On the probability that \( n \) and \( g(n) \) are relatively prime

by

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1. Introduction. It is a well-known theorem of Čebysjov that the probability of the relation \((n, m) = 1\) is \(6\zeta^{-2}\). One can expect this still to remain true if \( m = g(n) \) is a function of \( n \), provided that \( g(n) \) does not preserve arithmetic properties of \( n \). In this paper we consider the case when \( g(x) \) is the integral part of a smooth function \( f(x) \), which increases slower than \( x \). More exactly, let \( Q(x) \) be the number of \( n \leq x \) with the property \( (n, g(n)) = 1 \). The probability that \( n \) and \( g(n) \) are relatively prime is then by definition the limit \( \lim_{x \to \infty} \frac{Q(x)}{x} \).

Our main result is that if \( f(x) \) satisfies some mild smoothness assumptions, has the property (A) \( f(x) = o(a) \log \log a \) and satisfies condition (B) of § 2, then the probability in question exists and is equal to \( 6\zeta^{-2} \). Condition (B) means roughly that \( f(x) \) increases faster than the function \( \log \log \log x \).

In § 3 we show that condition (B) is the best possible. Condition (A) may be perhaps relaxed; but it cannot be replaced by \( f(x) = O(a) \log \log \log x \).

We also consider the average number of divisors of \((n, g(n))\). This is the limit \( \lim_{x \to \infty} \frac{S(x)}{x} \), where \( S(x) \) is the sum of the numbers of divisors of all numbers \((n, g(n))\), \( n \leq x \). We assume throughout that \( f(x) \) is a monotone increasing positive function with a piecewise continuous derivative; \( F(y) \) will denote the inverse of \( f(x) \).

By \( \varphi, \mu, \sigma, d \) we denote the standard number-theoretic functions, by \( \log \log, \log, \log \ldots \) the iterated logarithms of \( x \).

We begin with some elementary identities. Let \( \Omega_k(x) \) be the number of integers \( n \leq x \) such that \( n \) and \( g(n) \) have no common factors \( \leq k \).

If \( S(x, d) \) is the number of \( n \leq x \) with \( d|n, g(n) \), then

\[
\sum_{d|n} \mu(d) S(x, d) = \sum_{d|n} \mu(d) \sum_{d|\varphi(n)} 1 = \sum_{d|\varphi(n)} \sum_{d|\varphi(n)} \mu(d).
\]

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By the properties of the function \( \mu \), the inner sum is 1 if \( \{k!, n, g(n)!\} = 1 \)
and otherwise is 0. Hence
\[
Q_k(x) = \sum_{d \leq x} \mu(d) S(x, d).
\]
In particular, if \( k = n \), then, since \( S(x, d) = 0 \) for \( d > g(x) \), we obtain
\[
Q(x) = \sum_{d \leq x} \mu(d) S(x, d).
\]
There are similar but obvious formulas for \( S(x) \) and \( S_k(x) \) — the sum of the numbers of divisors, not exceeding \( k \), of all numbers \( n, g(n) \) with \( n \leq x \), namely
\[
S_k(x) = \sum_{d \leq x} S(x, d),
\]
\[
S(x) = \sum_{d \leq x} S(x, d).
\]
A function \( f(x) \) will be called homogeneously equidistributed modulo 1 (or shortly h. e.) if for each integer \( d \),
\[
h(x) = \frac{1}{d} f(d x)
\]
is equidistributed modulo 1. This means that for each subinterval \( I \) of \((0, 1)\), the density of \( n \)'s for which \( h(n) - [h(n)] \) belongs to \( I \), is equal to the length of \( I \).

**Theorem 1.** If \( f(x) \) is homogeneously equidistributed, then
\[
\lim_{x \to \infty} \frac{Q(x)}{x} < 6 \pi^{-2}, \quad \lim_{x \to \infty} \frac{S(x)}{x} > \frac{1}{6 \pi^2}.
\]

**Proof.** It follows from the definition of \( S(x, d) \) that this is the number of integers \( k \) with \( kd \leq x \) and \( dg(kd) \); or the number of \( k \leq x d^{-1} \) so that
\[
\frac{1}{d} f(kd) = \frac{1}{d} f(kd) \in I,
\]
is in the interval \((0, 1] \). Since \( f(x) \) is h. e., \( \lim_{x \to \infty} S(x, d)/x = d^{-1} \). Taking now into consideration the relations (1), (3) and the inequalities \( Q_k(x) \geq Q(x) \), \( S_k(x) \leq S(x) \), we obtain (5), since
\[
\sum_{d \leq x} d^{-1} \mu(d) = 6 \pi^{-1}, \quad \sum_{d \leq x} d^{-1} = \frac{1}{6 \pi^2}.
\]

All known simple criteria for \( f(x) \) to be equidistributed modulo 1 (by Weyl, Pólya-Szegö, see Kokama [2], p. 88) guarantee also that \( a f(x) \)
is equidistributed for arbitrary positive constants \( a, b \). The simplest set of conditions is

\[(A_1) \quad f(x) = o(x) \quad \text{for} \quad x \to \infty,
\]
\[(B_1) \quad a f(x) \to \infty \quad \text{for} \quad x \to \infty,
\]
and the additional hypothesis that \( f'(x) \) decreases. We shall mention here that the last assumption and \( B_1 \) can be replaced by

\[(C_1) \quad \int_0^y |F''(u)| du = o(F'(y))
\]

\((F'(y) \) is assumed here to have a piecewise continuous second derivative). If \( f'(x) \) decreases, the last integral is equal to \( 1/f'(x)+\text{const} \) with \( x = F'(y) \), and hence \((C_2)\) is implied by \( B_1 \). Further natural conditions which in the presence of \( B_1 \) imply \( C_1 \) are

\[
\lim_{y \to \infty} \int_0^y |F''(u)| du = 0 \quad \text{or} \quad \int_0^y |F''(u)| du = O(F'(y)).
\]

To establish our statement it is sufficient to show that \( f(x) \) is equidistributed mod 1 if it satisfies \((A_1)\) and \((B_1)\). Let \( I = (a, a + \delta) \subseteq (0, 1] \), then the number of \( n \)'s for which \( [f(n)] - k \) and \( f(n) - [f(n)] \) belongs to \( I \), is \( \Delta F_k + O(1) \), where \( \Delta F_k = F'(a_b) - F'(a_k), a_b = k + \alpha + \beta, a_k = k + n, \) except if \( k + \alpha + n > f(n) = y \) when \( \alpha_k = y \). Because of \((A_1)\), the total number of \( n \leq x \) with \( f(x) - [f(n)] \) \( I \) is

\[
N = \sum_{k + \alpha \leq x} \Delta F_k + o(x).
\]
Now
\[
\left| \frac{\Delta F_k}{\delta} - [F(x_k) - F(x_{k-1})] \right| = \left| F'(y_k) - F'(y_{k-1}) \right| \leq \sum_{k=1}^{n_k} \int_{x_{k-1}}^{x_k} |F''(u)| du,
\]

hence
\[
\frac{1}{x} N = \sum_{k + \alpha \leq x} \left| F(x_k) - F(x_{k-1}) \right| + \frac{1}{x} \int_{x_{k-1}}^{x_k} |F''(u)| du = \delta + o(1),
\]
by \((C_1)\).
2. The Main Theorem. Our main result is the following:

**Theorem 2.** Let \( f(a) \) be h. e. and let

\[
\begin{align*}
\text{(A)} & \quad f(x) = o(x^{1/\log a}), \\
\text{(B)} & \quad \frac{x f(x)}{\log f(x)} \to \infty, \\
\text{(C)} & \quad f'(y) \leq M f'(a) \text{ for some constant } M \text{ for all } y \geq x > 0.
\end{align*}
\]

Then

\[
\lim_{x \to \infty} \frac{Q(x)}{x} = \frac{6}{\pi^2}.
\]

Proof. Let \( Q_k(x) \) be defined as in § 1. Then by (1),

\[
\lim_{x \to \infty} \frac{Q_k(x)}{x} = \sum_{d \leq x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + \delta_k,
\]

where \( \delta_k \to 0 \) for \( k \to \infty \). To prove the theorem it is therefore sufficient to show that

\[
\lim_{x \to \infty} \frac{R_k(x)}{x} = \infty
\]

is arbitrarily small if \( h \) is sufficiently large. Here \( R_k(x) = Q_k(x) - Q(x) \) is the number of \( n \leq x \) such that for some prime \( p \) with \( k < p \leq g(x) \) we have \( p|n, g|g(n) \). It follows that

\[
R_k(x) \leq \sum_{k < p \leq g(x)} S(x, p).
\]

We consider the contribution to the sum (8) of the part of the curve

\[
y = g(x) \text{ given by } g(n) = n; \text{ these } n \text{ satisfy } F(m) < n < F(m+1).
\]

We put \( k_m = F(m+1) - F(m) \), except when \( m+1 > x \), in which case

\[
\text{we put } k_m = F(x) - F(m).
\]

The contribution to \( S(x, p) \) is zero if \( p|k_m \), otherwise it does not exceed

\[
\frac{1}{p} [F(m+1) - F(m)] + 1 = \frac{k_m}{p} + 1.
\]

Hence

\[
H_k(x) = \sum_{k < p \leq g(x)} \sum_{k_m \leq x} \frac{k_m}{p} \leq \frac{1}{p} \sum_{k < p \leq g(x)} \left( \frac{k_m}{p} + 1 \right) \leq \sum_{k < p \leq g(x)} \frac{g(x)}{p} + \sum_{k < p \leq g(x)} \frac{k_m}{p} + \sum_{k < p \leq g(x)} \frac{k_{m+1}}{p} = \Sigma_1 + \Sigma_2 + \Sigma_3.
\]

On the probability that \( n \) and \( g(n) \) are relatively prime

say, where \( k_i = [g(x)/p] \). For \( x \to \infty \) we have by (A)

\[
\sum_{p \leq x} \frac{1}{p} = O(g(x) \log x) = o(x).
\]

For \( l > m \) we have with properly chosen \( \xi, \xi_1, \xi_2 \),

\[
k_l = \frac{1}{f'(\xi)}, \quad k_m = \frac{1}{f'(\xi)},
\]

hence by (C),

\[
k_m \leq M k_l, \quad l > m.
\]

Therefore,

\[
k_m + k_{m+1} + \ldots + k_{m+1} \leq M \frac{M}{p} (k_p + k_{p+1} + \ldots + k_{p+1}) \leq \frac{M}{p},
\]

so that for an arbitrary \( \varepsilon > 0 \),

\[
\Sigma_2 \leq \sum_{p \leq x} \frac{M}{p} \leq \varepsilon x,
\]

if \( k \) is sufficiently large.

The sum \( \Sigma_3 \) we split into two parts \( \Sigma_3, \Sigma'_3 \), the first sum being extended over all \( p \) for which

\[
\Sigma'_3 \leq \sum_{p \leq x} \frac{1}{p} \leq 2\varepsilon x.
\]

and where \( A = M^{-1} \), and the second corresponding to \( p \) for which the opposite inequality holds. In the first case by (10) and (C),

\[
\Sigma_2 \leq \frac{M}{g(x)} - \frac{M}{p} \leq \frac{M}{g(x)} - \frac{M}{g(x) \log g(x)} = \frac{M}{x} \log g(x),
\]

hence for large \( x \),

\[
\sum_{p \leq x} \frac{1}{p} \leq 2 \varepsilon x.
\]

In the second case, \( g(x) > 1 - A \log g(x) \leq p \leq g(x), \) hence \( p \)

divides one of the consecutive numbers \( g(x) + 1 - A \log g(x), \ldots, g(x) \), hence also their product \( N \). Clearly,

\[
N \leq f(x) f'(x) g(x).
\]
We use the relation (1)
\[ \sum_{\nu} \frac{1}{p} \leq C \log \nu \]
and obtain
\[ \sum_{\nu} \frac{1}{p} \leq \max_{\nu \in \nu(\nu)} \sum_{\nu} \frac{1}{p} \leq \max_{\nu \in \nu(\nu)} \frac{1}{\nu} \log f(\nu) \leq o(1) \]
for large \( \nu \), by (C) and (B). Substituting our estimates into (9), we obtain that (7) does not exceed 5\( \varepsilon \) for large \( x \).

**Theorem 3.** Let \( f(x) \) be h. e. and satisfy (C), moreover
(A') \[ f(x) = o(x \log x), \]
(B') \[ x f'(x) / \log f(x) \to \infty. \]
Then the average order of the number of divisors of \( (n, g(n)) \) is \( \frac{1}{2} \pi^2 \):
\[ \lim_{n \to \infty} \frac{S(n)}{x} = 1 \frac{\pi^2}{2}. \]

Instead of (8) we have now
\[ S(x) = \sum_{k < n \leq n(k)} S(x, n), \]
where \( n \) runs through all integers, prime or not. The proof is similar to that of theorem 2, but simpler.

3. **Counterexamples.** To show that condition (B) is the best possible in Theorem 2, we shall use the following fact. There is an absolute constant
\[ C \]
such that for each \( \varepsilon_1 > 0 \), there is an \( \varepsilon_2 > 0 \) and infinitely many values of \( n \) with the property
\[ \frac{\varphi(m)}{m} < \varepsilon \] for all \( m \) with \( n \leq m \leq n + \varepsilon_2 \log n \).

See [1], p. 129, where \( \sigma(m)/m > 2 \) is shown to be possible for \( n \leq m \leq n + C_1 \log n \). The same proof establishes \( \sigma(m)/m > 1/\varepsilon_1 \) in intervals \( n \leq m \leq n + \varepsilon_2 \log_n n \), and the known connections between \( \varphi \) and \( \sigma \) give (12).

**Theorem 4.** Let \( f(x) \) be increasing and let
(B') \[ x f'(x) \log^2 f(x) \to \infty. \]
Then
\[ \lim_{x \to \infty} \frac{Q(x)}{x} < \frac{6}{\pi^2}. \]

**Proof.** From (B') we obtain by integration \( f(x) \leq \log^a x \) for all large \( x \).
It follows also that \( f'(x) \to 0 \), hence that \( g(x) \) takes all large integral values. From (3), using the argument and notations of § 2 we have, if \( d(m) \) is the number of divisors of \( m \),
\[ Q(x) = \sum_{d=1}^{\varphi(m)} \mu(d) S(x, d) = \sum_{d=1}^{\varphi(m)} \mu(d) \frac{\psi(d)}{d^2} + O(1) \]
\[ = \sum_{d=1}^{\varphi(m)} \frac{\psi(d)}{d^2} \log d + \sum_{d=1}^{\varphi(m)} O(1) \]
\[ = \sum_{d=1}^{\varphi(m)} \frac{\psi(m)}{m} + O(x \log^2 x). \]

We take \( x \) such that \( g(x) = n \) is one of the \( n \) for which (12) holds. Let \( x_1 = (1 + \delta M) x \), \( \delta > 0 \), \( n_1 = g(x_1) \). Then we have by (B') for some \( \varepsilon < \delta < n_1 \),
\[ n_1 - n \leq 1 + f(x)(x_1 - x) \leq 1 + \frac{\delta}{M} x f'(x) \]
\[ \leq 1 + \delta \log^2 f(x) \leq 1 + \delta \log_2 n_1. \]
hence \( a_1 - n \leq \text{const} \cdot \log \log n \). By (14) and (12),

\[
Q(x_1) - Q(x) = \sum_{a \leq x < x_1} \frac{\sigma(n)}{n} + O(\log^2 x) < \varepsilon(x_1 - x) + O(\log^4 x),
\]

for an arbitrary \( \varepsilon > 0 \), if \( \delta \) is sufficiently small. This gives

\[
\frac{Q(x)}{x_1} = \frac{Q(x)}{x} + \frac{x_1 - x}{x_1} + o(1),
\]

and if \( s \) denotes the constant \( s = (1 + \delta)M^{-1} < 1 \), we obtain by Theorem 1,

\[
\lim_{x \to \infty} Q(x) = \frac{6}{\pi^2} \cdot \frac{s}{x} + o(1) = \frac{6}{\pi^2}.
\]

A simple computation shows that \( f(x) = \frac{\sigma(x) \log \log x}{x} \) satisfies (B') as stated in the introduction.

In the same way we can prove \( \lim Q(x) = 0 \), if instead of (B') we have \( \sigma(x) \log f(x) \to 0 \).

Similar statements hold for the condition (B') of Theorem 3. If \( f(x) \) is increasing and

\[
\sigma(x) \log f(x) \leq M,
\]

then

\[
\lim_{x \to \infty} \frac{s(x)}{x} = \sigma(x) \log f(x)
\]

and if even \( \sigma(x) \log f(x) \to 0 \), then \( \lim s(x) / x = +\infty \).

To prove for example (15), we note that there are arbitrary large \( n \) with \( \sigma(n)/n > C \log \log n \); if \( n \) has this property, we put \( f(x) = x \) and \( \sigma_1 = x + M^{-1} x \log f(x) \); then \( \sigma_1 / x \to 1 \) and

\[
f(x_1) - f(x) = f'(x) \frac{x}{x_1} \leq \frac{1}{M} \frac{x}{x_1} \xi \log f(x) \leq 1
\]

by (B'). Hence \( \sigma_1 > a_1 - x \). As in (14) we obtain

\[
S(x_1) - S(x) = \sigma_1 \log f(x) \leq \sigma(M^{-1} x + O(\log^2 x)),
\]

therefore by Theorem 1,

\[
\lim_{x \to \infty} \frac{s(x)}{x} \geq \lim_{x \to \infty} \frac{S(x)}{x} + O(\log^2 x) + \frac{1}{x} \cdot \frac{1}{M} \cdot \sigma(M^{-1} x + O(\log^2 x)),
\]

which proves our assertion.

Similarly, there are functions \( f(x) \) satisfying \( f(x) = O(\log \log x) \) for which (15) holds. We take in (16), \( L = \sigma_0 \log^2 x \) and \( n \) such that \( \sigma(n)/n > C \log \log n \). Then the sum of the number of divisors of the numbers \( n = 2 \ldots n \) is greater than

\[
\sum_{n=2}^{n} \sigma(n, n + s) = \sum_{d=1}^{n} \sigma(n) = \sigma(n) \geq \log \log n \cdot C \log n > C_1 N_n
\]

with large \( C_1 \). Therefore

\[
S(2N_n) - S(N_n) \geq C_1 N_n,
\]

and (15) follows.

At present we can not decide whether condition (A) of Theorem 2 can be weakened to \( \sigma(x) \log \log x \).
On a question of additive number theory

by

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1. Let \( A = \{ a \}, B = \{ b \}, \ldots \) denote sets of non-negative integers containing the number zero;

\[
\sum_{\lambda} A_\lambda = \left\{ \sum_{\lambda} a_{\lambda \lambda} \mid a_{\lambda \lambda} \in A, \lambda = 1, 2, \ldots, k \right\}.
\]

Thus \( \sum A_{\lambda} \) consists of all the numbers \( a_1 + a_2 + \ldots + a_k \) where each \( a_\lambda \) lies in the corresponding \( A_\lambda \). For a given integer \( n \) let \( [A] \) denote the number of positive elements of \( A \) up to and including \( n \). \( A \) denotes the set of the integers \( \leq n \) which do not belong to \( A \).

It is well known and easy to see that \( n \notin A + B \) implies \([A] + [B] \leq n - 1\). The corresponding problem for three or more sets does not lead to anything new. For then

\[
n \notin \sum_{\lambda} A_{\lambda}
\]

implies \( n \notin A_1 + A_\mu \) and thus \([A_1] + [A_\mu] \leq n - 1\); \( 1 \leq \lambda < \mu \leq k \). Adding these \( \frac{1}{k} (k-1) \) inequalities we readily obtain

\[
\sum_{\lambda} [A_\lambda] \leq \frac{1}{k} (n-1).
\]

That (2) cannot be improved can be seen by taking \( A_1 = A_2 = \ldots = A_k = \{0\} \) set of integers between \([A_1] + 1\) and \( n - 1\) together with 0.

This question becomes more interesting if we require \( n \) to be the smallest number not in \( \sum A_\lambda \). For \( k = 3 \) and \( n < 15 \) one can show\(^{(1)}\) that

\([A_1] + [A_2] + [A_3] \leq n - 1\).

\(^{(1)}\) Written communication from Professor H. B. Mann.