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Reçu par la Rédaction le 27. 1. 1958.

ACTA ARITHMETICA
V (1958)

On the distribution of the solutions of diophantine equations with many unknowns

by

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To the solutions of a diophantine equation with r unknowns correspond geometrically — as we know — in the r-dimensional space R^r the points with entire coordinates of an $(r-1)$-dimensional hypersurface. From this geometrical interpretation follows immediately for every diophantine equation with r unknowns the following problem of a very general character, which can be formulated also merely arithmetically: how the lattice points representing the solutions of the diophantine equation in question are distributed in the space R^r. Of course this problem is interesting principally in the case when the diophantine equation has infinitely many solutions.

Let r and P be positive integers, $\Phi(x_1, \ldots, x_r)$ a polynomial of r variables with entire coefficients, in respect to which we do not make, for the moment, any restrictions.

The distribution of the solutions in positive integers of the equation

$$\Phi(x_1, \ldots, x_r) = 0 \tag{1}$$

can be described with the aid of the solution function $R(P)$ defined in the following manner: let $R(P)$ denote the number of all the points with entire coordinates of the hypersurface (1) which are placed inside the cube $1 \leqslant x_1 \leqslant P, \ldots, 1 \leqslant x_r \leqslant P$.

Purely arithmetically formulated, $R(P)$ means the number of all the positive entire solutions of the diophantine equation (1) in respect to which $x_1 \leqslant P, \ldots, x_r \leqslant P$.

As each of the variables $x_1, \ldots, x_r$ can assume only the values $1, \ldots, P$, for the $R(P)$ solution function we have in every case the trivial upper estimation

$$R(P) \leqslant P^r.$$

But in very many cases the upper estimation can be considerably improved. So for instance if $n \geqslant 2$, and $C = C(\varepsilon, n)$ is a positive constant depending exclusively on ε and n, then for the diophantine equation with four unknowns

$$x^n+y^n = u^n+v^n$$

we have

$$R(P) < CP^{2+\varepsilon}$$

(see [2] on page 139). Another example is the famous equation of Fermat

$$x^p+y^p = z^p$$

(p is an odd prime-number). For this equation the following non-trivial upper estimation has been given by Dénes and Turán in respect to the $R_p(P)$ function of the positive co-prime solutions ($D = D(p)$ is a positive constent depending only on p):

$$R_p(P) < D\frac{P^{2/p}}{\log^{2-2/p}P}$$

(see [1] on page 28).

In this paper I will give an upper estimation of the solution function $R(P)$ in respect to diophantine equations having in comparison with their degree — in a certain sense — "many" unknowns. For these equations the estimation can be carried out with the aid of the tools of analysis relatively easily and quickly.

In the following we shall use the following notations: $c_1, \ldots, c_{18}$ are positive constants whose numerical value may depend only on some of the parameters $\varepsilon, r, n,$ and m; $\nu = 1/n$; $\sigma = 2^{-n+1}$.

We shall prove the following

THEOREM. *Let be* $n \geqslant 3$; $r \geqslant r_0(n) = n2^{n-1}+1$,

(2) $$\Phi(x_1, \ldots, x_r) = a_1x_1^n+\ldots+a_rx_r^n+\varphi(x_1, \ldots, x_r)$$

where the coefficients $a_1, \ldots, a_r$ *are integers different from* 0, *and* $\varphi(x_1, \ldots, x_r)$ *is an arbitrary polynomial with integer coefficients of at most* $(n-3)$*-th degree. Further let* $\varepsilon > 0$, $m = \max(|a_1|, \ldots, |a_r|)$, $\mu = \min(|a_1|, \ldots, |a_r|)$,

$$\varrho = \frac{2(\sqrt{2})^r+2r\nu-2}{\mu(r\nu-1)},$$

$P > c_1(\varepsilon, r, n, m)$, $(a, q) = 1$, $q > 0$,

$$S_{a,q} = \sum_{u_1=1}^{q}\cdots\sum_{u_r=1}^{q}\exp\left(2\pi i\frac{a}{q}\Phi(u_1, \ldots, u_r)\right)$$

and

$$\mathfrak{S} = \sum_{q=1}^{\infty}A_q; \quad A_q = q^{-r}\sum_a S_{a,q}$$

where a *runs over a reduced residue-system* mod q. *We assert that in this case*

(*) $$R(P) \leqslant (\varrho+\varepsilon)|\mathfrak{S}|P^{r-n}.$$

The infinite series occuring in (*) resembles very much the so-called singular series, introduced by Hardy-Littlewood in the theory of the Waring problem (see e. g. [4], p. 268). It is worth remarking that the sum of the "singular" series occuring here does not depend on P (the convergence of the "singular" series will be proved afterwards).

Before beginning with the demonstration of the theorem we will show by a rather characteristic example that the obtained upper estimation in the case of equations with many unknowns is — in general — not liable to further essential improvements.

Namely let $n \geqslant 12$, $r \geqslant [10n^2\log n]+1$ and let us consider the diophantine equation

(3) $$x_1^n+\ldots+x_{r-1}^n = x_r^n.$$

If we denote the number of solutions in positive integers of the equation $x_1^n+\ldots+x_{r-1}^n = N$ by $I(N)$, for the equation (3) the identity

(4) $$R(P) = \sum_{k=1}^{R}I(k^n)$$

evidently holds. We know the real order of the function $I(N)$: the most important conclusion of the Hardy-Littlewood-Vinogradov theory of the Waring conjecture (in respect to this theory see [4]) is that for every sufficiently great positive N

$$c_2N^{(r-1)\nu-1} < I(N) < c_3N^{(r-1)\nu-1}, \quad c_2 = c_2(n, r), \quad c_3 = c_3(n, r).$$

But then it follows from (4) for sufficiently great P that

$$c_4(1^{r-n-1}+\ldots+P^{r-n-1}) < R(P) < c_5(1^{r-n-1}+\ldots+P^{r-n-1}),$$

$$c_4 = c_4(n, r), \quad c_5 = c_5(n, r),$$

whence we get easily

$$c_6P^{r-n} < R(P) < c_7P^{r-n}, \quad c_6 = c_6(n, r), \quad c_7 = c_7(n, r).$$

After this digression we begin with the demonstration of our main theorem. The demonstration follows in general Vinogradov's extraordinarily simplified method of the demonstration of the asymptotical formula of Hardy-Littlewood in connection with the Waring problem (see [4], chapters III and VII) with some more or less essential variations. For the demonstration we shall need the following four auxiliary lemmas.

LEMMA 1 (Lemma of van der Corput). *If M and M_1 are integers, $M < M_1$ and the twice differentiable real function $f(x)$ defined in the interval $M \leqslant x \leqslant M_1$ satisfies the conditions*

$$0 \leqslant f'(x) \leqslant \tfrac{1}{2}, \quad f''(x) \geqslant 0,$$

then, taking on both sides simultaneously the signs + or −, we have

$$\sum_{x=M}^{M_1} \exp(\pm 2\pi i f(x)) = \int_M^{M_1} \exp(\pm 2\pi i f(x))\,dx + 2\Theta \qquad (|\Theta| \leqslant 1).$$

For the demonstration of the lemma see e. g. [4], p. 261.

LEMMA 2. *If P is a positive integer and u a real number,*

$$I = \int_0^P \exp(2\pi i u x^n)\,dx,$$

then

$$|I| \leqslant U, \qquad U = \begin{cases} P & \text{if} \quad |u| \leqslant P^{-n}, \\ \sqrt{2}\,|u|^{-\nu} & \text{if} \quad |u| > P^{-n}. \end{cases}$$

For the demonstration of this lemma see e. g. [4], p. 262.

LEMMA 3 (Vinogradov). *If $n \geqslant 3$, r is an arbitrary positive integer, $\Phi(x_1, \ldots, x_r)$ a polynomial of the form (2), P integer > 1, y real, $\varepsilon > 0$,*

$$\left|y - \frac{a}{q}\right| < \frac{1}{q^2}, \quad (a, q) = 1, \quad 1 \leqslant q \leqslant P^n,$$

$$S = \sum_{x_1=1}^{P} \ldots \sum_{x_r=1}^{P} \exp(2\pi i y \Phi(x_1, \ldots, x_r)),$$

then

$$|S| < c_8 P^{r+\varepsilon} \left(\max\left(\frac{1}{q}, \frac{1}{P}, \frac{q}{P^n}\right)\right)^{r\sigma}, \qquad c_8 = c_8(\varepsilon, r, n, m).$$

For the demonstration of the lemma see [3], p. 145. We remark that Vinogradov in his above mentioned work puts as a condition that the coefficients $a_1, \ldots, a_r$ of the polynomial are positive, but in the demonstration of the present lemma he uses only the condition that these coefficients are different from 0.

LEMMA 4. *If $n \geqslant 3$, $r \geqslant r_0(n) = n2^{n-1}+1$, $\Phi(x_1, \ldots, x_r)$ is a polynomial of the form (2), P a positive integer, y real,*

$$\mu = \min(|a_1|, \ldots, |a_r|), \qquad m = \max(|a_1|, \ldots, |a_r|), \qquad \tau = 2mnP^{n-1},$$

$$y = \frac{a}{q} + z, \qquad (a, q) = 1, \qquad 0 < q \leqslant P, \qquad |z| \leqslant \frac{1}{q\tau},$$

$$S = \sum_{x_1=1}^{P} \ldots \sum_{x_r=1}^{P} \exp(2\pi i y \Phi(x_1, \ldots, x_r)),$$

$$S_{a,q} = \sum_{u_1=1}^{q} \ldots \sum_{u_r=1}^{q} \exp\left(2\pi i \frac{a}{q} \Phi(u_1, \ldots, u_r)\right),$$

$$Z = \begin{cases} P & \text{if} \quad |z| \leqslant \mu^{-1}P^{-n}, \\ \sqrt{2}\,\mu^{-\nu}|z|^{-\nu} & \text{if} \quad |z| > \mu^{-1}P^{-n}, \end{cases}$$

$$I_s = \int_0^P \exp(2\pi i a_s z x^n)\,dx \qquad (s = 1, \ldots, r),$$

then

$$S = q^{-r} S_{a,q} I_1 \ldots I_r + O(q^{-3}P^{r+n-3}|z| + q^{-2}Z^{r-1}).$$

Demonstration of Lemma 4. With the substitution

$$x_s = qt_s + u_s \qquad (s = 1, \ldots, r)$$

where u_s can assume the values $1, \ldots, q$ and for every fixed value of u_s, t_s runs over all the integers of the interval

$$(5) \qquad 0 \leqslant t_s \leqslant \omega_s, \qquad \omega_s = [(P-u_s)q^{-1}]$$

we get

(6) $S =$

$$= \sum_{u_1=1}^{q} \ldots \sum_{u_r=1}^{q} \sum_{t_1=0}^{\omega_1} \ldots \sum_{t_r=0}^{\omega_r} \exp\left(\frac{2\pi i a}{q} \Phi(u_1, \ldots, u_r) + 2\pi i z \Phi(qt_1+u_1, \ldots, qt_r+u_r)\right)$$

$$= \sum_{u_1=1}^{q} \ldots \sum_{u_r=1}^{q} \exp\left(\frac{2\pi i a}{q} \Phi(u_1, \ldots, u_r)\right) \Omega_{u_1,\ldots,u_r}$$

where

$$\Omega_{u_1,\ldots,u_r} = \sum_{t_1=0}^{\omega_1} \ldots \sum_{t_r=0}^{\omega_r} \exp(2\pi i z \Phi(qt_1+u_1, \ldots, qt_r+u_r)).$$

From the assumptions referring to the form of the polynomial $\Phi(x_1, \ldots, x_r)$ it follows immediately that

$$(7)\qquad \Omega_{u_1,\ldots,u_r} = \sum_{t_1=0}^{\omega_1}\cdots\sum_{t_r=0}^{\omega_r}\exp\left(2\pi i z[a_1(qt_1+u_1)^n+\ldots+a_r(qt_r+u_r)^n]\right)+ + O(P^{r+n-3}q^{-r}|z|),$$

namely, on the one hand if t_s falls into the interval (5) for $s = 1, \ldots, r$, it follows from the inequality

$$|1-\exp(2\pi i a)| \leqslant 2\pi|a|$$

holding for every real a that

$$\left|\exp\left(2\pi i z\Phi(qt_1+u_1, \ldots, qt_r+u_r)\right)- -\exp\left(2\pi i z[a_1(qt_1+u_1)^n+\ldots+a_r(qt_r+u_r)^n]\right)\right| \leqslant c_9(m, r)P^{n-3}|z|,$$

on the other hand the number of the members of the sum $\Omega_{u_1,\ldots,u_r}$ is not greater than $q^{-r}P^r$.

In the interval (5) the inequality

$$\frac{d}{dt_s}|a_s||z|(qt_s+u_s)^n \leqslant \tfrac{1}{2} \qquad (s = 1, \ldots, r)$$

evidently holds, therefore from the lemma 1 we have

$$\sum_{t_s=0}^{\omega_s}\exp\left(2\pi i a_s z(qt_s+u_s)^n\right) = \int_0^{\omega_s}\exp\left(2\pi i a_s z(qt_s+u_s)^n\right)dt_s+2\Theta = q^{-1}I_s+O(1).$$

But from this, applying lemma 2 and the easily verifiable inequality $Zq^{-1} \geqslant 1$, we immediately obtain

$$(8)\qquad \sum_{t_1=0}^{\omega_1}\cdots\sum_{t_r=0}^{\omega_r}\exp\left(2\pi i z[a_1(qt_1+u_1)^n+\ldots+a_r(qt_r+u_r)^n]\right) = q^{-r}I_1\ldots I_r+O(q^{-r+1}Z^{r-1}).$$

From (7) and (8)

$$\Omega_{u_1,\ldots,u_r} = q^{-r}I_1\ldots I_r+O(P^{r+n-3}q^{-r}|z|+q^{-r+1}Z^{r-1})$$

Substitung that into (6) we get

$$(9)\qquad S = q^{-r}S_{a,q}I_1\ldots I_r+O(P^{r+n-3}q^{-r}|z||S_{a,q}|+q^{-r+1}Z^{r-1}|S_{a,q}|).$$

From lemma 3

$$(10)\qquad |S_{a,q}| < c_{10}q^{r-n} \leqslant c_{10}q^{r-3}, \qquad c_{10} = c_{10}(r, m, n).$$

Substituting this into (9) we get the assertion of lemma 4.

Demonstration of the theorem. We can represent the solution function in the following integral form:

$$R(P) = \int_{-\tau^{-1}}^{-\tau^{-1}+1} S\,dy.$$

(The meaning of S and τ is the same as in lemma 4.)

In order to give an upper estimate of the integral standing on the right side, we shall divide the interval $-\tau^{-1} \leqslant y \leqslant -\tau^{-1}+1$ into "main" and "complementary" intervals. In the main intervals are placed those numbers y for which (with $c_{11}(n) < \sigma(n+\sigma)^{-1}$)

$$y = \frac{a}{q}+z, \qquad (a, q) = 1, \qquad 0 < q \leqslant P^{1-c_{11}}, \qquad 0 \leqslant a < q,$$
$$-\frac{1}{q\tau} \leqslant z \leqslant \frac{1}{q\tau}.$$

It can easily be proved that two different main-intervals cannot have any points in common. Namely, we should have

$$\frac{a}{q}+z = \frac{a'}{q'}+z', \qquad \frac{a}{q} \gtrless \frac{a'}{q'}, \qquad |z| \leqslant \frac{1}{q\tau}, \qquad |z'| \leqslant \frac{1}{q\tau};$$

from this would result

$$\left|\frac{aq'-a'q}{qq'}\right| \leqslant \frac{2}{\tau}, \qquad \frac{1}{qq'} \leqslant \frac{2}{\tau}, \qquad \frac{1}{P^2} < \frac{1}{3P^2}.$$

Complementary intervals are those which remain of the interval $-\tau^{-1} \leqslant y \leqslant -\tau^{-1}+1$ after removing the main-intervals. From Dirichlet's theorem results immediately that, if y_1 falls into a complementary interval,

$$(11)\qquad y_1 = \frac{a_1}{q_1}+z_1, \qquad (a_1, q_1) = 1, \qquad P^{1-c_{11}} < q_1 \leqslant \tau, \qquad 0 \leqslant a_1 < q_1,$$
$$|z_1| \leqslant \frac{1}{q_1\tau}.$$

Let $R_1(P)$ denote that part of the integral representing $R(P)$ which corresponds to the main intervals, and $R_2(P)$ the part corresponding to the complementary intervals. Then we have

$$R(P) = R_1(P)+R_2(P).$$

First step. We estimate the absolute value of $R_1(P)$. Let $(a, q) = 1$, $0 < q \leqslant P^{1-c_{11}}$, $0 \leqslant a < q$ and let us regard the main interval $aq^{-1}-(q\tau)^{-1}$

$\leqslant y \leqslant aq^{-1}+(q\tau)^{-1}$. Denoting by $H_{a,q}$ the integral of the sum S on this interval, we have

$$H_{a,q} = \int_{-(q\tau)^{-1}}^{(q\tau)^{-1}} S\,dz.$$

On the basis of lemma 4 in the interval $aq^{-1}-(q\tau)^{-1} \leqslant y \leqslant aq^{-1}+(q\tau)^{-1}$,

$$S = q^{-r}I_1\ldots I_r+O(q^{-3}P^{r+n-3}|z|+q^{-2}Z^{r-1})$$

and so

$$H_{a,q} = q^{-r}S_{a,q}\int_{-(q\tau)^{-1}}^{(q\tau)^{-1}} I_1\ldots I_r\,dz+F$$

where

$$F = O\Big(\int_0^{(q\tau)^{-1}} q^{-3}P^{r+n-3}z\,dz+\int_0^{\mu^{-1}P^{-n}} q^{-2}P^{r-1}dz+\int_{\mu^{-1}P^{-n}}^{(q\tau)^{-1}} q^{-2}z^{-r\nu+\nu}dz\Big)$$
$$= O(q^{-2}P^{r-n-1}).$$

But from this, making use of lemma 2 and of (10), we immediately obtain

$$H_{a,q} = q^{-r}S_{a,q}J(P)+O(q^{-1}P^{r-n-1}), \qquad J(P) = \int_{-\infty}^{\infty} I_1\ldots I_r\,dz.$$

Summing this equation at first with fixed q for every a less than q and relative prime to q, and then for $q = 1,\ldots,[P^{1-c_{11}}]$ we get

(12) $$R_1(P) = J(P)\sum_{q=1}^{[P^{1-c_{11}}]} A_q+O(P^{r-n-c_{11}}).$$

Making use of lemma 2 we obtain

(13) $|J(P)| =$

$$= \Big|\int_{-\infty}^{-\mu^{-1}P^{-n}}+\int_{-\mu^{-1}P^{-n}}^{\mu^{-1}P^{-n}}+\int_{\mu^{-1}P^{-n}}^{\infty}\Big| \leqslant 2(\sqrt{2})^r\mu^{-r\nu}\int_{\mu^{-1}P^{-n}}^{\infty} z^{-r\nu}dz+2\int_0^{\mu^{-1}P^{-n}} P^r dz$$
$$= \frac{2(\sqrt{2})^r}{\mu(r\nu-1)}P^{r-n}+\frac{2}{\mu}P^{r-n} = \varrho P^{r-n}.$$

From (10)

(14) $$\sum_{q>P^{1-c_{11}}} |A_q| < c_{12}\sum_{q>P^{1-c_{11}}} q^{1-n} < c_{13}P^{\sigma-1},$$
$$c_{12} = c_{12}(n, r, m), \qquad c_{13} = c_{13}(n, r, m).$$

From this follows directly the absolute convergence of the "singular" series. Further, from (12), (13) and (14)

$$|R_1(P)| \leqslant \varrho|\mathfrak{S}|P^{r-n}+O(P^{r-n-c_{11}}).$$

Second step. We estimate $R_2(P)$. The total length of complementary intervals is evidently $\leqslant 1$ and in every complementary interval, by reason of (11) it follows from lemma 3 that

$$|S| < c_{14}P^{r+\varepsilon-n-\sigma+c_{11}(n+\sigma)} < c_{14}P^{r-n-c_{15}}, \qquad c_{14} = c_{14}(\varepsilon, r, n, m),$$
$$c_{15} = c_{15}(n).$$

Therefore

$$R_2(P) = O(P^{r-n-c_{15}}).$$

Confronting the results of the first and the second step, we immediately get the statement of the theorem.

To conclude we make two more remarks: At first, we remark that if, instead of the partition used in the above demonstration, we make use of a considerably more complicated partition described in [3] and we estimate the integral on the "complementary" intervals as it is presented there, we shall extend the validity of the theorem also to the case when the polynomial $\varphi(x_1,\ldots,x_r)$ is of $(n-2)$ or $(n-1)$th degree. It is very probable that the (*) upper estimation is true also for a polynomial $\Phi(x_1,\ldots,x_r)$ with many variables of a quite general form.

Secondly, it is worth remarking that most probably $r_0(n)$ may be reduced also in order of magnitude. From the above demonstration it appears clearly that the question of order of magnitude of $r_0(n)$ is in close connection with the upper estimation of the absolute value of the trigonometrical sum S.

The upper estimation of the absolute value of the sum S, as it is stated in lemma 3, has been obtained by the so-called Weyl estimation method. As we know, some years afterwards, Vinogradov succeeded in elaborating a considerably more effective method than that of Weyl. Unfortunately Vinogradov's estimation method for the upper estimation of the absolute value of the sum S seems to present extraordinary difficulties in the general case, because of the complication of this method. In the case, however, when $\varphi(x_1,\ldots,x_r) = P_1(x_1)+\ldots+P_r(x_r)$, where $P_1(x_1),\ldots,P_r(x_r)$ are polynomials with one variable, we can immediately apply Vinogradov's estimation of trigonometrical sums of the form

$$T = \sum_{x=1}^{P}\exp\big(2\pi i g(x)\big), \qquad g(x) = a_n x^n+\ldots+a_1 x$$

(see [4], p. 291) and we immediately infer that if with suitably chosen $c_{16} = c_{16}(n)$

$$\left|y - \frac{a}{q}\right| < \frac{1}{q\tau}, \quad (a, q) = 1, \quad P^{1-c_{16}} < q \leqslant \tau,$$

then

$$|S| < c_{17} P^{r + c_{18} - r/3(n-1)^2 \log 13n(n-1)},$$

$$c_{17} = c_{17}(r, n, m), \quad c_{18} = \frac{rc_{16}}{3(n-1)^2 \log 13n(n-1)}.$$

When instead of the upper estimation stated in lemma 3 we make use of the upper estimation as presented above, we are able to reduce $r_0(n)$ immediately without any further consideration to

$$3n(n-1)^2 \log 13n(n-1) + 1.$$

It seems very probable that — at least in this special case — the order of magnitude of $r_0(n)$ can be reduced to n.

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Reçu par la Rédaction le 15. 3. 1958.

ACTA ARITMETICA
V (1958)

Remarks on number theory I
On primitive α-abundant numbers

by

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Denote by $\sigma(n)$ the sum of divisors of n. It is well known that $\sigma(n)/n$ has a continuous distribution function, i. e. for every c the density of integers satisfying $\sigma(n)/n \leqslant c$ exists and is a continuous function of c whose value $\to 1$ as $c \to \infty$. This result was first proved by Davenport [1], Behrend and Chowla. Thus in particular the density of abundant numbers exists (a number is abundant if $\sigma(n)/n \geqslant 2$). I [2] have proved the existence of this density by proving that the sum of the reciprocals of the primitive abundant numbers converges (a number m is called *primitive abundant* if $\sigma(m)/m \geqslant 2$ but for every proper divisor d of m, $\sigma(d)/d < 2$). More generally we shall say that m is *primitive α-abundant* if $\sigma(m)/m \geqslant \alpha$ but, for every proper divisor d of m, $\sigma(d)/d < \alpha$. I observed some time ago that it is not true that the sum of the reciprocals of the primitive α-abundant numbers converges for every α. It will be clear from our proof that if α can be approximated very well by numbers of the form $\sigma(n)/n$ then the sum of the reciprocals of the primitive α-abundants will diverge.

Let $p_1, p_2, \ldots$ be an infinite sequence of primes satisfying $p_{k+1} > e^{e^{p_k^2}}$. Put

$$\alpha = \prod_{k=1}^{\infty}\left(1 + \frac{1}{p_k}\right) = \lim_{k\to\infty} \frac{\sigma(p_1 p_2 \ldots p_k)}{p_1 p_2 \ldots p_k}.$$

A simple computation shows that for every k the integers

$$p_1 p_2 \ldots p_k p, \quad p_k < p < p_{k+1}$$

are primitive α-abundant. From

$$\sum_{p \leqslant x} \frac{1}{p} = (1 + o(1)) \log\log x$$