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Minimal bases and powers of 2

by

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Dedicated to Paul Erdős on his 75th birthday

1. Introduction. Let A be a set of nonnegative integers. Let $h \geqslant 2$. Denote by hA the set of all numbers n of the form $n = a_1 + a_2 + \ldots + a_h$, where $a_1, a_2, \ldots, a_h$ are elements of A and are not necessarily distinct. The set A is an *asymptotic basis of order* h if hA contains all sufficiently large integers.

The *counting function* of the set A is the function $A(x)$ defined as the number of positive elements of A not exceeding x. An elementary combinatorial argument shows that if A is an asymptotic basis of order h, then $A(x) > c_1 x^{1/h}$ for some constant $c_1 > 0$ and all x sufficiently large. An asymptotic basis A of order h is called *thin* if there is a constant $c_2 > 0$ such that $A(x) < c_2 x^{1/h}$ for all x sufficiently large.

Let $A^{(h)}$ be the set of all nonnegative integers of the form $\sum_{f \in F} 2^f$, where F is a finite set of nonnegative integers such that $f \equiv f' \pmod{h}$ for all $f, f' \in F$. Choosing the empty set for F shows that $0 \in A^{(h)}$ for all $h \geqslant 2$. Raikov [6] and Stöhr [7] proved that $A^{(h)}$ is a thin asymptotic basis of order h.

An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. This means that, for any $a \in A$, the set $E_a = hA \backslash h(A \backslash \{a\})$ is infinite. Härtter [4] gave a nonconstructive proof of the existence of minimal asymptotic bases. Erdős [1] and Erdős and Härtter [2] obtained other early results on minimal bases. Nathanson [5] proved that $A^{(2)} \backslash \{0\}$ is a minimal asymptotic basis of order 2. Erdős and Nathanson [3] have recently published a survey of open problems on minimal bases in additive number theory.

Let A be an asymptotic basis of order h. Let $E_a(x)$ be the counting function of the set E_a. If $n \in E_a$ and $n \leqslant x$, then every representation of n as a sum of h elements of A is of the form $n = a + a_1 + \ldots + a_{h-1}$, where $a_i \in A$ and $0 \leqslant a_i \leqslant x$ for $i = 1, \ldots, h-1$. Since there are at most $A(x)+1$ choices for each a_i, it follows that $E_a(x) \leqslant (A(x)+1)^{h-1}$. Let us call an asymptotic basis A of order h *strongly minimal* if $E_a(x) > c(A(x))^{h-1}$ for some constant $c = c(a) > 0$ and all x sufficiently large. Nathanson's proof [5] that $A^{(2)} \backslash \{0\}$ is minimal is, in fact, a proof that $A^{(2)}\backslash\{0\}$ is a strongly minimal asymptotic basis of order 2. Zöllner [8] has obtained some results concerning the minimality of the sets $A^{(h)}$.

The object of this paper is to show that $A^{(h)}\backslash\{0\}$ is a strongly minimal asymptotic basis of order h for every $h \geqslant 2$, and also to discuss an analogous class of asymptotic bases constructed from the set of powers of 2.

2. Results. Let N denote the set of nonnegative integers, and let W be a subset of N. Denote by $\mathscr{F}^*(W)$ the set of all finite, nonempty subsets of W. Let $A(W)$ be the set of all numbers of the form $\sum_{f\in F} 2^f$, where $F\in\mathscr{F}^*(W)$. Note that $\emptyset\notin\mathscr{F}^*(W)$, hence $0\notin A(W)$.

LEMMA 1. (a) *If W_1 and W_2 are disjoint subsets of N, then $A(W_1)\cap A(W_2) = \emptyset$.*

(b) *If $W\subseteq N$ and $W(x) = \theta x+O(1)$ for some $\theta\in(0, 1]$, then there exist positive constants c_1 and c_2 such that*

$$c_1 x^\theta < A(W)(x) < c_2 x^\theta$$

for all x sufficiently large

(c) *Let $N = W_0\cup W_1\cup\ldots\cup W_{h-1}$, where $W_i\neq 0$ for $i = 0, 1, \ldots, h-1$. Then $A = A(W_0)\cup A(W_1)\cup\ldots\cup A(W_{h-1})$ is an asymptotic basis of order h. Indeed, $hA = \{n\in N|\ n\geqslant h\}$ and $h(A\cup\{0\}) = N$.*

Proof. (a) This follows from the fact that every positive integer is uniquely the sum of distinct powers of 2.

(b) Let $x\geqslant 1$. Choose $k\geqslant 0$ such that $2^k\leqslant x<2^{k+1}$. If $n\in A(W)$ and $n\leqslant x$, then $n = \sum_{f\in F} 2^f$ for some $F\in\mathscr{F}^*(W)$, and $2^f\leqslant n\leqslant x$ implies that $0\leqslant f\leqslant k$ for every $f\in F$. Thus, F is a nonempty subset of $\{0, 1, \ldots, k\}\cap W$. Since the cardinality of $\{0, 1, \ldots, k\}\cap W$ is at most $W(k)+1$, it follows that there are at most $2^{W(k)+1}-1$ nonempty subsets of $\{0, 1, \ldots, k\}\cap W$. Therefore,

$$A(W)(x)\leqslant 2^{W(k)+1}-1 < 2^{W(\log x/\log 2)+1} < c_2 x^\theta.$$

Similarly, let F be a nonempty subset of $\{1, 2, \ldots, k-1\}\cap W$. Then

$$\sum_{f\in F} 2^f\leqslant \sum_{i=1}^{k-1} 2^i < 2^k\leqslant x$$

and so

$$A(x)\geqslant 2^{W(k-1)}-1 = 2^{\theta(\log x/\log 2)+O(1)}-1 > c_1 x^\theta$$

for some $c_1>0$ and all x sufficiently large.

(c) Let n be a positive integer. Then $n = \sum_{f\in F} 2^f$, where $F\in\mathscr{F}^*(N)$.

Define $F_i = F\cap(W_i\backslash\bigcup_{j=0}^{i-1} W_j)$ and $n_i = \sum_{f\in F_i} 2^f$ for $i = 0, 1, \ldots, h-1$. Then $n_i\in A(W_i)\cup\{0\}$, and, if $F_i\neq\emptyset$, then $n_i\in A(W_i)$. Thus, $n = n_0+n_1+\ldots+n_{h-1}\in h(A\cup\{0\})$. If $F_i\neq\emptyset$ for all i, then $n\in hA$.

Suppose that some of the sets F_i are empty. If $|F|\geqslant h$, then it is possible to partition those sets F_i that are nonempty so that F is a union of h nonempty sets $G_1, \ldots, G_h$, where each G_j is a subset of some F_i. It follows that $n\in hA$.

If $n = \sum_{f\in F} 2^f$ and $1\leqslant |F|\leqslant h-1$, then n is a sum of $|F|$ elements of A. If $g\geqslant 1$ for some $g\in F$, then

$$n = \sum_{f\in F} 2^f = \sum_{\substack{f\in F\\ f\neq g}} 2^f+2^{g-1}+2^{g-1}$$

is a sum of $|F|+1$ elements of A. We can continue in this way to divide powers of 2 until n is a sum of exactly h powers of 2, with repetitions allowed, or until n is a sum of at most $h-1$ ones. Thus, $n\notin hA$ if and only if $n\leqslant h-1$, and so $hA = \{n\in N|\ n\geqslant h\}$. This proves the lemma.

LEMMA 2. *Let $w_1, \ldots, w_s$ be s distinct nonnegative integers. If*

$$\sum_{i=1}^{s} 2^{w_i} = \sum_{j=1}^{t} 2^{x_j}$$

where $x_1, \ldots, x_t$ are nonnegative integers that are not necessarily distinct, then there is a partition of $\{1, 2, \ldots, t\}$ into s nonempty sets $J_1, \ldots, J_s$ such that

$$2^{w_i} = \sum_{j\in J_i} 2^{x_j}$$

for $i = 1, \ldots, s$.

Proof. By induction on t. Since the numbers w_i are distinct, it follows that, if $t = 1$, then $s = 1$ and $x_1 = w_1$, and so we can let $J_1 = \{1\}$.

Now assume that the lemma is true for all $t'<t$. Reorder the w_i and x_i so that $w_1<\ldots<w_s$ and $x_1\leqslant\ldots\leqslant x_t$. If $x_1<x_2$, then 2^{x_1} must occur in the 2-adic representation $\sum 2^{w_i}$, and so $w_1 = x_1$. Then $\sum_{i=2}^{s} 2^{w_i} = \sum_{j=2}^{t} 2^{x_j}$, and the result follows by induction.

Similarly, if $x_1 = x_2$, then

$$\sum_{i=1}^{s} 2^{w_i} = 2^{x_1+1}+\sum_{j=3}^{t} 2^{x_j}.$$

Since there are now only $t-1$ summands on the left side of the equation, the result again follows by induction. This proves the lemma.

LEMMA 3. *Let $h\geqslant 2$.*

(a) *If $1+2+2^2+\ldots+2^{h-1} = c_0+c_1+\ldots+c_{k-1}$, where $k\leqslant h$ and c_i*

$\in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\}$ *for* $i = 0, 1, \ldots, k-1$, *then* $k = h$ *and, after suitable rearrangement,* $c_i = 2^i$ *for* $i = 0, 1, \ldots, h-1$.

(b) *Let* $r \in \{0, 1, \ldots, h-1\}$. *If*

$$\sum_{\substack{i=0 \\ i \neq r}}^{h-1} 2^i = c_0 + c_1 + \ldots + c_{h-1}$$

and $c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\}$ *for* $i = 0, 1, \ldots, h-1$, *then, after suitable rearrangement, either* $c_r = 0$ *and* $c_i = 2^i$ *for* $i \neq r$, *or there is an integer* $s \neq 0, r$ *such that* $c_i = 2^i$ *for* $i \neq r, s$ *and* $c_r = c_s = 2^{s-1}$.

Proof. (a) This follows immediately from Lemma 2.

(b) By Lemma 2, there are at least $h-1$ nonzero terms c_i. If $c_i = 0$ for some i, then, after suitable rearrangement, $c_r = 0$ and $c_i = 2^i$ for $i \neq r$. If $c_i \neq 0$ for all i, then Lemma 2 implies that there is an $s \neq 0, r$ such that $c_i = 2^i$ for $i \neq r, s$ and $2^s = c_r + c_s = 2^j + 2^k$ for some j and k. This is possible only if $j = k = s-1$ and $c_r = c_s = 2^{s-1}$. This proves the lemma.

THEOREM 1. *Let* $h \geq 2$. *For* $i = 0, 1, \ldots, h-1$, *let* $W_i = \{n \in N \mid n \equiv i \pmod h\}$. *Let* $A = A(W_0) \cup \ldots \cup A(W_{h-1})$. *Then* A *is a thin, strongly minimal asymptotic basis of order* h.

Proof. Since the sets W_i partition N, Lemma 1 implies that A is an asymptotic basis of order h. Since $W_i(x) = (1/h)x + O(1)$ for $i = 0, 1, \ldots, h-1$, Lemma 1 also implies that there is a constant $c > 0$ such that $A(W_i)(x) < cx^{1/h}$ for all i and all x sufficiently large. Thus, $A(x) < hcx^{1/h}$ for large x, and A is a thin asymptotic basis of order h.

For $a \in A$, let $E_a = hA \setminus h(A \setminus \{a\})$. The set A is minimal if E_a is infinite for every $a \in A$, and strongly minimal if there is a real number $c = c(a) > 0$ such that $E_a(x) > cA(x)^{h-1}$ for all x sufficiently large. Since A is thin, it suffices to prove that there is a number $c = c(a) > 0$ such that $E_a(x) > cx^{(h-1)/h}$ for all x sufficiently large.

Let $a \in A$. Then $a \in A(W_u)$ for some u, and so a has a unique representation in the form

$$a = 2^{gh+u} + \sum_{f \in F_u} 2^{fh+u}$$

where $g \geq 0$ and F_u is a finite, possibly empty set of integers such that $f > g$ for all $f \in F_u$. Define $a_u = a$.

For $r = 0, 1, \ldots, h-1$ and $r \neq u$, let F_r be a finite, possibly empty set of integers such that $f > g$ for all $f \in F_r$. Define a_r by

$$a_r = \sum_{i=0}^{g} 2^{ih+r} + \sum_{f \in F_r} 2^{fh+r}. \tag{1}$$

Let $n = a_0 + a_1 + \ldots + a_{h-1}$. I claim that $n \notin h(A \setminus \{a\})$.

The proof is by induction on g. If $g = 0$, then $n = 1 + 2 + \ldots + 2^{h-1} + 2^h m$ for some $m \geq 0$. Let $n = b_0 + b_1 + \ldots + b_{h-1}$ be any representation of n as a sum of h elements of A. Define $c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\}$ by $c_i \equiv b_i \pmod{2^h}$. If $c_i \neq 0$, then $c_i = 2^{x(i)}$ for some $x(i) \in \{0, 1, \ldots, h-1\}$, and so $b_i \in A(W_{x(i)})$. It follows from Lemmas 2 and 3 that for some $k \leq h$ and $0 \leq t(0) < t(1) < \ldots < t(k-1) \leq h-1$ we have

$$1 + 2 + 2^2 + \ldots + 2^{h-1} = c_{t(0)} + \ldots + c_{t(k-1)} = 2^{x(t(0))} + \ldots + 2^{x(t(k-1))}.$$

Thus, $k = h$ and, after suitable renumbering, $t(i) = i = x(i)$ and $c_i = 2^i$ for $i = 0, 1, \ldots, h-1$. Then $b_i \in A(W_i)$ for $i = 0, 1, \ldots, h-1$. Since the representation of an integer as the sum of distinct powers of 2 is unique, it follows that $b_i = a_i$ for all i. In particular, $b_u = a_u = a$, and so $n \in E_a$.

Now assume that $g \geq 1$ and that the result holds for $g-1$. For $i = 0, 1, \ldots, h-1$, define a_i' by $a_u = 2^h a_u'$ and $a_r = 2^r + 2^h a_r'$ for $r \neq u$. Then

$$n = \sum_{\substack{r=0 \\ r \neq u}}^{h-1} 2^r + 2^h n'$$

where $n' = a_0' + \ldots + a_{h-1}' > 0$. Let $n = b_0 + \ldots + b_{h-1}$ be any representation of n as a sum of h elements of A. Define $c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\}$ by $c_i \equiv b_i \pmod{2^h}$. Then $b_i = c_i + 2^h b_i'$, where $b_i' \in A \cup \{0\}$ for $i = 0, 1, \ldots, h-1$. Just as in the case $g = 0$, Lemmas 2 and 3 imply that either

$$\sum_{\substack{r=0 \\ r \neq u}}^{h-1} 2^r = \sum_{i=0}^{h-1} c_i$$

or

$$\sum_{\substack{r=0 \\ r \neq u}}^{h-1} 2^r = \sum_{\substack{i=0 \\ i \neq s}}^{h-1} c_i$$

for some s. In the latter case,

$$2^h n' = c_s + 2^h(b_0' + \ldots + b_{h-1}').$$

Since $0 \leq c_s < 2^h$, it follows that $c_s = 0$ and so

$$\sum_{\substack{r=0 \\ r \neq u}}^{h-1} 2^r = \sum_{i=0}^{h-1} c_i.$$

In both cases, therefore, we can conclude that

$$n' = a_0' + \ldots + a_{h-1}' = b_0' + \ldots + b_{h-1}'.$$

Since

$$a_u' = 2^{-h} a = 2^{(g-1)h+u} + \sum_{f \in F_u} 2^{(f-1)h+u},$$

the induction hypothesis is satisfied, and, after suitable rearrangement, $b'_i = a'_i$ for all i. Then $b'_i \in A(W_i)$, and so $b_i \in A(W_i)$ for all i. Therefore, $a_i = b_i$ for all i. In particular, $b_u = a_u = a$. Thus, $n \in E_a$.

To find a lower bound for $E_a(x)$, choose an integer v such that $v > g$ and $v > f$ for all $f \in F_u$. Let $x > 2^{(v+1)h}$. Define $w \geq v$ by $2^{(w+1)h} \leq x < 2^{(w+2)h}$. For $r = 0, 1, \ldots, h-1$ and $r \neq u$, let F_r be any subset of $\{g+1, g+2, \ldots, w\}$. There are $2^{(w-g)(h-1)}$ choices of the $h-1$ subsets F_r. Define a_r by (1), and let $n = a_0 + \ldots + a_{h-1}$. Then

$$n \leq \sum_{r=0}^{h-1} \sum_{f=0}^{w} 2^{fh+r} = \sum_{i=0}^{wh+h-1} 2^i < 2^{(w+1)h} \leq x$$

and so n is counted in $E_a(x)$. Therefore,

$$E_a(x) \geq 2^{(w-g)(h-1)} > cx^{(h-1)/h}$$

where $c = 1/2^{(g+2)(h-1)}$. This completes the proof of the theorem.

Note that the thin, strongly minimal asymptotic basis A constructed in Theorem 1 is precisely the set $A^{(h)} \backslash \{0\}$, where $A^{(h)}$ was defined in the Introduction.

Because of the uniqueness of the representation of an integer as the sum of distinct powers of 2, it would be reasonable to conjecture that the asymptotic basis $A = \bigcup_{i=0}^{h-1} A(W_i)$ is minimal for any partition $N = W_0 \cup \ldots \cup W_{h-1}$. The following example shows that this is false even for $h = 2$.

THEOREM 2. *Let $N = V \cup W$, where $V \cap W \neq \emptyset$. Suppose that V contains no two consecutive integers, that $0 \in W$, and that $w_1, \ldots, w_s$ are distinct elements of W such that $w_i - 1 \in W$ for $i = 1, \ldots, s$. Define $a^* \in A(W)$ by $a^* = \sum_{i=1}^{s} 2^{w_i}$. Let $A = A(V) \cup A(W)$. Then $A \backslash \{a^*\}$ is an asymptotic basis of order 2. In particular, A is not a minimal asymptotic basis of order 2.*

Proof. It suffices to show that if $a \in A$ and $n = a^* + a$, then $n \in 2(A \backslash \{a^*\})$ for all but at most finitely many integers a.

There are two cases: Either $a \in A(V)$ or $a \in A(W)$. If $a \in A(V)$, then $a = \sum_{j=1}^{t} 2^{v_j}$, where $v_1 < \ldots < v_t$ and $v_j \in V$ for $j = 1, \ldots, t$. Since V contains no two consecutive integers, it follows that $v_j - 1 \in W$ for $j = 1, \ldots, t$. Therefore,

$$n = a^* + a = \sum_{i=1}^{s} 2^{w_i} + \sum_{j=1}^{t} 2^{v_j} = 2\Big(\sum_{i=1}^{s} 2^{w_i - 1} + \sum_{j=1}^{t} 2^{v_j - 1}\Big)$$
$$\in 2(A(W) \backslash \{a^*\}) \subseteq 2(A \backslash \{a^*\}).$$

If $a \in A(W)$, then $a = \sum_{j=1}^{t} 2^{z_j}$ where $z_1 < \ldots < z_t$ and $z_j \in W$ for $j = 1, \ldots, t$. Let us omit the finite number of integers a such that $\{z_1, \ldots, z_t\} \subseteq \{w_1, \ldots, w_s\}$. Then there is an exponent z_k such that $z_k \neq w_i$ for all $i = 1, \ldots, s$. Let $t \geq 2$. If $a \neq a^* + 2^{z_k}$, that is, if $\{z_1, \ldots, z_t\} \neq \{w_1, \ldots, w_s\} \cup \{z_k\}$, let $a_1 = a^* + 2^{z_k}$ and $a_2 = a - 2^{z_k}$. If $\{z_1, \ldots, z_t\} = \{w_1, \ldots, w_s\} \cup \{z_k\}$ and $z_k \neq w_1 - 1$, let $a_1 = a^* - 2^{w_1 - 1} = 2^{w_1 - 1} + \sum_{i=2}^{s} 2^{w_i}$ and let $a_2 = a + 2^{w_1 - 1}$. In both cases, $n = a_1 + a_2 \in 2(A(W) \backslash \{a^*\}) \subseteq 2(A \backslash \{a^*\})$. If $t = 1$ and $s \geq 2$, then we let $a_1 = a^* - 2^{w_s}$ and $a_2 = a + 2^{w_s}$. Then $n = a_1 + a_2 \in 2(A \backslash \{a^*\})$. If $s = t = 1$, then $a^* = 2^{w_1}$ and $a = 2^{z_1}$, where $z_1 \neq w_1$. If $z_1 \neq w_1 - 1$, let $a_1 = 2^{w_1 - 1}$ and $a_2 = 2^{z_1} + 2^{w_1 - 1}$. Then $n = a_1 + a_2 \in 2(A \backslash \{a^*\})$. This proves the theorem.

3. Open problems. The results above suggest several new problems.

1. Characterize the partitions $N = W_0 \cup \ldots \cup W_{h-1}$ such that $A = \bigcup_{i=0}^{h-1} A(W_i)$ is a minimal asymptotic basis of order h. Is A minimal for "almost all" partitions?

2. Let $N = W_0 \cup \ldots \cup W_{h-1}$ be a partition such that each set W_i is a union of intervals; that is, if $w \in W_i$, then W_i contains either $w-1$ or $w+1$ or both. Is $A = \bigcup_{i=0}^{h-1} A(W_i)$ minimal?

3. Under what conditions is an asymptotic basis A of the form $A = \bigcup_{i=0}^{h-1} A(W_i)$ strongly minimal?

4. Let θ satisfy $1/h < \theta \leq 1/(h-1)$. Does there exist a strongly minimal asymptotic basis A of order h such that $A(x) > cx^{\theta}$ for all x sufficiently large?

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