \( p_{n} = p_{0} = \ldots = p_{n-1} \) for otherwise \( \sum p_i x_i \) would not be of the type (5.13). From any one equation (5.14) it now follows that all \( p_i = 0 \) and the quadratic form (5.13) vanishes identically. Hence \( Q_m \) is perfect.

If \( 8 \leq m \leq 10 \), the result (5.15) still holds whenever \( a + b = a' + b' \), although the above simple counting argument breaks down; thus \( Q_m \) is perfect for \( m \geq 8 \).

\( Q_m \) is not perfect for \( m \leq 7 \), having fewer than \( \frac{1}{2} m(m+1) \) minimal vectors.

6. The refinements \( A^{\text{ref}} \), \( B^{\text{ref}} \). We define \( B^{\text{ref}} \) to be the form whose lattice is the sublattice of \( P_m \) given by

\[
(6.1) \quad x_{q+1} = x_{q+2} = \ldots = x_{q+16} \quad (i = 0, \ldots, r-1),
\]

\[
(6.2) \quad x_{q+1} = x_{q+2} = \ldots = x_{m} = 0 \quad (m \equiv 0 \pmod{24}),
\]

\[
(6.3) \quad x_{q+1} + x_{q+2} + \ldots + x_{m} = 0 \quad (m \equiv 0 \pmod{24}),
\]

\[
(6.4) \quad \sum x_i = 0 \quad (m \equiv 0 \pmod{24}).
\]

\( A^{\text{ref}} \) (with \( n = m - 1 \)) has the same definition, save that (6.4) is replaced by

\[
(6.5) \quad \sum x_i = 0.
\]

Here the positive integral parameters \( m, t, q, r \) are to satisfy \(^{(1)}\)

\[
t \geq 2, \quad r \geq 2, \quad q \geq r^t, \quad m \equiv r q \geq 8
\]

(and (6.2) is vacuous if \( m = r q \)).

Since the congruences (6.1), (6.2), (6.3) are independent and imply that

\[
\sum x_i = 0 \quad (m \equiv 0 \pmod{24}),
\]

so that (6.4) is only a condition modulo 2, the determinants of \( A(B^{\text{ref}}) \) and \( A(A^{\text{ref}}) \) are

\[
\begin{align*}
2^{t(r-1)(m-r+1)} - 2^{t(r-1)} & = 2^{t(r-1)}(m-r+1) \\
(2^{r(t-1)} - m^{r(t-1)})^2 & = 2^{r(t-1)}(m-r+1).
\end{align*}
\]

Hence

\[
D(B^{\text{ref}}) = 4^{2m-2r+2}, \quad D(A^{\text{ref}}) = m^{2m-2r}.
\]

\(^{(1)}\) The conditions \( t \geq 2, r \geq 2 \) merely ensure that the forms do not reduce to a multiple of \( B_m \) or \( A_m \).
The points of these lattices satisfying \( r_1 = r_2 = \ldots = r_m = 0 \) (mod \( t \)) clearly form sublattices which are the lattices of \( B_m(x) = \rho B_m(x) \), \( A_m(x) = \rho A_m(x) \) respectively. Since \( B_m \) and \( A_m \) are extreme for \( m \geq 8 \) and have minimum 2, Theorem 2.1 shows that \( B_m^{(2)} \) and \( A_m^{(2)} \) are extreme if they have minimum 2\( p \).

We now establish that in fact \( M = 2p \) for each form, and specify the minimal vectors. We begin with \( B_m^{(2)} \) and set

\[
x_{k+4} = \ldots = x_{k+1} = 0 \quad (i = 0, \ldots, r-1)
\]

Then, since \( x_1 \) is free (mod \( t \)) for \( f > rq \), we clearly have

\[
f(x) = \sum_{i=1}^{m} a_i^2 \geq q(a_1^2 + \ldots + a_{r-1}^2) \geq q(\ell a_r^2 + \ldots + a_1^2).
\]

Now, by (6.3), \( \sum a_i^2 = 0 \) (mod \( m \)), and so \( \sum a_i^2 \geq 2 \) unless either (i) \( a_i = \ldots = a_r = 0 \); or (ii) some two \( a_i \) are 1, \(-1\) and the rest are zero; or (iii) \( t = 2 \), some two \( a_i \) are 1, \(-1\) and the rest are zero. In the second and third cases \( \sum a_i^2 = 2 \) and \( f \geq 2q > 2p \), with equality only when \( q = \rho \). In the first case, all \( x_i = 0 \) (mod \( t \)) and the form assumes the same values as \( \rho B_m \); i.e. its least value is then \( 2p \), assumed at the points \( x_i = x_j \) (1 \( \leq i < j \leq m \)).

This shows that \( M = 2p \), as asserted. To calculate \( \sigma \), the number of pairs of minimal vectors, we note first that the minimal \( m(1, m-1) \) vectors \( e_i \) (1 \( \leq i < j \leq m \)), which exist in all cases; and these all are if \( q > \rho \). If \( q = \rho \) and \( t > 2 \), we obtain \( \ell r(1-r) \) further minimal vectors by choosing, in the above notation, \( a_i = 1 \), \( a_j = -1 \) for \( 1 \leq i < j \leq r \). Finally, if \( t = 2 \), \( q = \rho = 4 \), all additional minimal vectors are obtained by choosing any \( r \) of the \( 4 \) sets \( \{a_i, 1 \leq i \leq r \} \), \( \{a_i, 1 \leq i \leq r \} \), \( \{a_i, 1 \leq i \leq r \} \), \( \{a_i, 1 \leq i \leq r \} \) and taking the values of these variables to be any permutation of

\[
(1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4, 1_4)
\]

with all other variables zero. This yields in all

\[
\binom{r}{2} \left( 1 + \binom{r-1}{2} + \binom{r}{2} \right) = 32r(r-1)
\]

pairs of vectors. Thus for \( B_m^{(2)} \) we have

\[
s = \begin{cases}
m(m-1) & \text{if } q > \rho, \\
m(m-1) + \frac{1}{2} \rho^2(r-1) & \text{if } q = \rho, \\
m(m-1) + 32r(r-1) & \text{if } q = \rho = 4.
\end{cases}
\]

It now follows at once that \( A_m^{(2)} \) has also \( M = 2p \); its minimal vectors being those of \( B_m^{(2)} \) which satisfy (6.5). We thus find, for \( A_m^{(2)} \),

\[
s = \begin{cases}
\frac{1}{2}r(n+1) & \text{if } q > \rho, \\
\frac{1}{2}r(n+1) + \frac{1}{2}r(r-1) & \text{if } q = \rho, \\
\frac{1}{2}n(n+1) + \frac{3}{2}r(r-1) & \text{if } q = \rho = 4.
\end{cases}
\]

It is noteworthy that no two of the above forms are equivalent, despite the fact that different forms (either \( A \) or \( B \)) with \( q > \rho \) may agree in minimum, determinant, number of minimal vectors and even in the geometrical configuration of their minimal vectors. To see this, we need only consider forms with the same \( m, t, r \) and different \( q > \rho \), the inequivalence being otherwise trivial. Such forms take the values \( 2p^2, 4p^2, \ldots \) at precisely the same points, when all variables are congruent to zero (mod \( t \)). But the above analysis shows that the least value assumed at any other point is \( 2q \), so that forms with different values of \( q \) cannot be equivalent.

Finally we note that the absolutely extreme forms \( E_n \) are equivalent to \( A_2^{n+4}, A_2^{n+4} \) respectively; \( T_n \) is equivalent to \( B_2^{n+4} \); and \( J_n \) (found by Chaundy and given in [6]) is equivalent to \( B_2^{n+4} \).

7. The refinement \( \sigma \). For \( t > 2 \), we define \( \sigma \) to be the form

\[
f(x) = A_\alpha(x) = \sum_{i=1}^{m} a_i^2 \left( \frac{\alpha}{\rho} e_i \right)^2
\]

with lattice the sublattice \( \vartheta_n \) given by

\[
x_1 = x_2 = \ldots = x_m \text{ (mod } t)\).
\]

Coxeter [5] has defined a form \( A_\alpha \) under the condition \( t(n+1) \); it is easily verified that this agrees with the above definition when \( t(n+1) \).

Since \( D(A_m) = n+1 \) and the lattice (7.2) has determinant \( \rho \),

\[
D(A_\alpha) = (n+1)e^{(n+1)}
\]

Now the sublattice of (7.2) given by \( x_i = 0 \) (mod \( t \)) \( i = 1, \ldots, n \) is \( U_n \), and so corresponds to the form \( A_\alpha(x) = \vartheta_\alpha A_m(x) \), which is extreme with minimum \( 2p \). Hence \( A_\alpha \) is a refinement of \( \vartheta_\alpha A_m \) and, by Theorem 2.1, \( A_\alpha \) is extreme if its minimum is \( 2p \); in this case, we have

\[
A(A_\alpha) = \frac{n+1}{\rho}
\]
It is easy to show that $M(A_n^4) < 2^p$ if $n < 2t-1$. For if we choose
\[ x_1 = \ldots = x_{n-1} = 1, \quad x_n = 1-t \]
we obtain
\[ f(x) = (n-1)(1-t)^4 + (n-t)^4 < 2t - 2(1-t)^4 + (1-t)^4 = 2t-2. \]
For the application of Theorem 2.1, it suffices then to assume that
\[ m = n+1 > 2t. \]

**Theorem 7.1.** (i) We have
\[ M(A_n^4) = 2^t, \]
and $A_n^4$ is extreme, if and only if either
\[ m - Vm > 2t, \]
or
\[ m > 3t; \]
except for the (imperfect) forms $A_1^4, A_2^4$.

(ii) The corresponding number of minimal vectors $s$ is given by
\[ s = 2^t \quad \text{if} \quad m - Vm > 2t, \]
\[ s = \frac{1}{2}m(m+1) \quad \text{otherwise}, \]
with the following four exceptions:
\[ A_1^4 (s = 63), \quad A_2^4 (s = 71), \quad A_3^4 (s = 120), \quad A_4^4 (s = 129). \]

For the proof of these results, we denote by $\varphi(a)$ the minimum of $A_n(a)$ for
\[ a = a \mod 4 \quad (i = 1, \ldots, n), \quad x \neq O; \]
thus $M(A_n^4)$ is the minimum of $\varphi(a)$ for $0 \leq a \leq \frac{4}{3}$.

Clearly $\varphi(0) = 2^t = \min A_n(4t^2)$, and is attained at $s_n = \frac{1}{4}n(n+1)$ pairs of points $\pm e_i, e_i - e_j \ (i \neq j)$.

If $0 < a < \frac{4}{3}$, it is easily seen that $\varphi(a)$ is attained when each $a_i$ is $a$ or $a+1$ and if $k$ coordinates are $a-1$ and $n-k$ are $a$ we obtain
\[ f(x) = a(n+1)(na - 2ht) + (k^2 + n)^t. \]
Writing for convenience
\[ q = \frac{m}{t} > 2, \]
we may write this as
\[ f(x) = t^t - \frac{a}{2}q^3 + t^t(\frac{a}{2}q - 1)-ta^q. \]
$\varphi(a)$ is thus the least value of this expression for integral $k$ with $0 \leq k \leq n$, and so is attained when $k = [q]$.
\[ \varphi(a) = t^t([aq] + \frac{a}{2}q - 1) - ta^q \quad (0 < a < \frac{4}{3}). \]

We can now prove

**Lemma 7.1.** We have
\[ M = M(A_n^4) = \min [2^t, \varphi(1)], \]
and $\varphi(a) > M$ for $2 \leq a \leq \frac{4}{3}$.

**Proof.** If $a \geq 3$, then $t \geq 2a \geq 6$, and (7.9) gives
\[ \varphi(a) > t^t(aq - 1) - ta^q = ma(t-a) - \frac{1}{4}t^4 > 6(t-3) - \frac{1}{4}t^4 > 2^t. \]
If $a = 2$, then $t \geq 4$, $m \geq 2t \geq 8$, and
\[ \varphi(2) > t^t(2q - 1) - 4aq = 2m(t-2) - \frac{1}{4}t^4 > 4(t-2) - \frac{1}{4}t^4. \]
Hence certainly $\varphi(2) > 2^t$ if $t \geq 5$. If however $t = 4$, then
\[ \varphi(2) > 2m(t-2) - \frac{1}{4}t^4 = 4m - 4, \]
whereas, by (7.9),
\[ \varphi(1) \leq \frac{1}{4}t^4 + t^t(q-\frac{1}{2}) - \frac{1}{4}t^4 = 3m; \]
thus $\varphi(2) > \varphi(1)$, since $m > 8$.

The lemma follows at once, since $M = \min \varphi(a)$ ($0 \leq a \leq \frac{4}{3}$).

**Lemma 7.2.** Let $s_i$ denote the number of pairs $\pm x$ satisfying
\[ f(x) = \varphi(1), \quad s_i = 1 \mod 4 \quad (i = 1, \ldots, n). \]
Then
\[ s_i = \begin{cases} \frac{m}{q} & \text{if } q \text{ is not integral}, \\ \frac{m+1}{q} & \text{if } q \text{ is integral and } t > 2, \\ \frac{m}{q} & \text{if } q \text{ is integral and } t = 2. \end{cases} \]

**Proof.** $\varphi(1)$ is the least value of (7.8), with $a = 1$, and is attained only when $k = [q]$ if $q$ is not integral; this gives the first result of (7.10).
If $q$ is integral, $\varphi(1)$ is attained when $k = q$ or $q - 1$, and we obtain
\[ \left( \frac{n}{q} \right) + \left( \frac{n-1}{q} \right) = \left( \frac{n+1}{q} \right) \]
representations if $t \neq 2$. If however $t = 2$, the vectors with $k = q - 1$ are simply the negatives of those with $k = q$, since now
\[ 1 - t = -1 \quad (\text{mod} \, t) \]

The proof of Theorem 7.1 is now easily completed. By (7.4), $q > 2$; if now (7.7) is not satisfied we have $2 < q < 3$ and so, by (7.9),
\[ \varphi(1) = \left( \frac{n}{q} - 1 \right) ^2 + \left( \frac{n-1}{q} \right) ^2 - n = m^2 \quad \text{(mod) } q | n t^2. \]
By Lemma 7.1, (7.5) holds if and only if $\varphi(1) \geq 2t^2$, i.e.,
\[ (m - 2t)^2 - m \geq 0, \quad \text{or} \quad m - 2t^2 \geq 0, \]
which is (7.6).
If now (7.6) holds with inequality, we have $\varphi(1) > 2t^2$, whence
\[ s = n - 1 = \frac{1}{2} n(n+1) \]
and, using (5.9),
\[ s = s_0 + s_1 = \frac{1}{2} n(n - 1) + \frac{1}{2} n = n^2. \]
This establishes Theorem 7.1 when $q < 3$, i.e., when (7.7) does not hold.
Suppose now that (7.7) holds, so that $q > 3$. By (5,8),
\[ \varphi(1) > \left( q - \frac{3}{2} \right) ^2 - n \geq \frac{4}{9} t^2 - 3t, \]
so that certainly $\varphi(1) > 2t^2$ if $t \geq 5$. Thus
\[ M = 2t^2, \quad s = s_0 = \frac{1}{2} n(n + 1) \quad \text{if} \quad t \geq 5. \]
We now consider separately the values 3, 4, 4 of $t$.
(a) If $t = 2$, (7.9) gives
\[ \varphi(1) = 4 \left( \frac{n}{q} - q + \frac{1}{2} t^2 + 2q - 1 \right), \]
whence $\varphi(1) > 9$ if $q \geq 5$. The remaining possible values 3, 3, 4, 4 of $q = \frac{1}{2} m$ give respectively $\varphi(1) = 6, 6, 8, 8$. Thus
\[ M = \varphi(1) < 2t^2 \quad \text{for} \quad A_1, A_2, \]
\[ M = \varphi(1) = 2t^2 \quad \text{for} \quad A_3, A_4; \]
\[ M = 2t^2 < \varphi(1) \quad \text{for} \quad A_5, u \geq 9. \]
Here the forms $A_1, A_2$ are easily seen to be imperfect; for $A_3, A_4$, we have $s = s_0 + s_1 = 63, 71$ respectively; for $A_5$ ($u \geq 9$), $v = s_0 = \frac{1}{2} u(n+1)$.

(b) If $t = 3$, (7.9) gives
\[ \varphi(1) = 9 \left( \frac{n}{q} - q + \frac{1}{2} t^2 + 6q - \frac{3}{2} \right), \]
whence $\varphi(1) > 18$ if $q \geq 4$. The remaining possible values 3, 3, 4, 4 of $q = \frac{1}{4} m$ give respectively $\varphi(1) = 18, 18, 20, 20$. Thus
\[ M = \varphi(1) = 2t^2 \quad \text{for} \quad A_1, A_2; \]
\[ M = 2t^2 < \varphi(1) \quad \text{for} \quad A_3, u \geq 19. \]
For $A_4$ ($u \geq 19$) we therefore have $s = s_0 = \frac{1}{2} u(n+1)$; for $A_5, A_6, s = s_0 = 120, 129$ respectively.
(c) If $t = 4$, (7.9) gives
\[ \varphi(1) = 16 \left( \frac{n}{q} - q + \frac{1}{2} t^2 + 12q - 4 \right), \]
whence $\varphi(1) > 2t^2 = 32$ for all $q \geq 3$. Thus here $M = 2t^2, s = s_0 = \frac{1}{2} n(n+1)$.

This completes the proof of Theorem 7.1.
To settle the possible equivalence of a form $A_1$ with one of the forms $A_{3(\alpha)}$ of § 6, we prove:

**Theorem 7.2.** The only equivalences among the extreme forms $A_n, A_{3(\alpha)}$ are

\[ (7.11) \quad A_{14-n} \sim A_{12(n-1)}, \quad A_{14} \sim A_{13(\alpha)} \quad (q \geq 4). \]

**Proof.** The relations (7.11) are easily verified, the forms being in fact identical. Comparison of the values of $s$ and $\lambda$, viz.
\[ A(A_{14}) = \frac{n+1}{t}, \quad A(A_{13(\alpha)}) = \frac{n+1}{t^{-2}}, \]
shows that the only other possible equivalence is

\[ (7.12) \quad A_4 \sim A_{3(\alpha)}, \quad q > t \]
(each form having $A = \frac{1}{2}(n+1)^2, s = \frac{1}{2} n(n+1)$).
Now both $A_4, A_{3(\alpha)}$ take the values $2t^2, 4t^2, \ldots$ of $A_6$ (or $A_7$) at precisely the same points (when all variables are zero modulo 4). As was shown in § 6, the least value assumed by $A_{3(\alpha)}$ at any other point is $2q$. For $A_4$, the above analysis shows that the least value assumed at any other point is
\[ s = \min \{ s(a) \} \quad (1 \leq a \leq 4), \]
Now (7.9) gives
\[ v \geq \frac{f(a)}{t} \left( \frac{m}{t} - \frac{1}{4} \right) = a^b - ma(t-a) - \frac{1}{4}t^2, \]
\[ \geq m(t-1) - \frac{1}{4}t^2. \]
Since \( m > 2q > 2t^2 \), we obtain
\[ v > m(t-1) > 2q(t-1), \]
whence \( v > 2q \) if \( t \geq 3 \).

Thus the equivalence (7.12) cannot hold if \( t \geq 3 \). For \( t = 2 \), (7.12) reduces to (7.11) (with \( q = 4 \)). This proves the theorem.

We note finally the possibility that \( A_n \) may be perfect or extreme with \( M < 2t^2 \) (for which Theorem 2.1 would not apply). A slight extension of the above analysis shows that this occurs only when \( m = 2t \).

This case has been dealt with by Coxeter [5] and so we merely state:

**Theorem 7.3.** If \( m = n+1 = 2t \), then \( A_n \) is extreme, with
\[ M = 2t(t-1), \quad s = \frac{1}{4}m(n+1), \]
and so
\[ A = (n+1)^{n=1}(t-1)^{n}. \]

8. The refinement \( B_n \). Continuing the analogy between forms \( A \) and \( B \), we define \( B_n \) as
\[ f(x) = \sum_{i=1}^{m} x_i^2 \]
with lattice the sublattice of \( f_n \) given by
\[ x_1 = x_2 = \ldots = x_n \pmod{t}, \]
\[ x_1 = x_2 = \ldots = x_n \equiv 0 \pmod{2t}. \]
Here \( t \geq 2 \) and, as always, \( m = n+1 \). Clearly
\[ D(B_n^*) = 4t^{n-1}, \]
and \( B_n \) is a refinement of \( B_n(2x) = f^2 B_n(x) \). Hence, by Theorem 2.1, \( B_n \) is extreme provided that \( M(B_n^*) = 2t^2 \); in this case, \( A(B_n^*) = 4t^2 \).

It is easily verified that
\[ B_n^* = B_n^* B_n^*; \quad B_n^* = B_n^* B_n^*. \]

so that these forms (which are extreme for \( q \geq 4 \)) have been dealt with in \( \S \). 6.

We therefore suppose henceforward that \( t \geq 3 \), and define a unique integer \( k \) by
\[ n = \pm k \pmod{2t}, \quad 0 \leq k \leq t. \]

**Theorem 8.1.** For \( t \geq 3 \), \( B_n^* \) is extreme with \( M = 2t^2 \), if and only if
\[ n \geq 2t^2 - k^2 \quad \text{when} \quad 0 \leq k \leq t-1, \]
\[ n \geq t(t+2) \quad \text{when} \quad k = t. \]

The number \( s \) of minimal vectors is then given by
\[ s = m(m+1), \]
unless either
(i) \( n = t(t+2) \) and \( t \) is odd, when \( s = m^2 - 1 \); or
(ii) \( n = t(t+1) - 1 \) and \( t \) is even, when \( s = m^2 - 1 \); or
(iii) \( n = 2t^2 - k^2 \), \( 0 \leq k \leq t-2 \) and \( k^2 + k = 0 \pmod{2t} \), when \( s = m(m+1)+1 \).

As in \( \S \) 7, we define \( \varphi(a) \) to be the minimum of \( f(a) \) subject to
\[ a_1 = \ldots = a_n = a \pmod{t}, \quad 0 \leq a \leq \frac{1}{4} t \quad (a \neq 0); \]
then \( M(B_n^*) = \min \varphi(a) \).

We have at once \( \varphi(0) = M(B_n^*) = 2t^2 \), attained at the \( s_1 = m(m+1) \) point \( x = t e_1 \pm t e_2 \) (\( i < j \)).

If \( a = 1 \), (8.3) and (8.4) give
\[ x_m = \sum_{1}^{n} x_i = -n = \pm k \pmod{2t}, \]
and the least possible value of \( |x_m| \) is therefore either \( k \) or \( t-k \). From this, and (8.3), it follows easily that \( \varphi(1) \) is the lesser of
\[ v_1 = n+k^2, \]
attained when \( x_1 = x_2 = \ldots = x_n = 1, x_m = \mp k \), and
\[ v_3 = n-1+(t-1)^2+(t-1)^2, \]
attained when \( x_1 = 1-t, x_2 = \ldots = x_n = 1, x_m = \pm k \). Since
\[ v_2 > v_1 \quad \text{for} \quad 0 \leq k \leq t-2, \quad v_1 = v_3 \quad \text{for} \quad t = 1, \quad v_2 < v_3 \quad \text{for} \quad t = k, \]
we have
\[ \varphi(1) = \begin{cases} n+k^2 & \text{if} \quad 0 \leq k \leq t-1, \\ n-1+(t-1)^2 & \text{if} \quad k = t. \end{cases} \]
The conditions (8.5), (8.6) are therefore necessary for \( M(B_n^a) = 2^d \). Also, they imply that always
\[
n \geq 2^n - (t-1)^d = t^d - 2^t - 1;
\]
for \( 2 \leq a \leq \frac{1}{2}l \) we therefore have crudely
\[
\varphi(a) \approx 2^n \geq 4 \cdot (t^d - 2^t - 1) > 2^d.
\]
It follows that the inequalities (8.5), (8.6) are necessary and sufficient to ensure that \( M(B_n^a) = 2^d \).

To determine \( s \), the number of (pairs of) minimal vectors, we observe first that \( s = s_0 = m(m-1) \), unless equality holds in (8.5) or (8.6), when \( s = s_1 = s_2 \).

If equality holds in (8.6) we have, with (8.4),
\[
n = t(t+2) = t \mod(2),
\]
whence \( t \) is odd; and the representations of \( v_4 \) above give \( s_i = n \).

If equality holds in (8.5) and \( t = t+1 \) we have, with (8.4),
\[
n = t(t+2) - 1 = t \mod(2),
\]
whence \( t \) is odd; and the representations of \( v_1 \) and \( v_2 \) above give \( s_i = n+1 = m \).

If equality holds in (8.5) and \( 0 \leq s \leq 0 \) we obtain (iii) of Theorem 8.1; and \( s_i = 1 \) from the single representation of \( v_i \) above.

This completes the proof of Theorem 8.1.

By an analysis very similar to that of \( \S \) 7 we may establish:

**Theorem 8.2. The only equivalences among the extreme forms \( B_n^a \) are given by**

\[
B_n^{a_1} \sim B_n^{a_2}, \quad B_n^{a_3} \sim B_n^{a_4} \sim B_n^{a_5},
\]

\[
B_n^{a_2} \sim B_n^{a_3} \text{ if } t \geq 3 \text{ and } t \mid q \text{ or } t \mid (q-1),
\]

\[
B_n^{a_2} \sim B_n^{a_5} \text{ if } t \geq 3 \text{ and } t \mid q.
\]

We note that (8.7) has been noted above, while the equivalences in (8.8) and (8.9) arise simply from a change of sign of the variables \( a_{x+1} \), \( a_{y+1} \) or \( a_{x+1}, a_{y+1} \).

The simplest new forms \( B_n^a \) are:

\[
B_n^{19} \text{ with } s = 19^2 = 361,
\]

\[
B_n^{23} \text{ with } s = 26^2 = 676,
\]

\[
B_n^{35} \text{ with } s = 36^2 = 1296,
\]

\[
B_n^{47} \text{ with } s = 49^2 = 2401.
\]

We therefore have \( \varphi(a) \equiv 0 \mod(2) \).

9. **Refinements of \( L_n^a \).** We may obtain several types of refinement of \( L_n^a \) by the same devices as were used above to refine \( B_n^a \) and \( A_n \), beginning with the basic form

\[
f(x) = \sum_{t=1}^n (x_t^2 - x_t x_{t+1} + x_{t+1}^2) + \sum_{k=0}^{n-1} x_k^2 \quad (m \geq 2r)
\]

and taking sublattices of \( f(x) \). The analysis becomes rather complicated if done with complete generality, and most of the resulting forms require large values of \( n \).

We shall therefore consider here only one type of refinement, which yields new perfect or extreme forms for all \( n \geq 11 \).

We define \( M_2^a \) to be the form (9.1) with integral \( a \) subject to

\[
\sum_{t=1}^n x_t = 0,
\]

\[
x_{1} = x_2 = \ldots = x_n \mod(2).
\]

Since the sublattices of \( M_2^a \) with all \( x_t = 0 \mod(2) \) clearly gives the form \( M_0^a \), then \( M_2^a \) is a refinement of \( 2M_0^a \). By Theorem 4.1 and \( \S \) 4, \( M_2^a \) is perfect if \( m \geq 2r \geq 6 \) and extreme if also \( m \leq 4r-1 \) provided that it has minimum 8. In this case, we have

\[
\Delta(M_2^a) = 3^{r(n+1)}(n+1)^{2m+2}.
\]

Let \( \varphi(a) \) be the minimum of \( f(x) \) for \( a \neq 0 \) subject to (9.2) and \( a_t = a \mod(2) \) \((1 \leq t \leq n)\); then \( M = \min \{\varphi(0), \varphi(a)\} \).

If \( a = 0 \), (9.2) shows that also \( m_0 = 0 \mod(2) \); hence, by \( \S \) 4, \( \varphi(0) = 8 \), with \( a_t = 1 \mod(n+1)+4 \mod(r-3) \) representations.

If \( a = 1 \), (9.2) gives

\[
x_t = -x_{t+1} = n \mod(2).
\]

We set for convenience \( \mu = m \) or \( m-1 \) as \( m \) is even or odd, thus \( \mu \) is even and \( \mu \geq 2r \). Now, by (9.5), \( m_0 \) is even when \( m \) is even, and even when \( m \) is odd, and \( a_{n_0} \) is least for \( m_0 = \pm 1 \) or \( m_0 = 0 \). Since each \( \varphi(a_t, a_{t+1}) \geq 1 \), with equality only when \( (a_t, a_{t+1}) = (\pm 1, 1) \), we therefore have

\[
\varphi(1) \geq r+\mu(\mu-2r) = \mu-r.
\]

It is easy to see that the equality sign holds unless \( \mu = 2r \) and \( r \) is odd, since (9.2) may be satisfied by suitably choosing \( (a_t, a_{t+1}) = (\pm 1, 1), a_{n_0} = \pm 1 \) \((k = 2r+1, \ldots, n)\). If however \( \mu = 2r \) and \( r \) is odd, we have \( \varphi(1) = \mu-r+2 \), and is attained, for example, when \( (a_t, a_{t+1}) = (1, 1) \) for \( 1 \leq t \leq \frac{r}{2}(r+1) \), \((a_t, a_{t+1}) = (-1, -1) \) for \( \frac{r}{2}(r+1) \leq t \leq \frac{r}{2}(r-1) \), and \((a_t, a_{n_0}) = (1, -1) \).
Summing up, we have

\[ \varphi(1) = \mu = -r + 2 \quad \text{if} \quad \mu = 2r \quad \text{and} \quad r \quad \text{is odd,} \]

\[ \varphi(1) = \mu = -r \quad \text{otherwise;} \]

and \( M^{\varphi}_{2} \) has minimum 8 if and only if \( \varphi(1) \geq 8 \), i.e., if

\[ \mu \geq r + 6 \quad \text{if} \quad \mu = 2r \quad \text{and} \quad r \quad \text{is odd,} \]

\[ \mu \geq r + 8 \quad \text{otherwise.} \]

To determine the value of \( s \), we note that \( s = s_{2} \), if inequality holds in (9.7); also the equality sign cannot hold in (9.7), since it would give \( r = 6 \) and 6 is not odd. Equality in (9.7) implies that \( r \) is even and \( r \leq 8 \);

\[ \mu \geq 2r, \quad r \leq 8; \]

thus equality holds in (9.7) only for the extreme forms \( M^{(1)}_{2}, M^{(1)}_{3}, M^{(2)}_{5}, M^{(2)}_{6}, M^{(2)}_{8}, M^{(2)}_{11}, M^{(2)}_{14} \). For these we find respectively

\[ s_{1} = 22, 23, 29, 30, 32, 33, 35, 35 \]

representations of \( \varphi(1) = 8 \); and \( s = s_{2} + s_{1} \), where \( s_{1} = s(M^{(2)}_{2}) = \frac{1}{2}(m+1)+s(r-3) \).

We may inquire, as with \( A^{*}_{4} \), whether \( M^{(2)}_{2} \) can be perfect or extreme even when \( M^{(2)}_{2} \) is not, i.e., when \( r \leq 2 \). This can happen only when \( M^{(2)}_{2} \) has more minimal vectors, i.e., when equality holds in (9.7). Excluding \( r = 0 \) (which reduces us to \( A^{*}_{2} \)), we have as the only possibility \( r = 2, \mu = 10, \) i.e., \( m = 10 \) or 11.

The corresponding forms \( M^{(2)}_{2}, M^{(2)}_{3} \) are however not perfect (in spite of their relatively large number of minimal vectors, viz. 70 and 80 respectively); all minimal vectors in fact satisfy the relation \( s_{1}, s_{2} - s_{1}, s_{1}, s_{2} - s_{1} = 0 \).

10. Conclusion. The method of "refinement" described in this article is clearly capable of much wider application, though it is unlikely that any new forms would appear for small values of \( n \). A list of the distinct forms discussed here for \( n = 7, 8 \) and 9 follows; it includes all previously known perfect or extreme forms for those values of \( n \), and will serve as a basis of reference for part II. The table gives the name of the form, its number \( s \) of minimal vectors, its value of \( A = (2)M^{(2)}_{2}D_{1} \), and indicates whether the form is extreme \((E)\) or perfect and non-extreme \((P)\).

<table>
<thead>
<tr>
<th>Form</th>
<th>( s )</th>
<th>( A )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A^{*}_{1} )</td>
<td>28</td>
<td>8</td>
<td>E</td>
</tr>
<tr>
<td>( B^{*}_{1} )</td>
<td>42</td>
<td>4</td>
<td>E</td>
</tr>
<tr>
<td>( I^{*}_{2} )</td>
<td>30</td>
<td>3/2</td>
<td>P</td>
</tr>
<tr>
<td>( J^{*}_{2} )</td>
<td>38</td>
<td>3/2</td>
<td>E</td>
</tr>
<tr>
<td>( M^{*}_{2} )</td>
<td>28</td>
<td>3/2</td>
<td>E</td>
</tr>
<tr>
<td>( P^{*}_{3} )</td>
<td>38</td>
<td>3/2</td>
<td>E</td>
</tr>
<tr>
<td>( A^{*}_{2} )</td>
<td>28</td>
<td>3/2</td>
<td>E</td>
</tr>
<tr>
<td>( B^{*}_{2} )</td>
<td>42</td>
<td>4</td>
<td>E</td>
</tr>
</tbody>
</table>

References


University of Sydney, Australia

Reçu par la Rédaction le 26. 3. 1938