Remarks on number theory I
On primitive \(\sigma\)-abundant numbers

by

P. Erdős (Toronto)

Denote by \(\sigma(n)/n\) the sum of divisors of \(n\). It is well known that \(\sigma(n)/n\) has a continuous distribution function, i.e. for every \(c\) the density of integers satisfying \(\sigma(n)/n \leq c\) exists and is a continuous function of \(c\) whose value \(\rightarrow 1\) as \(c \rightarrow \infty\). This result was first proved by Davenport [1], Behrend and Chowla. Thus in particular the density of abundant numbers exists (a number is abundant if \(\sigma(n)/n \geq 2\)). I [2] have proved the existence of this density by proving that the sum of the reciprocals of the primitive abundant numbers converges (a number \(m\) is called primitive abundant if \(\sigma(m)/m \geq 2\) but for every proper divisor \(d\) of \(m\), \(\sigma(d)/d < 2\). More generally we shall say that \(m\) is primitive \(\sigma\)-abundant if \(\sigma(m)/m \geq a\) but, for every proper divisor \(d\) of \(m\), \(\sigma(d)/d < a\). I observed some time ago that it is not true that the sum of the reciprocals of the primitive \(\sigma\)-abundant numbers converges for every \(a\). It will be clear from our proof that if \(a\) can be approximated very well by numbers of the form \(\sigma(n)/n\) then the sum of the reciprocals of the primitive \(\sigma\)-abundants will diverge.

Let \(P_1, P_2, \ldots\) be an infinite sequence of primes satisfying \(P_{k+1} > e^{P_{k}}\).

Put

\[
\alpha = \prod_{k=1}^{\infty} \left(1 + \frac{1}{P_k}\right) = \lim_{k \to \infty} \sigma(P_1 P_2 \ldots P_k) / P_1 P_2 \ldots P_k
\]

A simple computation shows that for every \(k\) the integers

\(P_1 P_2 \ldots P_k \alpha, \quad P_k < P < P_{k+1}\)

are primitive \(\sigma\)-abundant. From

\[
\sum_{P \leq x} \frac{1}{P} = (1 + o(1)) \log \log x
\]
we have
\[ \sum_{p < P \in \mathbb{P}} \frac{1}{p} > \frac{1}{2} \log P, \]
further by the definition of the \( P \)'s \( p > p_1 p_2 \cdots p_{k-1} \). Thus
\[ \sum_{p < P \in \mathbb{P}} \frac{1}{p p_1 p_2 \cdots p_k} > \frac{1}{2}, \]
which clearly implies that the sum of the reciprocals of the primitive \( a \)-abundant numbers diverges. A simple argument shows that the \( a \)'s for which the sum of the reciprocals of the primitive \( a \)-abundant numbers form an everywhere dense \( G_0 \) in \((1, \infty)\), i.e. they are the countable intersection of dense open sets. But it is not difficult to show that they have measure 0, in fact they must be Liouville numbers (a number \( y \) is called a Liouville number if \( y = \frac{a}{b} \) with \( |y - \frac{a}{b}| < 1/b^n \) is solvable in integers \( a \) and \( b \) for every \( n \)). We shall not give a proof of this result.

Denote by \( N_a(x) \) the number of the primitive \( a \)-abundant not exceeding \( x \). I [3] have proved that (exp = \( c \))
\[ (1) \quad \frac{x}{\exp \left( 25(\log x)^{1/5} \right)} < N_a(x) < \exp \left( \frac{x}{\exp \left( \log x \right)^{1/2}} \right). \]
I can show that for every \( a \) (\( c_1, c_2, \ldots \) denote suitable positive constants)
\[ (2) \quad N_a(x) > \frac{x}{\exp \left( c_1 (\log x)^{1/2} \right)}, \]
and that for every \( a \) and an infinite sequence \( a_n \to x \)
\[ (3) \quad N_a(a_n) < \frac{x}{\exp \left( c_2 (\log x)^{1/2} \right)}. \]
Also if \( a \) is not a Liouville number then for a certain \( c_3 = c_3(a) \)
\[ (4) \quad N_a(x) < \frac{x}{\exp \left( c_3 (\log x)^{1/2} \right)} \]
for all \( x > 0 \). I am not going to give the details of the proof of (2), (3) and (4) since the results do not seem to me to be very interesting and the proof is similar to that of (1).

I shall prove in full detail the following

**Theorem.**
\[ N_a(x) = o \left( \frac{x}{\log x} \right). \]

The proof of our Theorem will be in many ways similar to that of (1).

It is easy to see that (5) is best possible in the following sense: if \( g(x) \to \infty \) as slowly as we like, there always exists an \( a \) so that for infinitely many \( x \)
\[ (6) \quad N_a(x) > \frac{x}{g(x) \log x}. \]

The proof of (6) can be left to the reader since it is almost identical with the proof that for a suitable \( a \) the sum of the reciprocals of the primitive \( a \)-abundant numbers diverges.

Now we prove (5). Denote the primitive \( a \)-abundant numbers by \( m_0 < m_1 < \ldots \). First of all we shall show that it will suffice to consider the \( m_i \)’s not exceeding \( x \) which satisfy the following properties:

I. \[ \frac{a}{(\log x)^{2}} < m_i, \]
II. \[ v(m_i) < 10 \log \log x, \]
where \( v(m) \) denotes the number of distinct prime factors of \( m_i \),
III. if \( p^a m_i \) and \( a > 1 \), then \( p^a < (\log x)^{1/2} \),
IV. the greatest prime factor of \( m_i \) is greater than \( \exp(\log x)^{1/2} \).

To see this we shall show that the number of integers which does not satisfy any of these conditions is \( o(\log x) \). This is trivial for I. To show it for II we observe that
\[ \sum_{n=1}^{x} \frac{m_i}{n} < \sum_{n=1}^{x} \frac{x}{n} < 2 \log x. \]
Thus the number of integers not exceeding \( x \) which do not satisfy II is less than
\[ 2 \log x / 10 \log \log x = o(\log x). \]

If \( p^a m_i, a > 1 \), then \( p^a \) or \( p^{a-1} \) is a square and thus \( n \) is divisible by a square greater than \( (\log x)^{1/2} \). Hence every integer which does not satisfy III is divisible by a square not less than \( (\log x)^{2/3} \). Thus the number of integers which do not satisfy III is less than
\[ \sum_{k > (\log x)^{2/3}} \frac{1}{k^2} = o \left( \frac{x}{\log x} \right). \]

Let
\[ n = \prod_{p \leq x} p^k, \]

Assume that \( n \) does not satisfy IV. We can assume that \( n \) satisfies II and III, whence
\[
\begin{align*}
  n < (a^{1/\log \log a})^{\log \log a} = o\left( \frac{\pi}{\log x} \right),
\end{align*}
\]
which proves that the number of integers not satisfying IV is \( o(x/\log x) \).

Henceforth we shall assume that our primitive \( \alpha \)-abundant numbers \( m_i \) satisfy the conditions I, II, III and IV. Put
\[
\begin{align*}
  m_i = A_4 B_i,
\end{align*}
\]
where all prime factors of \( A_4 \) are \( < (\log x)^{16} \) and all prime factors of \( B_i \) are \( > (\log x)^{16} \). By II and III we have
\[
\begin{align*}
  A_4 < (\log x)^{1/\log \log a}
\end{align*}
\]
and by (8) and I we have \( B_i > 1 \).

Now we split the \( m_i \) into two classes. In the first class are the \( m_i \) for which \( B_i \) is not a prime. Write (by II, \( (p_{i0})^{1/2} B_i \))
\[
\begin{align*}
  B_i = p_{i0} p_{i1} \ldots p_{i_0},
\end{align*}
\]
where by IV
\[
\begin{align*}
  \log \log a < p_{i0} < \ldots < p_{i_0}, \quad p_{i0} > (\log x)^{1/\log \log a}.
\end{align*}
\]

Now we split the \( m_i \) of the first class into two subclasses. In the first subclass are the \( m_i \) with
\[
\begin{align*}
  p_{i0} < (\log x)^{1/\log \log a}.
\end{align*}
\]

We shall show that if (10) is satisfied then the integers
\[
\begin{align*}
  \frac{m_i}{p_{i0}^{\theta}}
\end{align*}
\]
are all different, and if this is accomplished then it will follow from (9) that the number of integers of the first subclass is less than \( \pi/((\log x)^{16}) = o(x/\log x) \).

If the integers (11) were not all different we should have
\[
\begin{align*}
  \frac{m_i}{p_{i0}^{\theta}} = \frac{m_j}{p_{j0}^{\theta}}, \quad p_{i0} \neq p_{j0} \quad \text{(assume say } p_{i0} < p_{j0}).
\end{align*}
\]
Thus
\[
\begin{align*}
  \sigma \left( \frac{m_i}{p_{i0}^{\theta}} \right) = \sigma \left( \frac{m_j}{p_{j0}^{\theta}} \right) = \sigma \left( \frac{m_i}{m_j} \right) \frac{p_{i0}^{\theta}}{p_{j0}^{\theta} + 1} = \sigma \left( \frac{m_i}{m_j} \right) \frac{p_{i0}^{\theta}}{p_{j0}^{\theta} + 1},
\end{align*}
\]
whence
\[
\begin{align*}
  \gamma_{ij} = \frac{p_{i0}^{\theta}(p_{j0}^{\theta} + 1)}{p_{j0}^{\theta}(p_{i0}^{\theta} + 1)} = \frac{\sigma(m_i)}{\sigma(m_j)} \left( \frac{m_i}{m_j} \right)^{-1}.
\end{align*}
\]
Now since the \( m_i \)'s are primitive \( \alpha \)-abundant we have
\[
\begin{align*}
  \sigma(m_i) = \alpha \frac{m_i}{p_{i0}^{\theta}}, \quad \sigma \left( \frac{m_i}{m_j} \right) = \alpha \frac{m_i}{m_j} \left( 1 + \frac{1}{p_{i0}^{\theta}} \right) < \alpha
\end{align*}
\]
or by (9)
\[
\begin{align*}
  \alpha \frac{m_i}{m_j} < \alpha(1 + \alpha^{-1}(\log \log a)^{2}),(13)
\end{align*}
\]
and the same holds with \( m_i \) replacing \( m_j \). Further from (9), (10) and (12)
\[
\begin{align*}
  \gamma_{ij} < 1 + \frac{1}{p_{i0}^{\theta}(p_{i0}^{\theta} + 1)} > 1 + \frac{1}{2} \alpha^{-1}(\log \log a)^{2}.
\end{align*}
\]
On the other hand from (12) and (13)
\[
\begin{align*}
  \gamma_{ij} < 1 + \alpha^{-1}(\log \log a)^{2}.
\end{align*}
\]
(15) clearly contradicts (14); this shows that the integers (11) are all different and this preserves the first subclass.

Now we deal with the numbers of the second subclass. For these numbers we have
\[
\begin{align*}
  m_i = A_4 B_i, \quad B_i \text{ not a prime, all prime factors of } B_i \text{ are } > (\log x)^{1/\log \log a}.
\end{align*}
\]
I now show that for all \( m_i \) of the second subclass
\[
\begin{align*}
  \sigma(A_4)/\sigma(A_4) = 0 < a.
\end{align*}
\]

Assume that (17) does not hold. \( \sigma(A_4)/\sigma(A_4) < a \) is clear since \( m_i \) is primitive \( \alpha \)-abundant. Assume thus that for some \( m_i \) and \( m_j \) of the second subclass we have \( m_i = A_4 B_1, m_j = A_4 B_2, \sigma(A_4)/\sigma(A_4) < \sigma(A_4)/\sigma(A_4) \). But then by (8)
\[
\begin{align*}
  \frac{\sigma(A_4)}{A_4} - \sigma(A_4) \geq \frac{1}{A_4} > (\log x)^{-\log \log a}
\end{align*}
\]
or
\[
\begin{align*}
  \sigma(A_4)/\sigma(A_4) < a - (\log x)^{-\log \log a}.
\end{align*}
\]
Now by (16) (the number of prime factors of \( B_i \) is less than \( 4(\log \log a)^{3} \)
\[
\begin{align*}
  \sigma(B_i)/B_i < (1 + (\log \log a)^{3}) < 1 + (\log \log a)^{3}.
\end{align*}
\]
But by (18) and (19)

\[ \sigma(m_k) = \sigma(A_k) \cdot \sigma(B_k) \]

\[ \frac{a}{m_i} = \frac{\sigma(A_k)}{A_k} \cdot \frac{\sigma(B_k)}{B_k} < a, \]

an evident contradiction thus \( \text{(17)} \) is proved.

Now let \( p_i \) be the smallest prime factor of all the \( B_k \)’s which belong to the \( m_k \)’s of the second subclass. We have to split these \( m_k \)’s again into two classes (sub-subclasses). In the first class are those \( m_k \)’s for which the least prime factor \( p_i \) of \( B_k \)

\[ p_i < \frac{\log \log x}{\log x}. \]  

The number of the \( m_k \)’s of the first class is clearly (by \( \text{(20)} \) and the prime number theorem, or a more elementary inequality) less than

\[ x \sum_{p_i < \log \log x / \log x} \frac{1}{p_i} < \frac{\log \log x^2}{\log x^2} = o \left( \frac{x}{\log x} \right). \]

For the \( m_k \)’s of the second class we have

\[ p_i > \frac{1 + \log \log x}{\log x}. \]

First we show that for every \( B_k \)

\[ \frac{\sigma(B_k)}{B_k} > 1 + \frac{1}{p_i}. \]

To see this we only have to remark that \( p_i \) is a prime factor of some \( m_i = A_k B_k \). Thus since \( B_k \) is not a prime

\[ \frac{\sigma(A_k) \cdot \sigma(B_k)}{A_k B_k} = \left( 1 + \frac{1}{p_i} \right) \frac{\sigma(A_k)}{A_k} \cdot \frac{\sigma(B_k)}{B_k} < a \leq \frac{\sigma(A_k)}{A_k} \cdot \frac{\sigma(B_k)}{B_k}, \]

which proves (23). From (23) and (16) we have

\[ \left( 1 + \frac{1}{p_i} \right)^{\log \log x} > \frac{\sigma(B_k)}{B_k} > 1 + \frac{1}{p_i}. \]

Thus

\[ p_i \left( 1 + \frac{1}{\log x} \right) < p_i < 10 \log \log x p_i. \]

Next we estimate \( p_i^0 (p_i^0 < p_i^0 < \ldots \text{ are the prime factors of } B_i) \). We have by (23) and (24)

\[ \left( 1 + \frac{1}{p_i} \right) < \frac{\sigma(B_k)}{B_k} < \left( 1 + \frac{1}{p_i^0} \right) \left( 1 + \frac{1}{p_i} \right)^{\log \log x}. \]

Thus by (24)

\[ \left( 1 + \frac{1}{p_i} \right)^{1 + \frac{1}{p_i} (1 + (\log x)^{-1})} < \left( 1 + \frac{1}{p_i} \right)^{1 + \frac{1}{p_i} (1 + (\log x)^{-1})} < \left( 1 + \frac{1}{p_i^0} \right)^{\log \log x}, \]

or by a simple computation (for sufficiently large \( x \))

\[ p_i^0 < 10 \log x (\log \log x)^2 p_i < (\log x)^2 p_i. \]

Thus \( B_i \) has at least two prime factors in the interval \( (p_i, (\log x)^2 p_i) \).

Hence the number of \( m_i \)’s of the second class is less than \( (q, r, s \text{ are primes}) \)

\[ \sum_{p_i < \log \log x / \log x} \left( \sum_{p_i < \log \log x / \log x} \frac{1}{p_i^0} \right)^{\frac{1}{q}} < \frac{\log \log x^2}{\log x^2} \sum_{p_i < \log \log x / \log x} \frac{1}{p_i^0} < \frac{\log \log x^2}{\log x^2} = o \left( \frac{x}{\log x} \right). \]

(21) and (25) shows that the number of integers of the second subclass is also \( o(\log x) \). In fact, the number of primitive \( \sigma \)-abundants we have considered so far is easily seen to be \( o(\log^{3/2} x) \).

Finally we consider the \( m_k \)’s of the second class. Here

\[ m_i = A_k B_k. \]

From (8) and (1) it follows that it suffices to consider the \( m_i \)’s satisfying \( p_i > x^{3/2} \), but then we can again assume that (17) holds, i.e., that \( \sigma(A_k) / A_k = O(a) \) (the proof is the same as previously). But then the number of integers of the second class equals (by (8))

\[ \sum_{A_k} \frac{x}{A_k} \sum_{A_k} \frac{1}{A_k} = \frac{x}{\log x} \sum_{A_k} \frac{1}{A_k} = o \left( \frac{x}{\log x} \right) \]

where the dash indicates that the summation is extended over the \( A_k \)’s satisfying \( \sigma(A_k) / A_k = O \). Thus to prove our Theorem it will suffice to show that

\[ \sum_{A_k} \frac{1}{A_k} = o(1). \]
Now denote by \( n_1 < n_2 < \ldots \) the sequence of all integers which satisfy \( \sigma(n_k)/n_k = C \). Clearly

\[
\sum' \frac{1}{n_k} \leq \sum \frac{1}{n_k}.
\]

Put \( \sigma(n_2)/n_1 = a/b = C \). First we show that as \( x \to \infty \), \( b \to \infty \) (\( C \) and therefore the \( n_k \)'s depend on \( x \)). If for infinitely many values of \( a/b \) assumed the same value, then since there are only a finite number of choices of \( a/b \), \( a \), we should have from (24)

\[
\frac{\sigma(n_1)}{n_1} = \frac{a}{b} < a, \quad \frac{a}{b} \left(1 + \frac{1}{p_i}ight) > a, \quad p_i > \log x^{19},
\]

(29)

(if no such \( p_i \) existed there would be no integers of the form (26), i.e., there would be no integers of the second class and the proof of our Theorem would be complete). But since \( p_i \to \infty \), (29) is clearly impossible, thus, \( b \to \infty \) as \( x \to \infty \) as stated.

Thus to complete our proof it will suffice to show that

\[
\sum \frac{1}{n_k} < \frac{c_0}{b^{1/2}}.
\]

We write

\[
\sum \frac{1}{n_k} = \sum' + \sum''
\]

where in \( \sum' \) all the prime factors of \( n_k \) are not greater than \( b \) and in \( \sum'' \) are the other \( n_k \). Clearly for all \( k, n_k = 0 \mod b \); thus

\[
\sum' \leq \frac{1}{b} \sum \frac{1}{i} = \frac{1}{b} \prod_{p \leq b} \left(1 + \frac{1}{p-1}\right) < a \log b
\]

where in \( \sum' \) all prime factors of \( t \) are not exceeding \( b \). Now let \( n_k \) be in \( \sum'' \) and let \( p_k \) be the greatest prime factor of \( n_k \). Clearly \( p_k > b \). But since \( \sigma(n_k)/n_k = a/b \), we must have \( \sigma(n_k) = 0 \mod p_k \). Therefore

\[
n_k = 0 \mod p_k, \quad \sigma(p_k) = 0 \mod p_k, \quad a \gg 1
\]

(29) or

\[
q_i > \frac{1}{2} p_i.
\]

(\( a \gg 1 \) follows from the fact that \( p_i \) was the greatest prime factor of \( n_k \)).

Thus

\[
\sum_x < \sum \frac{1}{p_i} \sum \frac{1}{q_i} \sum \frac{1}{t}
\]

where in \( \sum_x, q_i > \frac{1}{2} p_i \) and \( a > 1 \) and in \( \sum_x \) all prime factors of \( t \) are not greater than \( p \). Thus finally

\[
\sum_x < \sum \frac{1}{p_i} \sum \frac{1}{q_i} \prod_{p \leq p_i} \left(1 + \frac{1}{p-1}\right) < \sum \frac{\log p_i}{p_i} \sum \frac{1}{q_i}
\]

(31), (32) and (33) prove (30) and hence the proof of our Theorem is complete.

References


University of Toronto
Reçu par la Réunion le 15.3.1938