Minimal bases and powers of 2

by

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Dedicated to Paul Erdős on his 75th birthday

1. Introduction. Let \( A \) be a set of nonnegative integers. Let \( h \geq 2 \). Denote by \( hA \) the set of all numbers \( n \) of the form \( n = a_1 + a_2 + \ldots + a_h \), where \( a_1, a_2, \ldots, a_h \) are elements of \( A \) and are not necessarily distinct. The set \( A \) is an asymptotic basis of order \( h \) if \( hA \) contains all sufficiently large integers.

The counting function of the set \( A \) is the function \( A(x) \) defined as the number of positive elements of \( A \) not exceeding \( x \). An elementary combinatorial argument shows that if \( A \) is an asymptotic basis of order \( h \), then \( A(x) > c_1 x^{1/h} \) for some constant \( c_1 > 0 \) and all \( x \) sufficiently large. An asymptotic basis \( A \) of order \( h \) is called thin if there is a constant \( c_2 > 0 \) such that \( A(x) < c_2 x^{1/h} \) for all \( x \) sufficiently large.

Let \( A^{(1)} \) be the set of all nonnegative integers of the form \( \sum_{f \in F} 2^f \), where \( F \) is a finite set of nonnegative integers such that \( f = f' \pmod{h} \) for all \( f, f' \in F \). Choosing the empty set for \( F \) shows that \( 0 \in A^{(1)} \) for all \( h \geq 2 \). Raikov [6] and Stöhr [7] proved that \( A^{(1)} \) is a thin asymptotic basis of order \( h \).

An asymptotic basis \( A \) of order \( h \) is minimal if no proper subset of \( A \) is an asymptotic basis of order \( h \). This means that, for any \( a \in A \), the set \( E_a = hA \setminus (A \setminus \{a\}) \) is infinite. Härter [4] gave a nonconstructive proof of the existence of minimal asymptotic bases. Erdős [1] and Erdős and Härter [2] obtained other early results on minimal bases. Nathanson [5] proved that \( A^{(2)} \setminus \{0\} \) is a minimal asymptotic basis of order 2. Erdős and Nathanson [3] have recently published a survey of open problems on minimal bases in additive number theory.

Let \( A \) be an asymptotic basis of order \( h \). Let \( E_a(x) \) be the counting function of the set \( E_a \). If \( n \in E_a \) and \( n \leq x \), then every representation of \( n \) as a sum of \( h \) elements of \( A \) is of the form \( n = a_1 + a_2 + \ldots + a_h \), where \( a_i \in A \) and \( 0 \leq a_i \leq x \) for \( i = 1, \ldots, h-1 \). Since there are at most \( A(x)+1 \) choices for each \( a_i \), it follows that \( E_a(x) \leq (A(x)+1)^{h-1} \). Let us call an asymptotic basis \( A \) of order \( h \) strongly minimal if \( E_a(x) > c(A(x))^{h-1} \) for some constant \( c = c(a) > 0 \) and all \( x \) sufficiently large. Nathanson's proof [5] that \( A^{(2)} \setminus \{0\} \) is
minimal is, in fact, a proof that $A^{(2)} \setminus \{0\}$ is a strongly minimal asymptotic basis of order 2. Zollner [8] has obtained some results concerning the minimality of the sets $A^{(n)}$.

The object of this paper is to show that $A^{(h)} \setminus \{0\}$ is a strongly minimal asymptotic basis of order $h$ for every $h \geq 2$, and also to discuss an analogous class of asymptotic bases constructed from the set of powers of 2.

2. Results. Let $N$ denote the set of nonnegative integers, and let $W$ be a subset of $N$. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of $W$. Let $A(W)$ be the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathcal{F}^*(W)$. Note that $\emptyset \notin \mathcal{F}^*(W)$, hence $0 \notin A(W)$.

Lemma 1. (a) If $W_1$ and $W_2$ are disjoint subsets of $N$, then $A(W_1) \cap A(W_2) = \emptyset$.

(b) If $W \subseteq N$ and $W(\alpha) = \theta \alpha + O(1)$ for some $\theta \in (0, 1]$, then there exist positive constants $c_1$ and $c_2$ such that

$$c_1 x^\theta < A(W)(x) < c_2 x^\theta$$

for all $x$ sufficiently large.

(c) Let $N = W_0 \cup W_1 \cup \ldots \cup W_{h-1}$, where $W_i \neq \emptyset$ for $i = 0, 1, \ldots, h-1$. Then $A = A(W_0) \cup A(W_1) \cup \ldots \cup A(W_{h-1})$ is an asymptotic basis of order $h$. Indeed, $hA = \{n \in N \mid n \geq h\}$ and $hA(\cup \{0\}) = N$.

Proof. (a) This follows from the fact that every positive integer is uniquely the sum of distinct powers of 2.

(b) Let $x > 1$. Choose $k > 0$ such that $2^k < x < 2^{k+1}$. If $n \notin A(W)$ and $n \leq x$, then $n = \sum_{f \in F} 2^f$ for some $F \in \mathcal{F}^*(W)$, and $2^f \leq n \leq x$ implies that

$$0 \leq f \leq k$$

for every $f \in F$. Thus, $F$ is a nonempty subset of $\{0, 1, \ldots, k\} \cap W$. Since the cardinality of $\{0, 1, \ldots, k\} \cap W$ is at most $W(k)+1$, it follows that there are at most $2^{W(k)+1} - 1$ nonempty subsets of $\{0, 1, \ldots, k\} \cap W$. Therefore,

$$A(W)(x) \leq 2^{W(k)+1} - 1 < 2^{W(\log_2 x)+1} - 1 < c_2 x^\theta.$$

Similarly, let $F$ be a nonempty subset of $\{1, 2, \ldots, k-1\} \cap W$. Then

$$\sum_{f \in F} 2^f \leq \sum_{i=1}^{k-1} 2^i \leq 2^k \leq x$$

and so

$$A(x) \geq 2^{W(k-1)} - 1 = 2^{W(\log_2 x)+1} - 1 > c_1 x^\theta$$

for some $c_1 > 0$ and all $x$ sufficiently large.

(c) Let $n$ be a positive integer. Then $n = \sum_{f \in F} 2^f$, where $F \in \mathcal{F}^*(N)$.

Define $F_t = F \cap \bigcup_{i=0}^{t-1} W_i$ and $n_t = \sum_{i=0}^{t-1} 2^i$ for $i = 0, 1, \ldots, h-1$. Then $n_t \in A(W_t) \cup \{0\}$, and, if $F_t \neq \emptyset$, then $n_t \notin A(W_t)$. Thus, $n = n_0 + n_1 + \ldots + n_{h-1} \in h(A \cup \{0\})$. If $F \neq \emptyset$ for all $i$, then $n \in hA$.

Suppose that some of the sets $F_i$ are empty. If $|F| \geq h$, then it is possible to partition those sets $F_i$ that are nonempty so that $F$ is a union of $h$ nonempty sets $G_1, \ldots, G_h$, where each $G_j$ is a subset of some $F_i$. It follows that $n \in hA$.

If $|n| \leq 2^f$ and $1 \leq |F| \leq h-1$, then $n$ is a sum of $|F|$ elements of $A$. If $g \geq 1$ for some $g \in F$, then

$$n = \sum_{f \in F} 2^f = \sum_{f \in F} 2^f + 2^{g-1} = 2^{g-1}$$

is a sum of $|F|+1$ elements of $A$. We can continue in this way to divide powers of 2 until $n$ is a sum of exactly $h$ powers of 2, with repetitions allowed, or until $n$ is a sum of at most $h-1$ ones. Thus, $n \notin hA$ if and only if $n \leq h-1$, and so $hA = \{n \in N \mid n \geq h\}$. This proves the lemma.

Lemma 2. Let $w_1, \ldots, w_s$ be $s$ distinct nonnegative integers. If

$$\sum_{i=1}^s 2^{w_i} = \sum_{j=1}^t 2^{w_j}$$

where $w_1, \ldots, w_s$ are nonnegative integers that are not necessarily distinct, then there is a partition of $\{1, 2, \ldots, t\} \cup \{s+1, \ldots, t\}$ into $s$ nonempty sets $J_1, \ldots, J_s$ such that

$$\sum_{i \in J_i} 2^{w_i} = \sum_{j=1}^s 2^{w_j}$$

for $i = 1, \ldots, s$.

Proof. By induction on $t$. Since the numbers $w_i$ are distinct, it follows that, if $r = 1$, then $s = 1$ and $w_1 = w_1$, and so we can let $J_1 = \{1\}$.

Now assume that the lemma is true for all $t < t$. Reorder the $w_i$ and $x_i$ so that $w_1 < \ldots < w_s$, and $x_1 < \ldots < x_s$. If $x_1 < x_2$, then $2^s$ must occur in the 2-adic representation $\sum 2^{w_i}$, and so $w_1 = x_1$. Then $\sum_{i=1}^s 2^{w_i} = \sum_{j=1}^s 2^{w_j}$, and the result follows by induction.

Similarly, if $x_1 = x_2$, then

$$\sum_{i=1}^s 2^{w_i} = 2^{w_1+1} + \sum_{j=3}^s 2^{w_j}.$$

Since there are now only $t-1$ summations on the left side of the equation, the result again follows by induction. This proves the lemma.

Lemma 3. Let $h \geq 2$.

(a) If $1 + 2^2 + \ldots + 2^{k-1} = c_0 + c_1 + \ldots + c_{k-1}$, where $k \leq h$ and $c_i$
In \( \{0, 1, 2, 2^2, \ldots, 2^{h-1}\} \) for \( i = 0, 1, \ldots, k-1 \), then \( k = h \) and, after suitable rearrangement, \( c_i = 2^i \) for \( i = 0, 1, \ldots, h-1 \).

(b) Let \( r \in \{0, 1, \ldots, h-1\} \). If
\[
\sum_{i=0}^{h-1} 2^i = c_0 + c_1 + \ldots + c_{h-1}
\]
and \( c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\} \) for \( i = 0, 1, \ldots, h-1 \), then, after suitable rearrangement, either \( c_i = 0 \) and \( c_i = 2^i \) for \( i \neq r \), or there is an integer \( s \neq 0, r \) such that \( c_i = 2^i \) for \( i \neq r, s \) and \( c_s = 2^s \).

Proof. (a) This follows immediately from Lemma 2.

(b) By Lemma 2, there are at least \( h-1 \) nonzero terms \( c_i \). If \( c_i = 0 \) for some \( i \), then, after suitable rearrangement, \( c_i = 0 \) and \( c_i = 2^i \) for \( i \neq r \). If \( c_i \neq 0 \) for all \( i \), then Lemma 2 implies that there is an \( s \neq 0, r \) such that \( c_i = 2^i \) for \( i \neq r, s \) and \( c_s = 2^s \) for some \( j \) and \( k \). This is possible only if \( j = k = s-1 \) and \( c_s = 2^s \). This proves the lemma.

Theorem 1. Let \( h \geq 2 \). For \( i = 0, 1, \ldots, h-1 \), let \( W_i = \{n \in N \mid n \equiv i \pmod{h}\} \). Let \( A = A(W_0) \cup \ldots \cup A(W_{h-1}) \). Then \( A \) is a thin, strongly minimal asymptotic basis of order \( h \).

Proof. Since the sets \( W_i \) partition \( N \), Lemma 1 implies that \( A \) is an asymptotic basis of order \( h \). Since \( W_i(x) = (1/\theta)x + O(1) \) for \( i = 0, 1, \ldots, h-1 \), Lemma 1 also implies that there is a constant \( c > 0 \) such that \( A(W_i)(x) < cx^i \) for all \( i \) and \( x \) sufficiently large. Thus, \( A(x) < hnx^{1/n} \) for large \( x \), and \( A \) is a thin asymptotic basis of order \( h \).

For \( a \in A \), let \( E_a = hA \setminus hA(\{a\}) \). The set \( A \) is minimal if \( E_a \) is infinite for every \( a \in A \), and strongly minimal if there is a real number \( c = c(a) > 0 \) such that \( E_a(x) > cA(x)^{1/a} \) for all \( x \) sufficiently large. Since \( A \) is thin, it suffices to prove that there is a number \( c = c(a) > 0 \) such that \( E_a(x) > cA(x)^{1/h} \) for all \( x \) sufficiently large.

Let \( a \in A \). Then \( a \in A(W_i) \) for some \( i \), and so \( a \) has a unique representation in the form
\[
a = 2^{h+u} + \sum_{f \in F_u} 2^{f+h+u}
\]
where \( g \geq 0 \) and \( F_u \) is a finite, possibly empty set of integers such that \( f > g \) for all \( f \in F_u \). Define \( a_u = a \).

For \( r = 0, 1, \ldots, h-1 \) and \( r \neq u \), let \( F_r \) be a finite, possibly empty set of integers such that \( f > g \) for all \( f \in F_r \). Define \( a_r \) by
\[
a_r = \sum_{i=0}^{h-1} 2^{h+r} + \sum_{f \in F_r} 2^{f+h+r}
\]
Let \( n = a_0 + a_1 + \ldots + a_{h-1} \). I claim that \( n \notin h(A(\{a\})) \).

The proof is by induction on \( g \). If \( g = 0 \), then \( n = 1+2+\ldots+2^{h-1}+2^m \) for some \( m \geq 0 \). Let \( n = b_0 + b_1 + \ldots + b_{h-1} \) be any representation of \( n \) as a sum of \( h \) elements of \( A \). Define \( c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\} \) by \( c_i \equiv b_i \pmod{2^h} \). If \( c_i \neq 0 \), then \( c_i = 2^{k(i)} \) for some \( k(i) \in \{0, 1, \ldots, h-1\} \), and so \( b_i \in A(W_{k(i)}) \). It follows from Lemmas 2 and 3 that for some \( k \leq h \) and \( 0 \leq k(i) \leq k(i+1) \) and \( i < \ldots < i(k-1) < i(k) \leq h-1 \) we have
\[
1+2+2^2+\ldots+2^{h-1} = c_{i(0)} + \ldots + c_{i(k-1)} = 2^{k(i)} + \ldots + 2^{k(i+h-1)}
\]
Thus, \( k = h \) and, after suitable renumbering, \( t(k) = i(0) \) and \( c_i = 2^i \) for \( i = 0, 1, \ldots, h-1 \). Then \( b_i \in A(W_i) \) for \( i = 0, 1, \ldots, h-1 \). Since the representation of an integer as the sum of distinct powers of 2 is unique, it follows that \( b_i = a_i \) for all \( i \). In particular, \( b_r = a_r = a \), and so \( n \notin E_a \).

Now assume that \( g \geq 1 \) and that the result holds for \( g-1 \). For \( i = 0, 1, \ldots, h-1 \), define \( a'_i \) by \( a'_i = 2^i a'_i \) and \( a_r = 2^r + 2^r a_r \), for \( r \neq u \). Then
\[
n = \sum_{r=0}^{h-1} 2^{h+r} + \sum_{r=0}^{h-1} 2^{r+h'+r}
\]
where \( n' = a'_0 + \ldots + a'_{h-1} > 0 \). Let \( n = b_0 + \ldots + b_{h-1} \) be any representation of \( n \) as a sum of \( h \) elements of \( A \). Define \( c_i \in \{0, 1, 2, 2^2, \ldots, 2^{h-1}\} \) by \( c_i \equiv b_i \pmod{2^h} \). Then \( b_i \in A \cup \{0\} \) for \( i = 0, 1, \ldots, h-1 \). Just as in the case \( g = 0 \), Lemmas 2 and 3 imply that either
\[
\sum_{r=0}^{h-1} 2^{r+h'} = \sum_{r=0}^{h-1} 2^{r+h'}
\]
or
\[
\sum_{r=0}^{h-1} 2^{r+h'} = \sum_{r=0}^{h-1} 2^{r+h'}
\]
for some \( s \). In the latter case,
\[
2^{h'} = c_{s} + 2^{s} (b_0 + \ldots + b_{h-1})
\]
Since \( 0 < c_s < 2^h \), it follows that \( c_s = 0 \) and so
\[
\sum_{r=0}^{h-1} 2^{r+h'} = \sum_{r=0}^{h-1} 2^{r+h'}
\]
In both cases, therefore, we can conclude that
\[
n' = a'_0 + \ldots + a'_{h-1} = b_0 + \ldots + b_{h-1}
\]
Since
\[
a'_r = 2^{-h} a = 2^{h-1} a + \sum_{f \in F_r} 2^{f-1} a
\]
If $a \in A(W)$, then $a = \sum_{i=1}^{t} 2^{w_i} z_i$ where $z_1 < \ldots < z_t$ and $z_i \in W$ for $j = 1, \ldots, t$. Let us omit the finite number of integers $a$ such that $\{z_1, \ldots, z_t\} \subseteq \{w_i, \ldots, w_s\}$. Then there is an exponent $s$ such that $z_i \neq w_i$ for all $i = 1, \ldots, s$. Let $t \geq 2$. If $a = a^* + 2^{w_i}$, that is, if $\{z_1, \ldots, z_t\} \subseteq \{w_i, \ldots, w_s\} \cup \{z_1, \ldots, z_t\}$, let $a_1 = a^* + 2^{w_i}$ and $a_2 = a^* - 2^{w_i}$. If $\{z_1, \ldots, z_t\} = \{w_i, \ldots, w_s\} \cup \{z_1, \ldots, z_t\}$ and $z_i \neq w_i - 1$, let $a_1 = a^* - 2^{w_i-1} = 2^{w_i-1} + \sum_{i=1}^{t} 2^{w_i}$ and $a_2 = a + 2^{w_i-1}$. In both cases, $n = a_1 + a_2 = 2(A(W) \setminus \{a^*\}) \subseteq 2(A \setminus \{a^*\})$. If $t = 1$ and $s \geq 2$, then we let $a_1 = a - 2^{w_i}$ and $a_2 = a + 2^{w_i}$. Then $n = a_1 + a_2 = 2(A \setminus \{a^*\})$. If $s = t = 1$, then $a = 2^{w_i}$ and $a^* = 2^{w_i}$, where $z_i \neq w_i$. If $z_i \neq w_i - 1$, let $a_1 = 2^{w_i-1}$ and $a_2 = 2^{w_i-1}$. Then $n = a_1 + a_2 = 2(A \setminus \{a^*\})$. This proves the theorem.

3. Open problems. The results above suggest several new problems.

1. Characterize the partitions $N = W_0 \cup \ldots \cup W_{h-1}$ such that $A = \bigcup_{i=0}^{h-1} A(W_i)$ is a minimal asymptotic basis of order $h$. Is $A$ minimal for “almost all” partitions?

2. Let $N = W_0 \cup \ldots \cup W_{h-1}$ be a partition such that each set $W_i$ is a union of intervals; that is, if $w \in W_i$, then $W_i$ contains either $w - 1$ or $w + 1$ or both. Is $A = \bigcup_{i=0}^{h-1} A(W_i)$ minimal?

3. Under what conditions is an asymptotic basis $A$ of the form $A = \bigcup_{i=0}^{h-1} A(W_i)$ strongly minimal?

4. Let $0$ satisfy $1/h < 0 < 1/(h-1)$. Does there exist a strongly minimal asymptotic basis $A$ of order $h$ such that $A(x) > cx^\delta$ for all $x$ sufficiently large?

References:


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