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On new "explicit formulas" in prime number theory I

by

S. KNAPOWSKI (Poznań)

1. The distribution of prime numbers is closely bound up with the distribution of the complex zeros of the Riemann zeta-function. The following "explicit formula", due to Riemann, was first rigorously proved by von Mangoldt (see e. g. [1], p. 77, Theorem 29) and gives perhaps the most striking evidence of this connection.

If

$$x > 1, \quad \psi(x) = \sum_{n \leqslant x} \Lambda(n),$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^m,\ p \text{ prime},\ m = 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_0(x) = \tfrac{1}{2}\big(\psi(x-0) + \psi(x+0)\big),$$

then

$$\psi_0(x) = x - \sum_{\varrho} \frac{x^{\varrho}}{\varrho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right); \tag{1.1}$$

$\varrho = \beta + i\gamma$ denoting all complex zeta-zeros and $\sum_{\varrho}(x^{\varrho}/\varrho)$ being the limit of $\sum_{|\gamma| \leqslant T}(x^{\varrho}/\varrho)$ as $T \to \infty$. Replacing the infinite series $\sum_{\varrho}(x^{\varrho}/\varrho)$ by its partial sum we can deduce from (1.1) an approximate formula (see e. g. [1], p. 77, Theorem 29).

$$\psi_0(x) = x - \sum_{|\gamma| \leqslant x^2} \frac{x^{\varrho}}{\varrho} + O(\log^2 x) \qquad \text{for} \qquad x \geqslant 2.$$

Formula (1.1) suggest at once the well-known fact that the error $\max_{1 \leqslant x \leqslant T} |\psi_0(x) - x|$ in the prime number formula lies in the interval $\langle T^{\theta-\varepsilon}, T^{\theta+\varepsilon} \rangle$, for large T, θ being the upper bound of the real parts of all zeta-zeros. This illustrates the significance of Riemann's conjecture $\theta = \frac{1}{2}$.

It was Turán who recognized that it is of interest in the prime number theory not only to investigate the zeta-zeros but also the zeros of

$$U_N(S) = \sum_{n\leqslant N} \frac{1}{n^s}.$$

He has proved the following theorem:

If for integer $N > n_0$ the partial-sums $U_N(s)$ do not vanish in the half-plane

$$\sigma > 1+\frac{\log^{100} N}{\sqrt{N}}, \qquad s = \sigma+it,$$

then Riemann's hypothesis is true(¹).

In this paper I prove a new "explicit formula", similar to (1.1) but depending on the zeros of $U_N(s)$ instead of the zeta-zeros.

The result is as follows:

For $2 \leqslant x \leqslant N$, $N \geqslant N_0$, we have

$$\psi_0(x) = \frac{\log N!}{N} - \sum_{\varrho} \frac{x^\varrho}{\varrho}; \tag{1.2}$$

$\varrho = \beta+i\gamma$ denoting the zeros $U_N(s)$ and $\sum_{\varrho}(x^\varrho/\varrho)$ being the limit of $\sum_{|\gamma|\leqslant T}(x^\varrho/\varrho)$ as $T\to\infty$(²).

Further I prove some estimates for $\sum_{|\gamma|>T}(x^\varrho/\varrho)$, which give information about the behaviour of the series $\sum_{\varrho}(x^\varrho/\varrho)$ analogous to that in the classical case.

The formula (1.2) implies the following corollary, which gives a new approximation for the "remainder term" $\psi_0(x)-x$:

If $x \geqslant 2$ then

$$\psi_0(x) = x - \sum_{\substack{\beta\geqslant -1\\ |\gamma|\leqslant x^{16}}} \frac{x^\varrho}{\varrho} + O(\log x),$$

$\varrho = \beta+i\gamma$ being the zeros of $U_N(S)$ for $N = [e^x]$ (³).

(¹) See [4], p. 4. The quoted result is an easy combination of theorems II and III of the paper.

(²) $\sum_{|\gamma|\leqslant T}(x^\varrho/\varrho)$ could be understood as $\lim_{q\to\infty} \sum_{|\gamma|\leqslant T, \beta\geqslant -q} (x^\varrho/\varrho)$, but it is not necessary in virtue of a certain theorem of Pólya (see [2]) which implies that there is only a finite number of $U_N(s)$-zeros in every strip $-\infty < A \leqslant t \leqslant B < +\infty$.

(³) $[\alpha]$ denotes as usual the integral part of α.

I wish to express my deep gratitude to Professor Paul Turán, who has made a number of important suggestions.

2. LEMMA 1. (a) *The number of zeros of $U_N(s)$ in the rectangle*

$$0 \leqslant \sigma \leqslant 2, \quad n \leqslant t \leqslant n+1, \quad s = \sigma+it \quad (n = 0, 1, 2, \ldots)$$

is $\leqslant c_1 \log N$(⁴).

(b) *The number of zeros of $U_N(s)$ in the rectangle*

$$-m \leqslant \sigma \leqslant -m+1, \quad n \leqslant t \leqslant n+1, \quad s = \sigma+it$$

$(n = 0, 1, 2, \ldots, m = 1, 2, \ldots)$ is $\leqslant c_2 m \log N$.

Proof. Apply Jensen's inequality (see e. g. [1], p. 49, Theorem D). The number of zeros of $f(s)$ in the circle $|s-s_0| < R$ is

$$\leqslant \log \max_{|s-s_0|=Re} \left|\frac{f(s)}{f(s_0)}\right|.$$

Put, for (a), $s_0 = 2+(n+\frac{1}{2})i$, $R = \sqrt{5}$. Then

$$U_N(s) = O(N^{\sqrt{5}-1}), \qquad |U_N(s_0)| \geqslant 2-\tfrac{1}{6}\pi^2 > 0$$

and

$$\log \max_{|s-s_0|=Re} \left|\frac{U_N(s)}{U_N(s_0)}\right| \leqslant c_1 \log N.$$

For (b) put $s_0 = 2+(n+\frac{1}{2})i$, $R = \sqrt{(m+2)^2+1}$, whence

$$U_N(s) = O(N^{Re-1})$$

and

$$\log \max_{|s-s_0|=Re} \left|\frac{U_N(s)}{U_N(s_0)}\right| \leqslant c_2 \log N.$$

LEMMA 2. (a) *In the rectangle*

$$-\tfrac{1}{2} \leqslant \sigma \leqslant 2, \quad n \leqslant t \leqslant n+1, \quad s = \sigma+it \quad (n = 0, 1, 2, \ldots)$$

we have

$$\left|\frac{U_N'}{U_N}(s) - \sum_{\varrho} \frac{1}{s-\varrho}\right| \leqslant c_3 \log N,$$

where ϱ runs through the zeros of $U_N(s)$ lying in the rectangle

$$-1 \leqslant \sigma \leqslant 2, \quad n-\tfrac{1}{2} \leqslant t \leqslant n+\tfrac{3}{2}, \quad s = \sigma+it.$$

(⁴) Throughout this paper $c_1, c_2, \ldots$ always denote positive numerical constants.

(b) *In the rectangle*

$$-m-\tfrac{1}{2}\leqslant\sigma\leqslant -m+1,\quad n\leqslant t\leqslant n+1,\quad s=\sigma+it$$

$(n=0,1,2,\ldots;\ m=1,2,\ldots)$ *we have*

$$\left|\frac{U_N'}{U_N}(s)-\sum_{\varrho}\frac{1}{s-\varrho}\right|\leqslant c_4 m\log N,$$

where ϱ *runs through the zeros of* $U_N(s)$ *lying in the rectangle*

$$-m-1\leqslant\sigma\leqslant -m+2,\quad n-\tfrac{1}{2}\leqslant t\leqslant n+\tfrac{3}{2},\quad s=\sigma+it.$$

Proof. I will prove only (b).

Put $s_0=2+(n+\frac{1}{2})i$ and consider the function $G(z)=U_N(z+s_0)$ in the circle $|z|\leqslant 2R$, $R=\sqrt{1+(m+3)^2}$. Denote by z_k all the zeros of $G(z)$ in the circle $|z|<\frac{3}{2}R$. The function

$$G(z)\left(\prod_{z_k}2R\frac{z-z_k}{4R^2-z\bar{z}_k}\right)^{-1}\equiv G_1(z)$$

is regular and $\neq 0$ in the circle $|z|<\frac{3}{2}R$. We showed above that

$$\left|\frac{G(z)}{G(0)}\right|\leqslant\exp(c_5 m\log N)\quad\text{if}\quad |z|\leqslant 2R.$$

Denote by $G_2(z)$ the branch of $\log\big(G_1(z)/G_1(0)\big)$ in the circle $|z|<\frac{3}{2}R$, determined by $G_2(0)=0$. We have for $|z|=2R$

$$\left|\frac{G_1(z)}{G_1(0)}\right|\leqslant\left|\frac{G(z)}{G(0)}\right|\leqslant\exp(c_5 m\log N)\quad\text{(compare e. g. [1], p. 49).}$$

Hence

$$\Re G_2(z)\leqslant c_5 m\log N\quad\text{for}\quad |z|<\tfrac{3}{2}R$$

and

$$\left|\frac{G_1'}{G_1}(z)\right|\leqslant\frac{2\cdot\frac{3}{2}R}{(\frac{1}{2}R)^2}c_5 m\log N\leqslant c_6\log N\quad\text{for}\quad |z|\leqslant R$$

(apply [1], p. 50, Theorem E with $\nu=1$). That is

$$\left|\frac{G'}{G}(z)-\sum_{z_k}\left(\frac{1}{z-z_k}+\frac{1}{4R^2/\bar{z}_k-z}\right)\right|\leqslant c_6\log N\quad\text{for}\quad |z|\leqslant R.$$

But

$$\left|\frac{4R^2}{\bar{z}_k}-z\right|\geqslant\frac{4R^2}{\frac{3}{2}R}-R=\tfrac{5}{3}R,$$

and the number of the zeros z_k is $\leqslant c_7 m\log N$, whence

$$\left|\sum_{z_k}\frac{1}{4R^2/\bar{z}_k-z}\right|\leqslant\frac{3}{5R}c_7 m\log N\leqslant c_8\log N.$$

Finally

$$\left|\frac{U_N'}{U_N}(s)-\sum_{s_k}\frac{1}{s-s_k}\right|\leqslant c_9\log N$$

after the transformation $s=z+s_0$. If s_k lies outside the rectangle

$$-m-1\leqslant\sigma\leqslant -m+2,\quad n-\tfrac{1}{2}\leqslant t\leqslant n+\tfrac{3}{2},\quad s=\sigma+it,$$

then $|s-s_k|\geqslant\frac{1}{2}$ for

$$-m-\tfrac{1}{2}\leqslant\sigma\leqslant -m+\tfrac{3}{2},\quad n\leqslant t\leqslant n+1,\quad s=\sigma+it$$

and the result follows.

LEMMA 3.

(a) *There exists a sequence of numbers* $T_0, T_1, T_2, \ldots$ *such that*

1. $n\leqslant T_n\leqslant n+1$,

2. $\left|\dfrac{U_N'}{U_N}(s)\right|\leqslant c_{10}\log^2 N$ *for* $-\frac{1}{2}\leqslant\sigma\leqslant 2$, $t=T_n$, $s=\sigma+it$.

(b) *For every* $m=1,2,\ldots$ *there exists a sequence* $T_0^{(m)}, T_1^{(m)}, T_2^{(m)}, \ldots$ *such that*

1. $n\leqslant T_n^{(m)}\leqslant n+1$,

2. $\left|\dfrac{U_N'}{U_N}(s)\right|\leqslant c_{11}m^2\log^2 N$ *for* $-m-\frac{1}{2}\leqslant\sigma\leqslant -m+1$, $t=T_n^{(m)}$, $s=\sigma+it$.

(c) *For every* $m=1,2,\ldots$ *there exists a sequence* $S_0^{(m)}, S_1^{(m)}, S_2^{(m)}, \ldots$ *such that*

1. $-m+\frac{1}{2}\leqslant S_n^{(m)}\leqslant -m+1$,

2. $\left|\dfrac{U_N'}{U_N}(s)\right|\leqslant c_{12}m^2\log^2 N$ *for* $\sigma=S_n^{(m)}$, $n\leqslant t\leqslant n+1$, $s=\sigma+it$.

(d) *For every* $m=1,2,\ldots$ *there exists a sequence* $\tilde{S}_0^{(m)}, \tilde{S}_1^{(m)}, \tilde{S}_2^{(m)}, \ldots$ *such that*

1. $-m\leqslant\tilde{S}_n^{(m)}\leqslant -m+1$,

2. $\left|\frac{U_N'}{U_N}(s)\right| \leqslant c_{13} m^2 \log^2 N$ *for* $\sigma = \tilde{S}_n^{(m)}$, $n \leqslant t \leqslant n+\frac{3}{2}$, $s = \sigma + it$.

(e) *For every* $m = 1, 2, \ldots$ *there exists a sequence* $\tilde{T}_0^{(m)}, \tilde{T}_1^{(m)}, \tilde{T}_2^{(m)}, \ldots$ *such that*

1. $n \leqslant \tilde{T}_n^{(m)} \leqslant n+\frac{1}{2}$,

2. $\left|\frac{U_N'}{U_N}(s)\right| \leqslant c_{14} m^2 \log^2 N$ *for* $-m \leqslant \sigma \leqslant -m+1$, $t = \tilde{T}_n^{(m)}$,

$$s = \sigma + it.$$

Proof. Since all the proofs are analogous, it is clearly enough to prove, say, (b).

Divide the interval $\langle n, n+1\rangle$ into $Q+1$ equal parts, where Q denotes the number of zeros of $U_N(s)$ in the rectangle

$$-m-\tfrac{1}{2} \leqslant \sigma \leqslant -m+1, \quad n \leqslant t \leqslant n+1, \quad s = \sigma + it.$$

At least one of the rectangles so obtained is free of $U_N(s)$-zeros. Denote the ordinate of the centre of this rectangle by $T_n^{(m)}$. Let ϱ be any $U_N(s)$-zero lying in the rectangle

$$-m-1 \leqslant \sigma \leqslant -m+2, \quad n-\tfrac{1}{2} \leqslant t \leqslant n+\tfrac{3}{2}, \quad s = \sigma + it$$

and let s^* lie on the line $t = T_n^{(m)}$. Then

$$|s^* - \varrho| \geqslant \tfrac{1}{2} \cdot \frac{1}{Q+1},$$

whence

$$\frac{1}{|s^*-\varrho|} \leqslant 2(Q+1) \leqslant c_{15} m \log N$$

by Lemma 1. Hence and from Lemma 2 we obtain

$$\left|\frac{U_N'}{U_N}(s)\right| \leqslant c_{11} m^2 \log^2 N$$

for

$$-m-\tfrac{1}{2} \leqslant \sigma \leqslant -m+1, \quad t = T_n^{(m)}, \quad s = \sigma + it.$$

LEMMA 4. *The function* $U_N'(s)/U_N(s)$ *may be developed in a Dirichlet series* $\sum_{n=1}^{\infty}(a_n/n^s)$ *convergent in the half-plane*

$$\sigma > 1 + 2\frac{\log\log N}{\log N} \quad (N \geqslant N_0).$$

Further

(2.1) $$a_n = -\Lambda(n) \quad \textit{for} \quad n \leqslant N$$

and

(2.2) *if* $x \leqslant N$ *and* $n \leqslant \frac{3}{2}x$ *then* $|a_n| \leqslant c_{16} x \log x$.

Remark. It is sufficient to prove (2.2) only for $x = N$. In fact, if $x \leqslant \frac{2}{3}N$, then $n \leqslant N$ and $|a_n| \leqslant \log n \leqslant \log\frac{3}{2}x$ by (2.1). If $\frac{2}{3}N \leqslant x \leqslant N$, then $|a_n| \leqslant c_{16} N \log N \leqslant c_{16}' x \log x$, since we suppose that (2.2) holds for $x = N$.

Proof. The existence of a development of $U_N'(s)/U_N(s)$ in a Dirichlet series follows from Turán's remark (10.1.7) in [5], p. 121. The convergence of this series for

$$\sigma > 1 + 2\frac{\log\log N}{\log N}$$

follows from another theorem of Turán ([4], p. 20), stating that

(2.3) $$\left|\frac{1}{U_N(s)}\right| \leqslant c_{17} \frac{\log^2 N}{(\log\log N)^2} \quad \text{for} \quad \sigma > 1 + 2\frac{\log\log N}{\log N}.$$

This inequality, combined with a general theorem on Dirichlet series (see e. g. [3], § 9.44, p. 302), gives the required result. We note incidentally that (2.3) implies the inequality

(2.4) $$\left|\frac{U_N'}{U_N}(s)\right| \leqslant c_{18} \frac{\log^4 N}{(\log\log N)^3} \quad \text{for} \quad \sigma > 1 + 2\frac{\log\log N}{\log N},$$

which we will use afterwards. Turning now to the proof of (2.1) we have

$$U_N'(s) = U_N(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Hence, for $n \leqslant N$ we obtain

$$\sum_{k|n} a_k = -\log n,$$

whence by the Möbius inversion formula we obtain

(2.5) $$a_n = \sum_{d|n} \mu(d)\big(-\log(n/d)\big).$$

Since this defines a_n uniquely for $n \leqslant N$, from the well-known formula

$$\sum_{n=1}^{\infty} \frac{-\Lambda(n)}{n^s} = \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{-\log n}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{for} \quad \sigma > 1$$

we see at once that (2.1) holds.

Now we shall prove inequality (2.2) for $x = N$. It is sufficient to consider only $N < n \leqslant \frac{3}{2}N$. We then have

$$\sum_{k|n} a_k = 0\,.$$

If $k < n$, then $k \leqslant \frac{1}{2}n$ and consequently $k \leqslant N$. Hence

$$a_n = -\sum_{\substack{k|n\\k<n}} a_k\,, \qquad |a_n| \leqslant \sum_{\substack{k|n\\k<n}} |a_k| \leqslant \sum_{k|n} \log k \leqslant \log n! \leqslant c_{16} N \log N\,.$$

3. THEOREM. *Let $N > N_0$ be an integer. If $2 \leqslant x \leqslant N$, then*

$$\psi_0(x) = \frac{\log N!}{N} - \sum_{\varrho} \frac{x^\varrho}{\varrho},$$

$\varrho = \beta + i\gamma$ running through the zeros of $U_N(s) = \sum_{n\leqslant N} (1/n^s)$, and $\sum_{\varrho} (x^\varrho/\varrho) = S_N(x)$ denoting the limit of $S_N(x, T) = \sum_{|\gamma|\leqslant T} (x^\varrho/\varrho)$ as $T \to \infty$.

Writing further

$$R_N(x, T) = S_N(x) - S_N(x, T)$$

we have

$$|R_N(x, T)| \leqslant \begin{cases} c_{19} \dfrac{x^2}{T}\left(\dfrac{\log^{14} N}{(\log\log N)^6} + \dfrac{\log^6 N}{\xi}\right) & \text{if} \quad x \neq p^m, \\[2ex] c_{19} \dfrac{x^2}{T} \cdot \dfrac{\log^{14} N}{(\log\log N)^6} & \text{if} \quad x = p^m, \end{cases}$$

$$|R_N(x, T)| \leqslant c_{19}\left(\frac{x^2}{T} \cdot \frac{\log^{14} N}{(\log\log N)^6} + \log x\right) \qquad \textit{always}$$

where $\xi = \xi(x)$ is the distance of x from the nearest prime power p^m.

Proof. Let $T \geqslant 3$ and denote by T' the least T_n of Lemma 3 which is greater than T. Let $q \geqslant 2$ be any integer.

Denote by C_q^T the contour consisting of the segment

$$\left\langle 2 + 6\frac{\log\log N}{\log N} - iT',\; 2 + 6\frac{\log\log N}{\log N} + iT' \right\rangle$$

and of three broken lines given by Lemma 3 as follows:

First there is given an infinite broken line consisting of the horizontal segments

$$S_n^{(1)} \leqslant \sigma \leqslant 2 + 6\frac{\log\log N}{\log N}, \qquad t = T_n.$$

$$S_n^{(m+1)} \leqslant \sigma \leqslant S_n^{(m)}, \qquad t = T_n^{(m)} \qquad (m = 1, 2, \ldots)$$

joined by vertical ones. Secondly there is an infinite broken line consisting of the vertical segments

$$\sigma = \tilde{S}_\mu^{(q)}, \qquad \tilde{T}_\mu^{(q)} \leqslant t \leqslant \tilde{T}_{\mu+1}^{(q)} \qquad (\mu = 0, 1, \ldots)$$

joined by horizontal ones. By symmetrical mapping in the half-plane $t < 0$ we obtain three infinite broken lines. The contour C_q^T is formed by their intersection.

Put

$$a = 2 + 6\frac{\log\log N}{\log N} - iT', \qquad b = 2 + 6\frac{\log\log N}{\log N} + iT'.$$

Consider the integral

$$(3.1) \quad \frac{1}{2\pi i}\int_{C_q^T} \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right) ds = \frac{1}{2\pi i}\int_a^b \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right) ds + \frac{1}{2\pi i}(I_1 + I_2 + I_3)$$

where I_1, I_2, I_3 denote the integrals along respective upper, lower and left broken line. First of all, estimate these integrals:

$$(3.2) \quad |I_3| \leqslant x^{-q+1} c_{20} q^2 \log^2 N\left(\int_0^{T'} \frac{dt}{\sqrt{(q-1)^2 + t^2}} + \sum_{n\leqslant T} \frac{1}{n}\right) \leqslant c_{21} \frac{q^2 \log^2 N}{x^{q-1}} \log T,$$

$$(3.3) \quad |I_1| \leqslant c_{22}\left(\sum_{\nu=1}^{q} \int_{\nu-1}^{\nu} \frac{x^{-\sigma}}{\sqrt{\sigma^2 + (T-1)^2}} \nu^2 \log^2 N\, d\sigma + \frac{x^2 \log^8 N}{T} + \sum_{\nu=1}^{q} \frac{\nu^2 \log^2 N}{x^{\nu-1} T}\right)$$

$$\leqslant c_{23} \frac{\log^8 N}{T}\left(x^2 + \sum_{\nu=1}^{\infty} \frac{\nu^2}{2^{\nu-1}}\right) \leqslant c_{24} \frac{x^2 \log^8 N}{T}$$

and similarly:

$$(3.4) \qquad |I_2| \leqslant c_{24} \frac{x^2 \log^8 N}{T}.$$

Now apply Cauchy's theorem

$$(3.5) \qquad \frac{1}{2\pi i}\int_{C_q^T} \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right) ds = \frac{\log N!}{N} - \sum_{\varrho \text{ inside } C_q^T} \frac{x^\varrho}{\varrho}.$$

As $q \to \infty$, we obtain from (3.1), (3.2), (3.3), (3.4), (3.5)

$$\frac{1}{2\pi i}\int_a^b \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right)ds = \frac{\log N!}{N} - \sum_{\varrho}{}' \frac{x^{\varrho}}{\varrho} + O\left(\frac{x^2}{T}\log^8 N\right) \tag{3.6}$$

where $\sum_{\varrho}'$ denotes the summation over all zeros ϱ lying between the upper and the lower broken line. Now

$$\left|\sum_{\varrho}{}' \frac{x^{\varrho}}{\varrho} - S_N(x, T)\right| \leqslant \frac{c_{25}}{T}\left(x\log^3 N + \sum_{\nu=1}^{\infty}\frac{\nu \log N}{x^{\nu-1}}\right) \leqslant \frac{c_{26}}{T}\, x\log^3 N.$$

From this and from (3.6) we obtain

$$\frac{1}{2\pi i}\int_a^b \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right)ds = \frac{\log N!}{N} - S_N(x, T) + O\left(\frac{x^2}{T}\log^8 N\right). \tag{3.7}$$

Now put

$$I(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}ds, \qquad I(y, T) = \int_{c-iT}^{c+iT}\frac{y^s}{s}ds, \qquad \varDelta(y, T) = I(y) - I(y, T).$$

Then it is well known (see e. g. [1], p. 75, Theorem G) that for $T > 0$

$$|\varDelta(y, T)| < \begin{cases} \dfrac{y^c}{\pi T|\log y|} & \text{if} \quad y \neq 1, \\[2mm] \dfrac{c}{\pi T} & \text{if} \quad y = 1, \end{cases}$$

$$|\varDelta(y, T)| < y^c \quad \text{always.}$$

Using this notation and putting $c = 2 + 6\dfrac{\log\log N}{\log N}$ we have

$$\frac{1}{2\pi i}\int_{c-iT'}^{c+iT'}\frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right)ds = \sum_{n=1}^{\infty}\frac{-a_n}{2\pi i}\int_{c-iT'}^{c+iT'}\frac{(x/n)^s}{s}ds = \sum_{n=1}^{\infty} -a_n I\left(\frac{x}{n}, T'\right)$$

$$= \sum_{n=1}^{\infty} -a_n I\left(\frac{x}{n}\right) + \sum_{n=1}^{\infty} a_n \varDelta\left(\frac{x}{n}, T'\right)$$

$$= \psi_0(x) + \sum_{n=1}^{\infty} a_n \varDelta\left(\frac{x}{n}, T'\right).$$

Write

$$u_n = a_n \varDelta\left(\frac{x}{n}, T'\right), \qquad \sum_{n=1}^{\infty} u_n = X.$$

Then

$$\frac{1}{2\pi i}\int_a^b \frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right)ds = \psi_0(x) + X, \tag{3.8}$$

where

$$a = 2 + 6\frac{\log\log N}{\log N} - iT', \qquad b = 2 + 6\frac{\log\log N}{\log N} + iT'.$$

We now estimate X. Let

$$2 + 6\frac{\log\log N}{\log N} = d.$$

If $n \neq x$ then

$$|u_n| \leqslant |a_n|\cdot\frac{(x/n)^d}{\pi T'|\log(x/n)|} \leqslant \frac{x^2\log^6 N\cdot|a_n|}{\pi T n^d}\cdot\frac{n+x}{|n-x|}.$$

Denote by $\nu = \nu(x)$ the integer defined by $\nu - \frac{1}{2} < x \leqslant \nu + \frac{1}{2}$. Then it is not difficult to see that $|a_\nu| \leqslant \log\nu$ and we obtain analogously to [1] (p. 79)

$$|u_\nu| \leqslant \begin{cases} c_{27}\dfrac{x^2}{T\xi}\log^6 N & \text{if} \quad x \neq p^m, \\[2mm] c_{28}\dfrac{\log x}{T} & \text{if} \quad x = p^m, \end{cases} \tag{3.9}$$

$$|u_\nu| \leqslant c_{29}\log x \quad \text{always.}$$

Further (compare [1], p. 79)

$$|X - u_\nu| \leqslant \frac{x^2\log^6 N}{\pi T}\left(5\sum_{n=1}^{\infty}\frac{|a_n|}{n^d} + 2\sum_{r=1}^{[x]}\frac{\max\limits_{1\leqslant n\leqslant 3x/2}|a_n|\cdot\frac{5}{2}x}{(\frac{1}{2}x)^2\frac{1}{2}r}\right). \tag{3.10}$$

Clearly

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^d} \leqslant \sum_{n=1}^{\infty}\frac{|a_n|^2}{n^d} + \sum_{n=1}^{\infty}\frac{1}{n^d}. \tag{3.11}$$

By the mean-value theorem for Dirichlet series (see e. g. [5], p. 307) and inequality (2.4) we obtain

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^d} \leqslant c_{30}\frac{\log^8 N}{(\log\log N)^6} \quad \text{where} \quad d = 2 + 6\frac{\log\log N}{\log N}. \tag{3.12}$$

Further, by (2.2),

$$\sum_{r=1}^{[x]} \frac{\max\limits_{1\leqslant n\leqslant 3x/2} |a_n|\cdot \frac{5}{2}x}{(\frac{1}{2}x)^2 \frac{1}{2}r} \leqslant c_{31}\frac{x^2\log x}{x^2}\sum_{r=1}^{[x]}\frac{1}{r} \leqslant c_{32}\log^2 x. \tag{3.13}$$

(3.10), (3.11), (3.12), and (3.13) give

$$|X-u_\nu| \leqslant c_{33}\frac{x^2\log^{14}N}{T(\log\log N)^6}. \tag{3.14}$$

(3.7) and (3.8) give

$$\psi_0(x) = \frac{\log N!}{N} - S_n(x, T) + O\left(\frac{x^2}{T}\log^8 N\right) + O(|X|).$$

By (3.9) and (3.14) we obtain the required estimates for $R_N(x, T)$. The formula

$$\psi_0(x) = \frac{\log N!}{N} - \sum_\varrho \frac{x^\varrho}{\varrho}$$

follows on letting T tend to infinity.

Corollary. *Let $x \geqslant 2$. Then*

$$\psi_0(x) = x - \sum_{\substack{|\gamma|\leqslant x^{16}\\ \beta\geqslant -1}} \frac{x^\varrho}{\varrho} + O(\log x) \tag{3.15}$$

where $\varrho = \beta+i\gamma$ denote the zeros of $U_n(s) = \sum\limits_{n\leqslant N}\frac{1}{n^s}$ and $N = [e^x]$.

Obviously

$$\frac{\log N!}{N} = x + O(1).$$

Taking further

$$T = x^2\log^{16}N$$

we obtain

$$\Bigg|\sum_{\substack{|\gamma|\leqslant T\\ -m\leqslant\beta<-m+1}} \frac{x^\varrho}{\varrho}\Bigg| \leqslant c_{34}x^{-m+1}\sum_{n\leqslant x^2\log^{14}N}\frac{m\log N}{n}$$

$$\leqslant c_{35}x^{-m+1}m\log(x^2\log^{14}N)\log N \leqslant c_{36}m\frac{\log x}{x^{m-2}},$$

whence

$$\Bigg|\sum_{\substack{|\gamma|\leqslant T\\ \beta<-1}} \frac{x^\varrho}{\varrho}\Bigg| \leqslant c_{37}\log x\sum_{m=2}^{\infty}\frac{m}{x^{m-2}} \leqslant c_{37}\log x\sum_{m=2}^{\infty}\frac{m}{2^{m-2}} = O(\log x)$$

and the result follows.

Remark. It is not difficult to replace x^{16} in formula (3.15) by $x^{4+\varepsilon}$. This could have been achieved by taking

$$c = 2+6\frac{\log\log x}{\log x}$$

instead of

$$c = 2+6\frac{\log\log N}{\log N}$$

in the proof of Theorem and by more careful estimation.

4. Considering now the function

$$\frac{x^s}{s^{k+1}}\left(-\frac{U_N'}{U_N}(s)\right), \quad k\geqslant 1 \text{ integer},$$

instead of

$$\frac{x^s}{s}\left(-\frac{U_N'}{U_N}(s)\right)$$

we can obtain formulas involving $\sum\limits_\varrho \frac{x^\varrho}{\varrho^{k+1}}$ analogous to those already obtained.

Put, say, $x = N$. Then we prove

$$\sum_{n\leqslant N}\Lambda(n)\frac{\log^k(N/n)}{k!} = A_k(N) - \sum_{|\gamma|\leqslant T}\frac{N^\varrho}{\varrho^{k+1}} + O\left(\frac{N^2}{T^k}\cdot\frac{\log^{14}N}{(\log\log N)^6}\right) \tag{4.1}$$

where $\varrho = \beta+i\gamma$ denotes the zeros of $U_N(s)$ and

$$A_k(N) = \frac{1}{k!}\cdot\frac{d^k}{ds^k}\left(-N^s\frac{U_N'}{U_N}(s)\right)_{s=0}.$$

It can be noticed that we are in a position to deduce from (4.1) some information on the distribution of zeros of $U_N(s)$. As this subject seems to be of self-contained interest I will return to it somewhere else.

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On the distribution of the solutions of diophantine equations with many unknowns

by

L. VEIDINGER (Budapest)

To the solutions of a diophantine equation with r unknowns correspond geometrically — as we know — in the r-dimensional space R^r the points with entire coordinates of an $(r-1)$-dimensional hypersurface. From this geometrical interpretation follows immediately for every diophantine equation with r unknowns the following problem of a very general character, which can be formulated also merely arithmetically: how the lattice points representing the solutions of the diophantine equation in question are distributed in the space R^r. Of course this problem is interesting principally in the case when the diophantine equation has infinitely many solutions.

Let r and P be positive integers, $\Phi(x_1, \ldots, x_r)$ a polynomial of r variables with entire coefficients, in respect to which we do not make, for the moment, any restrictions.

The distribution of the solutions in positive integers of the equation

$$\Phi(x_1, \ldots, x_r) = 0 \tag{1}$$

can be described with the aid of the solution function $R(P)$ defined in the following manner: let $R(P)$ denote the number of all the points with entire coordinates of the hypersurface (1) which are placed inside the cube $1 \leqslant x_1 \leqslant P, \ldots, 1 \leqslant x_r \leqslant P$.

Purely arithmetically formulated, $R(P)$ means the number of all the positive entire solutions of the diophantine equation (1) in respect to which $x_1 \leqslant P, \ldots, x_r \leqslant P$.

As each of the variables $x_1, \ldots, x_r$ can assume only the values $1, \ldots, P$, for the $R(P)$ solution function we have in every case the trivial upper estimation

$$R(P) \leqslant P^r.$$

But in very many cases the upper estimation can be considerably improved. So for instance if $n \geqslant 2$, and $C = C(\varepsilon, n)$ is a positive constant de-