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Effective oscillation theorems for a general class of real-valued remainder terms

by

SZILÁRD GY. RÉVÉSZ (Budapest)

Dedicated to Professor Pál Erdős on the occasion of his 75th birthday

1. Introduction.

1.1. Denoting complex variables by $s = \sigma + it$ ($\sigma = \operatorname{Re} s$), and writing $\zeta(s)$ for Riemann's zeta function, it is well known that if the remainder term of the prime number formula is given by

$$\Delta(x) = \Psi(x) - x = \sum_{n<x} \Lambda(n) - x, \qquad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^a, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

then its Dirichlet–Laplace transform is

$$\int_1^\infty x^{-s}\, d\Delta(x) = -\frac{\zeta'}{\zeta}(s) - \frac{s}{s-1}. \tag{1.2}$$

This function is meromorphically continued to $\sigma > 0$ and its poles are the nontrivial zeros of $\zeta(s)$. The connection between these zeros $\varrho = \beta + i\gamma$ and $\Delta(x)$ is even more clear in the formula

$$\Delta(x) = -\sum_{\substack{\varrho = \beta + i\gamma \\ |\gamma| < x}} \frac{x^\varrho}{\varrho} + O(\log^2 x). \tag{1.3}$$

This explains that the best estimates of $\Delta(x)$ from above depend on zero-free regions of $\zeta(s)$. E.g. Riemann's Conjecture (i.e. $\beta = 1/2$ for every ϱ) implies $\Delta(x) = O_\varepsilon(x^{1/2+\varepsilon})$. In turn, if we know or suppose the existence of a nontrivial zero $\varrho_0 = \beta_0 + i\gamma_0$, then $\Delta(x)$ must show some oscillatory phenomena. A detailed history of early results in this direction can be found in [2]. For its generality we quote only a theorem of Phragmen [3] stating for any $\varepsilon > 0$ in this special case that

$$\Delta(x) = \Omega(x^{\beta_0 - \varepsilon}). \tag{1.4}$$

This also gives the equivalence of Riemann's Conjecture with $\Delta(x) = O_\varepsilon(x^{1/2+\varepsilon})$.

Despite the results listed in [2], an irritating ineffectivity occurs in all these methods, and Littlewood's [7] problem of finding an effective form is much harder. P. Turán [21], [22], was able to prove the first effective result(¹)

$$(1.5)\qquad \max_{x\le X}|\Delta(x)| > X^{\beta_0}\exp\left(-C(\varrho_0)\frac{\log X\log_3 X}{\log_2 X}\right).$$

This breakthrough was reached by using Turán's power-sum method. Subsequent developments aimed at the sharpening of (1.5), localizing the large value of x, and proving large oscillations in both directions, i.e. $\Omega_\pm$ results. A glance at (1.4) shows that if the interference of the term x^{ϱ_0}/ϱ_0 with the contribution of the other terms is not too dominant, then we should have a one-sided oscillation (i.e. oscillation in both directions) as large as $x^{\beta_0}/|\varrho_0|$. J. Pintz built up a new method, which combined Turán's power-sum theory with the technique of kernel functions, and obtained the following very satisfactory result.

THEOREM (J. Pintz [8], [11]). *Let $0<\varepsilon<1$ and $\zeta(\varrho_0)=0$ with $\varrho_0 = \beta_0+i\gamma_0$. Then for any $Y>c_1(\varepsilon,\varrho_0)$, in the interval*

$$(1.6)\qquad I=[Y,\ Y^{6\log|\varrho_0|+60}]$$

there are some x and y for which

$$(1.7)\qquad \Delta(x) > \frac{(1-\varepsilon)x^{\beta_0}}{|\varrho_0|+4},\qquad \Delta(y) < \frac{-(1-\varepsilon)y^{\beta_0}}{|\varrho_0|+4}.$$

Now, in what follows we investigate the question of the size of the oscillation, and leave out of consideration the sign. To that we shall return later and here we shall refer to (1.7) as if it merely stated that

$$|\Delta(x)| > (1-\varepsilon)x^{\beta_0}/(|\varrho_0|+4).$$

1.2. Landau [6] proved, in the case of the prime ideal distribution, a sharpening of the quoted theorem of Phragmen. Effective generalization needs a careful application of the analytic method, in combination with explicit estimates in the theory of the Dedekind zeta function. Turán's pioneering work was extended to algebraic number fields by Staś [17], and even Pintz's method could be adapted (see [12], or in a stronger form [13], Corollary 1).

1.3. Since the old result of Phragmen, quite general oscillation theorems were proved for certain classes of functions. We mention the only if part of Wiener's [23] Tauberian theorem as an example. But if we want some effective generalizations of the number field case, we have to introduce some conditions using parameters corresponding to the parameters of the fields. In this line, W. Staś and K. Wiertelak [19] introduced a class $\mathscr{A}$ of number-theoretic functions, which contained the special cases of interest. A much wider class of complex-valued functions was introduced in [13]. Slightly changing the definition given there, we may take $\mathscr{C}$ to be the class of functions satisfying conditions I, II, III and IV of Section 2.1, but with $\boldsymbol{R}$ in condition I replaced by $\boldsymbol{C}$ (i.e. in (2.1), α is a complex measure). Then (2.6) defines the corresponding "remainder term". With this notation and the notation of (3.7) the following holds:

THEOREM ([13]). *Let the remainder of some function in $\mathscr{C}$ (defined in (2.6)) be denoted by $r(x)$, and let $0<\varepsilon<1$ be given. If $Z(\varrho_0)=0$ with $\varrho_0=\beta_0+i\gamma_0$, $\beta_0\geqslant 1/2$ and*

$$(1.8)\qquad Y\geqslant \max\{c_2,\ e^{|\varrho_0|},\ \exp(12/\varepsilon^2),\ K_1,\ K_4,\ e^{(K_2+1)^2},\ K_6\},$$

then in the interval

$$(1.9)\qquad [Y,\ Y^{c_3(\log|\varrho_0|+K_6)}],$$

there is an x for which

$$(1.10)\qquad |r(x)|\geqslant(1-\varepsilon)x^{\beta_0}/|\varrho_0|.$$

This theorem gives back as a special case the corresponding result for algebraic number fields, and also Pintz's theorem with some other constants (and without the prescribed sign).

1.4. As easy examples show, this theorem is optimal for $\mathscr{C}$; $1-\varepsilon$ in (1.10) cannot be $1+\varepsilon$ in general. However, in all the quoted special cases we have real-valued functions, and so instead of $\mathscr{C}$ we can work in $\mathscr{R}$. By the reflection principle $Z(\varrho_0)=0$ implies $Z(\overline{\varrho_0})=0$ if $f\in\mathscr{R}$, and so (1.3) suggests that a larger oscillation occurs, perhaps even $2x^{\beta_0}/|\varrho_0|$. This problem of improving his theorem by the use of the conjugate zero was proposed by J. Pintz. The present work gives a somewhat surprising answer to this question, since 1 is improved to $\pi/2$ in Theorem 1, but Example 1 shows that in the very general class $\mathscr{R}$ we cannot prove more. Then we define a much restricted class $\mathscr{P}$, with properties very similar to the case of the prime number theorem, and we show in Theorem 2 that even in $\mathscr{P}$ we cannot state more than in Theorem 1. This means that for any further improvement we have to get much more information about the location of the zeros of the Riemann zeta function, e.g. we should exclude the configuration given in the proof of Theorem 2. As for the proof, we emphasize the underlying extremal problem, which seems to be new and nontrivial. In fact, the method of its solution can be adapted to other problems concerning Fourier series.

On the other hand, our Theorem 1 is quite new in all the special cases, and gives an improved estimate of the oscillation of the remainder in the prime

(¹) $\log_2 x$ and $\log_3 x$ denote $\log\log x$ and $\log\log\log x$, respectively.

ideal theorem too. That is the content of Corollary 1. Corollary 2 specializes to the original case of $\Psi(x)$. Applying the results to the zeros having the least imaginary part, we get some effectively localized oscillatory estimates (without any assumption on the existence of some hypothetical ϱ_0). These estimates are not optimal, but their effectiveness makes them interesting, and so we state them as Corollaries 3 and 4.

1.5. Finally we state some effective estimate from below for the mean value of the remainder. Since here we have a much stronger localization, Theorem 3 yields at once a general, effective, and localized form of (1.5) with even a smaller factor in the denominator. Here the special case of Corollary 6 has earlier been worked out by J. Pintz [9]; the proof in the general case is very similar, so we omit it. In fact, sharper results for the important special case of $\Psi(x)$ have recently been obtained by Pintz [10].

1.6. My thanks go to J. Pintz, who turned my attention to the problem and discussed the matter with me several times during this work. I am even more in debt to G. Halász, who communicated me several ideas on the problem — in fact, I regard this as a joint work with him, though he chose not to be named as a co-author.

2. The definition of the function class $\mathscr{R}$.

2.1. Let α be any locally finite real (or complex) Borel measure on $(1, \infty)$. We use the term "distribution function" for

$$f(x) := \alpha\big((1, x)\big), \tag{2.1}$$

whatever α is. Since by (2.1) a distribution function is left continuous, we fix the convention that $\int_a^b$ means $\int_{[a,b)}$. Also, $\int_{(\sigma)}$ means upward integration on the vertical line $\{\sigma+it,\ t\in \boldsymbol{R}\}$. We introduce for any complex z

$$\operatorname{Log} z := \max\{\log|z|, 2\}. \tag{2.2}$$

We denote explicit numerical constants by $c_1, c_2, \ldots$ and use the O and $\ll$ symbols as a substitute for those. Any dependence on the parameters $K_1, \ldots, K_5$ will be handled explicitly, while other constants depending on them will be denoted by $K_6, \ldots$ and will be defined at proper places.

Definition. We say that $f\in\mathscr{R}$ if the following conditions hold:

I. $f(x)\colon [1, \infty)\to \boldsymbol{R}$ is a distribution function, and

$$|f(x)| < K_1 x \log^{K_2} x \qquad (K_1 \geqslant 1,\ K_2 \geqslant 0).$$

II. The Dirichlet–Laplace transform of f,

$$F(s) = \int_1^\infty x^{-s}\, df(x) = -\int_1^\infty f(x)\, dx^{-s} \qquad (\sigma > 1)$$

can be represented as

$$F(s) = -\frac{Z'}{Z}(s),$$

where Z is regular for $\sigma > 0$ except for a possible pole at $s = 1$.

III. For $0 < \sigma \leqslant 4$

$$|Z(s)(s-1)| \leqslant K_3(|t|+1)^{K_4} \qquad (K_4 \geqslant 0).$$

IV. For $-\infty < t < \infty$

$$|Z(2+it)| > K_5 \qquad (0 < K_5 \leqslant K_3).$$

2.2. Define the order of the possible pole of Z at $s = 1$ as

$$\varkappa = \varkappa(f) = \begin{cases} 1 & \text{if } Z \text{ has a pole at } s = 1, \\ 0 & \text{if } Z \text{ is regular at } s = 1. \end{cases} \tag{2.3}$$

We remark that if in III we suppose to have the estimate only for $Z(s)(s-1)^{K_0}$ where K_0 is another parameter, then every item of this paper remains valid with the corresponding modification of (2.3). We take $K_0 = 1$ here only for simplicity, all the applications fitting to this.

For nonnegative f we infer from a classical theorem of Landau [5] that $\varkappa \neq 0$, whence $\varkappa = 1$. By the well-known method of de la Vallée Poussin it can be proved that for nondecreasing f

$$Z(1+it) \neq 0 \qquad (t\in \boldsymbol{R}). \tag{2.4}$$

Then by the Wiener–Ikehara theorem [23] we are led to

$$f(x) \sim x \qquad (x\to\infty). \tag{2.5}$$

When f is not so, we do not have the corresponding assymptotics as a consequence of our assumptions, but nevertheless we can prove the same oscillation for the "remainder term"

$$r(x) := f(x) - \varkappa x. \tag{2.6}$$

2.3. We present here some lemmas from the theory of prime and prime ideal distribution. As forecasted, we prove that $\Psi\in\mathscr{R}$, $\Psi_{\boldsymbol{K}}\in\mathscr{R}$ with explicit parameters.

Let $\boldsymbol{K}$ be an algebraic number field, with degree n and discriminant Δ. P denotes any prime ideal of $\boldsymbol{K}$ and NP is its norm. We write

$$\Lambda_{\boldsymbol{K}}(m) := \sum_{\substack{P,k \\ NP^k = m}} \log NP = \sum_{\substack{P,k \\ NP^k = m}} \frac{\log m}{k}. \tag{2.7}$$

The remainder in the prime ideal theorem is defined by

$$\Delta_{\boldsymbol{K}}(x) := \Psi_{\boldsymbol{K}}(x) - x := \sum_{m<x} \Lambda_{\boldsymbol{K}}(m) - x. \tag{2.8}$$

For the Dedekind zeta function of K we have

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} \frac{\Lambda_K(m)}{m^s}. \tag{2.9}$$

LEMMA 1. *For $n \geqslant 2$ we have $\Psi_K \in \mathcal{R}$, $\varkappa(\Psi_K) = 1$ and we can take*

$$K_1 = n/\log 2, \quad K_2 = 2, \quad K_3 = c_4^n |\Delta|, \quad K_4 = n+1, \quad K_5 = (6/\pi^2)^n.$$

Proof. These values of K_1, K_2 and K_5 can be found in [17], where there are also estimates of K_3 and K_4 of a similar type. For exactly these values of K_3 and K_4 take $\varepsilon = 1/2$ in [24], Lemma 2.

LEMMA 2. *$\Psi \in \mathcal{R}$, $\varkappa(\Psi) = 1$ and we can take*

$$K_1 = 2, \quad K_2 = 0, \quad K_3 = c_5, \quad K_4 = 5/2, \quad K_5 = 6/\pi^2.$$

Proof. These values of K_3, K_4 and K_5 are well known, see [20]. $K_2 = 0$ since $\Psi(x) \sim x$. The best value of $K_1 = 1.03883\ldots$ was computed in [14]. Also, the trivial $K_1 = K_2 = 1$ would suffice.

Now, write for the enumerating function of the nontrivial zeros

$$N_K(T) := \sum_{\substack{\varrho = \beta + i\gamma \\ \zeta_K(\varrho) = 0 \\ |\gamma| < T}} 1. \tag{2.10}$$

Then the following explicit version of an ineffective theorem of Landau [4] holds.

LEMMA 3.

$$N_K(T) = \frac{n}{\pi} T \log T + \frac{\log|\Delta| - n - n\log 2\pi}{\pi} T + O(n\log T + \log|\Delta|). \tag{2.11}$$

The proof of an even more general fact can be found in [18]. In [12] hints for a more transparent proof were given.

We collect all the special information we need about the zeros as:

LEMMA 4. *For all K, if $\varrho = \beta + i\gamma$ is a zero of ζ_K, then so are $\bar{\varrho} = \beta - i\gamma$, $1/2 - \varrho = (1/2 - \beta) - i\gamma$ and $1/2 - \bar{\varrho} = (1/2 - \beta) + i\gamma$. For $K = Q$, the least zeros are $\varrho_0 = 1/2 + i\,14.13\ldots$ and its conjugate, and there exists a constant c_6 such that for any K*

$$N_K(c_6) > 0. \tag{2.12}$$

Proof. That the location of the zeros of ζ_K is symmetric about the line $\sigma = 1/2$ follows from the functional equation [4], and the symmetry about the real axis follows by the reflection principle. The first zero of the Riemann zeta function is well known [20], and the last statement evidently follows from Lemma 3.([2])

3. Lemmas.

3.1. The following propositions hold in view of definition (2.2).

PROPOSITION 1. *For $D \geqslant 1$, $0 < \alpha < 1$ and $x \geqslant 1$ we have*

$$\mathrm{Log}^D x \leqslant e^{(D/\alpha) + D^2} x^{\alpha}. \tag{3.1}$$

PROPOSITION 2. *For arbitrary complex numbers u and v we have*

$$\mathrm{Log}(u+v) \leqslant \mathrm{Log}(u) \cdot \mathrm{Log}(v). \tag{3.2}$$

The following easily computable formulas will be used.

PROPOSITION 3. *For any $B \geqslant 1/2$*

$$\int_B^{\infty} e^{-x^2}\,dx < e^{-B^2}. \tag{3.3}$$

PROPOSITION 4. *For $a > 0$, b complex and $c > 0$ we have*

$$\frac{1}{2\pi i} \int_{(c)} e^{as^2 + bs}\,ds = \frac{1}{2\sqrt{\pi a}} \exp\left(-\frac{b^2}{4a}\right). \tag{3.4}$$

PROPOSITION 5. *For any complex z and $0 \leqslant \alpha \leqslant 1$ we have*

$$|\alpha e^{(1-\alpha)z} + (1-\alpha) e^{-\alpha z}| \leqslant e^{|z|^2}. \tag{3.5}$$

Proof. For $|z| > 1$ this is trivial, and for $|z| \leqslant 1$ the Taylor expansion around 0 gives the estimate $1 + |z|^2$ for the left-hand side.

PROPOSITION 6. *For $A > 0$ and real B we have*

$$\int_{-\infty}^{\infty} |\cos(Ay+B)|\, e^{-y^2}\,dy \leqslant \frac{2}{\sqrt{\pi}} + \frac{2\pi}{A}.$$

Proof. Integration by parts gives

$$\int_0^{\infty} |\cos(Ay+B)|\, e^{-y^2}\,dy = \int_0^{\infty} \int_0^{y} |\cos(At+B)|\,dt\, 2ye^{-y^2}\,dy$$
$$\leqslant \int_0^{\infty} \left(\frac{2}{\pi} y + \frac{\pi}{A}\right) 2ye^{-y^2}\,dy = \frac{\pi}{A} + \frac{2}{\pi} \int_0^{\infty} 2y^2 e^{-y^2}\,dy$$
$$= \frac{\pi}{A} + \frac{2}{\pi} \int_0^{\infty} e^{-y^2}\,dy = \frac{\pi}{A} + \frac{1}{\sqrt{\pi}}.$$

([2]) Thanks to a remark of W. Narkiewicz, I learned that before the appearence of [18], the corollary $N_K(T+c_6) - N_K(T) > 0$ was known, even in a more general form (see [16]).

Estimating $\int\limits_{-\infty}^{0}$ in the same way, we get the proposition.

3.2. In the proofs we will use the power-sum method.

Lemma 5 (The continuous form of Cassels' power-sum theorem [1]). *For $h > 1$ and α_j arbitrary complex numbers ($j = 1, \ldots, n$) we have*

$$\max_{h \leqslant u \leqslant (2n-1)h} \left| e^{-\alpha_1 u} \sum_{j=1}^{n} e^{\alpha_j u} \right| \geqslant 1.$$

Lemma 6 (A modified form of Cassels' power-sum theorem). *If $w_1 = \ldots \ldots = w_k = 1$, $w_{k+2j} = \bar{w}_{k+2j-1}$ ($j = 1, \ldots, (n-k)/2$) are n complex numbers with $w_l = r_l e^{i\alpha_l}$ ($|\alpha_l| \leqslant \pi$, $l = 1, \ldots, n$), then*

$$\max_{a \leqslant v \leqslant (n-k+1)a} \operatorname{Re}\Big(\sum_{l=1}^{n} r_l^v e^{i\alpha_l v} \Big) \geqslant k.$$

Proof. In the proof of Cassels' power-sum theorem (see Cassels [1]) it is deduced from the Newton formulas that if $z_1, \ldots, z_\nu$ are the roots of the polynomial

$$z^\nu + a_1 z^{\nu-1} + \ldots + a_\nu$$

with real coefficients, and the power-sums are denoted by

$$s_\mu^{(\nu)} := z_1^\mu + \ldots + z_\nu^\mu,$$

then for these real numbers we have

$$\max_{1 \leqslant \mu \leqslant \nu+1} s_\mu^{(\nu)} \geqslant 0.$$

For arbitrary complex $z_1, \ldots, z_\nu$ we can take $z_{\nu+j} = \bar{z}_j$ ($j = 1, \ldots, \nu$) and the real polynomial

$$\prod_{j=1}^{2\nu} (z - z_j) = \prod_{j=1}^{\nu} (z - z_j)(z - \bar{z}_j)$$

to obtain for the corresponding power-sums

$$\max_{1 \leqslant \mu \leqslant 2\nu+1} s_\mu^{(2\nu)} \geqslant 0.$$

Applying this to $\nu = (n-k)/2$, $z_j = w_{k+2j}$, $z_{\nu+j} = w_{k+2j-1}$ ($j = 1, \ldots, \nu$), we deduce

$$\max_{1 \leqslant \mu \leqslant (n-k)+1} s_\mu^{(n-k)} \geqslant 0.$$

Since $w_j^\mu = 1$ for $j = 1, \ldots, k$ and all μ, this gives

$$\max_{1 \leqslant \mu \leqslant (n-k)+1} \Big(\sum_{l=1}^{n} w_l^\mu \Big) \geqslant k.$$

This is a discrete form of the statement to be proved. Since we have a real number under the maximum, we can take real parts, and deduce the continuous form stated here in the same way as it is done in [21] for the case of Lemma 5.

3.3. In this subsection we collect some lemmas which are natural extensions of known facts of the theory of the ζ and ζ_K functions. Special emphasis is on the explicit handling of parameters. In the following, Z always denotes a "zeta-type" function, that is, $Z(s)$ is an analytic function in $\sigma > 0$ with a possible pole of order $\varkappa$ at $s = 1$, and satisfying our conditions III and IV.

For the zeta-type function Z we fix some notation as follows. The complex number $\varrho = \beta + i\gamma$ represents a zero of Z. $\sum_{\varrho}$ is extended over all zeros satisfying the conditions indicated under the summation symbol and counted with multiplicities. For any real numbers T_1, T_2, T and $0 < a < 1$

$$N(a, T_1, T_2) := \sum_{\substack{\varrho \\ \beta \geqslant a \\ T_1 \leqslant \gamma \leqslant T_2}} 1, \quad N(a, T) := N(a, -T, T). \tag{3.6}$$

For easier writing put

$$K_6 := \log \frac{K_3}{K_5} \quad (\geqslant 0). \tag{3.7}$$

Lemma 7. *For $0 < a < 1$ and T real*

$$N(a, T-1, T+1) \ll \frac{1}{a}(K_4 \log T + K_6).$$

Lemma 8. *For $0 < a < 1$, $l \geqslant 1$ and any real T we have*

$$N(a, T-l, T+l) \ll \frac{l}{a}(K_4 \operatorname{Log}(|T|+l) + K_6), \tag{3.8}$$

and

$$N(a, T) \ll \frac{|T|+1}{a}(K_4 \operatorname{Log} T + K_6). \tag{3.9}$$

Lemma 9. *For $0 < a$, $2a \leqslant b < 1$ we have uniformly in $b \leqslant \sigma \leqslant 4$*

$$\frac{Z'}{Z}(s) = \sum_{\substack{\varrho \\ |\varrho - s| \leqslant b-a}} \frac{1}{s-\varrho} - \frac{\varkappa}{s-1} + O\left(\frac{K_4 \operatorname{Log} t + K_6}{a(b-a)} \right).$$

Lemma 10. *For $0 < b < 1$ and real T there exists a T' with $|T - T'| < 1$ such that we have uniformly in $b \leqslant \sigma \leqslant 4$*

$$\left| \frac{Z'}{Z}(\sigma + iT') \right| \ll \frac{1}{b^2}(K_4 \operatorname{Log} T + K_6)^2.$$

LEMMA 11. *Let $0 < b < 0.25$. There exists a broken line L, symmetric about the real axis, whose upper part L_+ is given by*

$$L_+ = \bigcup_{k=1}^{\infty} L_k, \qquad \begin{array}{ll} L_k = [\sigma_k + it_{k-1}, \sigma_k + it_{k+1}] & \text{for } k \text{ odd}, \\ L_k = [\sigma_{k-1} + it_k, \sigma_{k+1} + it_k] & \text{for } k \text{ even}, \end{array}$$

where $t_0 := 0$, $t_{k-1} < t_{k+1} < t_{k-1} + 2$ and $b \leqslant \sigma_k \leqslant 2b$, and for which the following holds: Uniformly for all $s \in L$ and all $s \in [b + it_k, 4 + it_k]$ $(k = 1, 2, \ldots)$,

$$\left|\frac{Z'}{Z}(s)\right| \ll \frac{1}{b^2}(K_4 \operatorname{Log} t + K_6)^2.$$

As for the proofs, Lemma 7 uses Jensen's inequality $2[1/a]$ times on the circles centered at $2 + i(T + ja)$ $(j = \pm 1, \ldots, \pm [1/a])$ and with radii $R = 2$, $r = \sqrt{(2-a)^2 + a^2}$; Lemma 8 follows from Lemma 7, and Lemma 9 uses (3.8) and the Borel–Carathéodory Lemma with $s_0 := 2 + it$, $R = 2 - a$, $r = 2 - b$, $F(s) = Z(s)(s-1)^{\varkappa}$. Finally, Lemma 10 and the construction of L in Lemma 11 can be done in view of Lemmas 7 and 9 by staying as far off the zeros of Z as possible. Compare [19].

4. Large oscillation of the remainder term.

4.1. THEOREM 1. *Let $f \in \mathscr{R}$ and let r be its remainder term as defined in* (2.6). *Let $Z(\varrho_0) = 0$ with $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqslant 0.5$ and $\gamma_0 > 0$.*
If $0 < \varepsilon < 1$ is arbitrary and

$$\log Y > \max\{c_7, |\varrho_0|, 100/(\varepsilon^2 \gamma_0^2), K_2^2, \log K_1, \log K_4, \log K_6\}, \tag{4.1}$$

then there exists an x in the interval

$$I = [Y, Y^{c_8(K_4 \operatorname{Log} \gamma_0 + K_6)}], \tag{4.2}$$

for which

$$|r(x)| > \left(\frac{\pi}{2} - \varepsilon\right)\frac{x^{\beta_0}}{|\varrho_0|}. \tag{4.3}$$

4.2. Proof. We introduce the parameters

$$\begin{gathered} \eta := \varepsilon/10, \quad m \geqslant \log Y, \\ M = 16m, \quad \mu = 12m, \\ A = e^{M-\mu} = e^{4m}, \quad B = e^{M+\mu} = e^{28m}, \end{gathered} \tag{4.4}$$

so that only m remains to be chosen. We postpone it but suppose that it will be done so as to fulfill

$$[A, B] \subset I. \tag{4.5}$$

Define

$$C := \sup_{x \in I} \frac{|r(x)|}{x^{\beta_0}}, \tag{4.6}$$

so that (4.5) implies

$$|r(x)| \leqslant Cx^{\beta_0} \quad \text{for} \quad x \in [A, B]. \tag{4.7}$$

Having the Dirichlet–Laplace transform of f defined and meromorphically continued according to II, we also define

$$R(s) = F(s) - \frac{\varkappa s}{s-1} = -\int_1^{\infty} r(x)\, dx^{-s}. \tag{4.8}$$

For some $w = u + iv$, $0.5 \leqslant u \leqslant 1$ put

$$\begin{aligned} U = U(w) :&= \frac{1}{2\pi i}\int_{(2)} R(s+w)\, e^{ms^2 + Ms}\, ds \\ &= \frac{1}{2\pi i}\int_{(2)} \left(-\int_1^{\infty} r(x)\frac{d}{dx}(x^{-s-w})\, dx\right) e^{ms^2 + Ms}\, ds \\ &= -\int_1^{\infty} r(x)\frac{d}{dx}\left\{x^{-w}\frac{1}{2\pi i}\int_{(2)} e^{ms^2 + (M - \log x)s}\, ds\right\} dx \\ &= -\int_1^{\infty} r(x)\frac{d}{dx}\left\{x^{-w}\frac{1}{2\sqrt{\pi m}}\exp\left(-\frac{(\log x - M)^2}{4m}\right)\right\} dx \\ &= \frac{1}{2\sqrt{\pi m}}\int_1^{\infty}\frac{r(x)}{x}x^{-w}\left\{\frac{\log x - M}{2m} + w\right\}\exp\left(-\frac{(\log x - M)^2}{4m}\right) dx, \end{aligned} \tag{4.9}$$

where we have interchanged the order of the integrations and the derivation and applied the integral formula of Proposition 4. Split up the last integral as

$$U_1 = \int_1^A, \quad U_2 = \int_A^B, \quad U_3 = \int_B^{\infty}. \tag{4.10}$$

Suppose that

$$4m \geqslant |w| \tag{4.11}$$

and choose $\alpha = 0.5$, $D = K_2 + 1$ in (3.1) to infer from I that

(4.12)

$$\begin{aligned} |U_1| &\leqslant \frac{1}{2\sqrt{\pi m}}\int_1^A (K_1 \operatorname{Log}^{K_2} x + \varkappa)\, x^{-u}\left\{\frac{M - \log x}{2m} + |w|\right\}\exp\left(-\frac{(\log x - M)^2}{4m}\right) dx \\ &\leqslant \frac{2K_1(4m)^{K_2} 5m}{2\sqrt{\pi m}}\int_1^{e^{4m}} \exp\left(-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2 - u\log x\right) dx \end{aligned}$$

$$\leqslant K_7 e^{M/2} \int_{-8\sqrt{m}}^{-6\sqrt{m}} e^{-y^2-u(2\sqrt{m}y+M)} e^{2\sqrt{m}y+M} dy$$

$$< K_7 \exp\big(M(1.5-u)+m(1-u)^2\big) \int_{6\sqrt{m}+(1-u)\sqrt{m}}^{\infty} e^{-t^2} dt$$

$$< K_7 \exp\big(M(1.5-u)-12m(1-u)-36m\big) \leqslant K_7 e^{-20m},$$

where we have substituted $y = (\log x - M)/(2\sqrt{m})$, $t = y-(1-u)\sqrt{m}$, and used Proposition 3, $u \geqslant 0.5$, (4.4), and the abbreviation

$$K_7 := K_1 e^{2K_2^2} \quad (\geqslant K_1 e^{2(K_2+1)+(K_2+1)^2}). \tag{4.13}$$

Since for $x > B$ by (4.11) and (4.4)

$$\frac{\log x - M}{2m} + |w| < \log x,$$

we infer similarly that

$$|U_3| \leqslant \frac{1}{2\sqrt{\pi m}} \int_B^\infty 2K_1 \operatorname{Log}^{K_2}(x) x^{-u} \log x \exp\left(-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2\right) dx \tag{4.14}$$

$$\leqslant \frac{K_7}{2\sqrt{m}} \int_{e^{28m}}^\infty \exp\left\{-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2 + (1.5-u)\log x\right\} \frac{1}{x} dx$$

$$= K_7 \int_{6\sqrt{m}}^\infty \exp\{-y^2+(1.5-u)(2\sqrt{m}y+M)\} dy$$

$$= K_7 \exp\big(M(1.5-u)+m(1.5-u)^2\big) \int_{6\sqrt{m}-(1.5-u)\sqrt{m}}^\infty e^{-t^2} dt$$

$$< K_7 \exp\big(M(1.5-u)+12(1.5-u)m-36m\big) \leqslant K_7 e^{-8m}.$$

These calculations prove that if (4.11) holds, then we have

$$|U_1+U_3| < c_9 K_7 e^{-8m}. \tag{4.15}$$

Now (4.1) and (4.4) show that for $w = \varrho_0$ or $w = \bar{\varrho}_0$ the inequality (4.11) is valid, and so if we define

$$S := U(\varrho_0) + U(\bar{\varrho}_0), \tag{4.16}$$

then (4.15) applies to both quantities on the right-hand side. Splitting up S

according to (4.10) we can write in view of (4.7) and (4.9)

(4.17) $|S_2|$

$$\leqslant \frac{1}{2\sqrt{\pi m}} \int_A^B \frac{|r(x)|}{x} \left\{ x^{-\varrho_0}\varrho_0 + x^{-\bar{\varrho}_0}\bar{\varrho}_0 + 2x^{-\beta_0}\frac{|\log x - M|}{2m} \right\} \times \exp\left(-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2\right) dx$$

$$\leqslant \frac{C}{\sqrt{\pi}} \int_A^B \frac{|x^{-i\gamma_0} e^{i\arg\varrho_0} + x^{i\gamma_0} e^{i\arg\bar{\varrho}_0}|\,|\varrho_0| + \dfrac{|\log x - M|}{m}}{x\, 2\sqrt{m}} \exp\left(-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2\right) dx$$

$$= \frac{C}{\sqrt{\pi}} \int_A^B \frac{2|\varrho_0|\,|\cos(\gamma_0 \log x - \varphi_0)| + \dfrac{|\log x - M|}{m}}{x\, 2\sqrt{m}} \exp\left(-\left(\frac{\log x - M}{2\sqrt{m}}\right)^2\right) dx$$

$$\leqslant \frac{C}{\sqrt{\pi}} \left\{ \int_{-\infty}^{\infty} 2|\varrho_0|\,|\cos(2\sqrt{m}\gamma_0 y - \varphi_0 + \gamma_0 M)|\, e^{-y^2} dy + \frac{2}{\sqrt{m}} \int_{-\infty}^{\infty} |y|\, e^{-y^2} dy \right\},$$

where we have denoted $\arg\varrho_0$ by φ_0. Now by Proposition 6 we compute

$$|S_2| \leqslant \frac{2C}{\sqrt{\pi}} |\varrho_0| \left(\frac{2}{\sqrt{\pi}} + \frac{2\pi}{2\sqrt{m}\gamma_0}\right) + \frac{2C}{\sqrt{\pi m}} \leqslant \frac{4}{\pi} |\varrho_0|\, C \left\{1 + \frac{\sqrt{\pi}(\pi+1)}{2\gamma_0\sqrt{m}}\right\}. \tag{4.18}$$

By (4.4), m is large enough to satisfy

$$\frac{\sqrt{\pi}(\pi+1)}{2\sqrt{m}\gamma_0} = \sqrt{\frac{\pi(\pi+1)^2}{\gamma_0^2 4m}} < \sqrt{\frac{\pi(\pi+1)^2}{4}\eta^2} < 4\eta. \tag{4.19}$$

Now (4.1), (4.5), (4.16), (4.18) and (4.19) yield

$$|S| \leqslant \frac{4}{\pi} |\varrho_0|\, C(1+4\eta) + \eta. \tag{4.20}$$

4.3. Now we calculate S by using the first form of U in (4.9). We fix $b = 1/16$ and for $0.5 \leqslant u \leqslant 1$ we transform the line of the integration in (4.9). By the residue theorem, Lemma 11, and the estimate $e^{ms^2+Ms} = O(e^{-t^2})$ $(0 < \sigma \leqslant 4)$, we get

$$U(w) = \frac{1}{2\pi i} \int_{L-w} R(s+w)\, e^{ms^2+Ms} ds + \sum_{\varrho}{}^{*} e^{m(\varrho-w)^2+M(\varrho-w)} \tag{4.21}$$

where * indicates that only the zeros to the right of the broken line L are to be considered.

By Lemma 11, the definition of R in (4.8) and condition III the integral can be estimated as follows. Let

$$K_8 = K_8(w) := K_4 \operatorname{Log} v + K_6. \tag{4.22}$$

Then we have, using also Proposition 2

$$\left|\frac{1}{2\pi i}\int_{L-w}\right| \tag{4.23}$$

$$\ll \int_{-\infty}^{\infty} (K_4 \operatorname{Log}(t+v)+K_6)^2 \exp\{m((u-b)^2-t^2)+M(2b-u)\}\,dt$$

$$< K_8^2 e^{-5m} \int_{-\infty}^{\infty} \operatorname{Log}^2 t \cdot e^{-mt^2}\,dt < K_8^2 e^{-5m}.$$

As for the sum, the contribution of those zeros which lie far from w can also be estimated via Lemma 7 and Proposition 2 as

$$\Big|\sum_{\substack{\varrho\\ |\varrho-w|\geq 4}}^{*} e^{m(\varrho-w)^2+M(\varrho-w)}\Big| \tag{4.24}$$

$$\ll \sum_{l=4}^{\infty} e^{m((1-u)^2-l^2)+M(1-u)}\{N(b, v-l-1, v-l)+N(b, v+l, v+l+1)\}$$

$$\ll e^{9m}\sum_{l=4}^{\infty} K_8 \log l \cdot e^{-ml^2} < K_8 e^{-7m}.$$

Now, what remains of U is a finite sum with not more than

$$N(b, v-4, v+4) \ll K_8 \tag{4.25}$$

terms. Moreover, if we write

$$U_4(w) := \sum_{\substack{\varrho\\ |v-\gamma|<4}}^{*} e^{m(\varrho-w)^2+M(\varrho-w)} = \sum_{\substack{\varrho\\ |v-\gamma|<4}}^{*} \{e^{(\varrho-w)^2+16(\varrho-w)}\}^m, \tag{4.26}$$

we see at once that this essential part of U is a pure power-sum.

Now returning to S as it was defined by (4.16), we obtain from (4.21), (4.23), (4.24), (4.26) and (4.1), (4.4)

$$|S| \geq |U_4(\varrho_0)+U_4(\bar{\varrho}_0)|-\eta. \tag{4.27}$$

Here the power-sum on the right has the form described in the hypothesis of Lemma 6, and by (4.25) for the number of terms we have

$$n = \sum_{\substack{\varrho\\ |\varrho-\varrho_0|<4}}^{*} 1 + \sum_{\substack{\varrho\\ |\varrho-\bar{\varrho}_0|<4}}^{*} 1 < c_{10} K_8(\varrho_0). \tag{4.28}$$

Since we have at least two terms in the power-sum equal to 1 (those for $\varrho = \varrho_0$ in $U(\varrho_0)$ and $\varrho = \bar{\varrho}_0$ in $U(\bar{\varrho}_0)$), Lemma 6 immediately implies that in any interval of the form

$$[H, c_{10} K_8 H] \tag{4.29}$$

there exists some m for which

$$|S| \geq 2-\eta. \tag{4.30}$$

Now let $H = \log Y$, from which $m \geq \log Y$ follows at once, and put $c_8 = 28c_{10}$ so that

$$B = e^{28m} \leq e^{28c_{10}K_8H} = Y^{c_8K_8},$$

which means that our choice of m fits (4.4) and (4.5) as well. Now from (4.20) and (4.30) we deduce

$$C \geq \frac{2-2\eta}{|\varrho_0|(1+4\eta)}\frac{\pi}{4} = \frac{1}{|\varrho_0|}\frac{\pi}{2}\frac{1-\eta}{1+4\eta} > \left(\frac{\pi}{2}-\varepsilon\right)\frac{1}{|\varrho_0|},$$

whence the theorem.

5. On the sharpness of Theorem 1. The accuracy of Theorem 1 is shown by the relation

$$\inf_{f\in\mathcal{R}} \inf_{\substack{\varrho_0\\ Z(\varrho_0)=0}} \sup_{x>1} \frac{|f(x)-\varkappa\cdot x|}{x^{\beta_0}/|\varrho_0|} = \frac{\pi}{2}. \tag{5.1}$$

To show (5.1) choose $0<\varepsilon<1$. If $\varrho = \beta+i\gamma$, $0<\beta<1$ and γ is large, then $\arg\varrho$ must be around $\pi/2$, and

$$\frac{x^\varrho}{\varrho}+\frac{x^{\bar{\varrho}}}{\bar{\varrho}} \approx \frac{x^\beta}{\gamma} 2\sin(\gamma\log x). \tag{5.2}$$

Now, put

$$\operatorname{sgn}^*(y) := \begin{cases} 1, & 2n\pi < y \leq (2n+1)\pi, \quad n\in \mathbf{Z},\\ -1, & (2n+1)\pi < y \leq (2n+2)\pi, \quad n\in\mathbf{Z}. \end{cases} \tag{5.3}$$

This function has the Fourier series

$$\sum_{k=0}^{\infty} \frac{4}{\pi}\cdot\frac{\sin(2k+1)y}{2k+1}, \tag{5.4}$$

so the following construction works.

EXAMPLE 1. Take some $0.5 \leq B \leq 1$ and define for the given ε, $\Gamma = \Gamma(\varepsilon) = 10/\varepsilon$. Let

$$f_0(x) := -\sum_{k=0}^{\infty}\left\{\frac{x^{\varrho_k}}{\varrho_k}+\frac{1}{\varrho_k}+\frac{x^{\bar{\varrho}_k}}{\bar{\varrho}_k}+\frac{1}{\bar{\varrho}_k}\right\}, \qquad \varrho_k := B+(2k+1)\Gamma i. \tag{5.5}$$

Then for the function

$$(5.6)\qquad f(x)=\begin{cases}\lim\limits_{y\to x-0} f_0(y), & x>1,\\ 0 & x=1,\end{cases}$$

we have $f\in\mathscr{R}$ with $\varkappa(f)=0$, i.e. $r(x)=f(x)-\varkappa\cdot x=f(x)$, and

$$(5.7)\qquad |r(x)|\leqslant\left(\frac{\pi}{2}+\varepsilon\right)\frac{x^B}{|\varrho_0|}.$$

We omit the somewhat lengthy computations of the verification, since we will prove even more later on, but we give some hints. Note that (5.6) is only to make f left continuous and $f(1)=0$, which is unimportant here, but belongs to the definitions of a distribution function, and so is necessary for $f\in\mathscr{R}$. We will have

$$(5.8)\qquad Z(s)=\prod_{k=0}^{\infty}\left(1-\frac{s}{\varrho_k}\right)\left(1-\frac{s}{\bar{\varrho}_k}\right)=\prod_{k=0}^{\infty}\left(1-\frac{B^2}{|\varrho_k|^2}\right)\operatorname{ch}\left(\frac{\pi(s-B)}{2}\right),$$

and with some continuous function $g(x)$, for which $|g(x)|\leqslant 1$ for all x,

$$(5.9)\qquad f(x)=-\frac{\pi x^B}{2\Gamma}\operatorname{sgn}^*(\Gamma\log x)-\frac{x^B\pi^2}{2\Gamma^2}g(x)+\sum_{k=0}^{\infty}\frac{2B}{|\varrho_k|^2}\qquad (x>1),$$

so (5.7) follows immediately, since $\varkappa(f)=0$ is obvious. For

$$(5.10)\qquad K_1=1,\quad K_2=0,\quad K_3=6,\quad K_4=1,\quad K_5=\pi/(4\Gamma)$$

conditions I–IV can be computed too.

Now let us analyse a little what that all means in the case of Chebyshev's function $\Psi(x)$. If Riemann's Conjecture is true, then Theorem 1 is not optimal, since there are results of Littlewood with estimates

$$\Omega(\sqrt{x}\log_3 x)$$

in this case (see [3]). Similarly, if

$$(5.11)\qquad \theta:=\sup\{\beta:\ \zeta(\varrho)=0,\ \varrho=\beta+i\gamma\}$$

is not attained, especially if $\theta=1$ and the Quasi Riemann Hypothesis is false, then we can choose some ϱ_1 with $\beta_1>\beta_0$ and apply Theorem 1 to this ϱ_1 to obtain the larger estimate

$$\Delta(x)=\Omega(x^{\beta_1}).$$

Thus for $\Psi(x)$ the optimality of Theorem 1 would imply the existence of some ϱ_0 with

$$(5.12)\qquad \varrho_0=\theta+i\gamma_0,\quad \gamma_0=\min\{\gamma>0:\ \zeta(\theta+i\gamma)=0\}$$

and also

$$(5.13)\qquad 1/2<\theta<1.$$

In this case, however, the previous counterexample does not work, since choosing $B=\theta$, we find that for the zeros of Z

$$(5.14)\qquad N(\theta,T)\sim\frac{1}{\Gamma}T,$$

though for ζ any density estimate excludes this in the case of (5.13).

Note that the asymptotical formula for the number of the zeros of ζ could be satisfied by a small modification of Example 1. Indeed, we can take some fairly arbitrary $\varrho=1/2+i\gamma$, with changing $f(x)$ at most by $O(x^{1/2+\eta})$ where $\eta<B-1/2$, and by means of these zeros we can ensure the asymptotical formula for $N(T)$. Also, to include a pole at $s=1$ (i.e. $\varkappa=1$) is no problem, and the symmetrizing of the zeros about $s=1/2$ can be done easily. Finally, to include the trivial zeros is easy, and since (5.8) is a regular function on the whole plane, these modifications will lead to a somewhat more sophisticated example, where the corresponding $Z(s)$ has the same analytic character as $\zeta(s)$ should have (we may even take the first "few" zeros on $\sigma=0.5$ to be equal to the known zeros of ζ, and choose $\Gamma>10^{10}$). Since the concrete values of the residues of $1/Z(s)$, or the functional equation characterizes the entire function $(s-1)\zeta(s)$, we cannot do more without explicitly determining the exact location of all the zeros. Since all these considerations show that the only problem regarding the analytic character of our example is that (5.13) and (5.14) are excluded by density theorems, we proceed further in this direction. Let us introduce the conditions:

V. For the zeros of $Z(s)$ with the notation (3.6)

$$N(a,T)\leqslant K_9 T^{\lambda(a)}$$

where for $a>1/2$

$$\lambda(a)<1.$$

VI. The total number of the zeros of $Z(s)$ with real part exceeding 1/2 is finite.

Now we may say that $f\in\mathscr{Q}$ if $f\in\mathscr{R}$ and satisfies V, and also $f\in\mathscr{P}$ if it satisfies VI too. By (5.12), (5.13) and VI it can be deduced (cf. Lemmas 7–10 and the standard proof of (1.3)) that for any $f\in\mathscr{Q}$ and $2\eta<\theta-1/2$

$$(5.15)\qquad r(x)=-\sum_{\substack{\varrho\\ \beta>\theta-2\eta\\ \gamma<x}}\frac{x^\varrho}{\varrho}+O_{f,\eta}(x^{\theta-\eta}),$$

where in these considerations we use the O and $\ll$ symbols with nonexplicit and nonabsolute constants too. Now defining θ in the analogy to (5.11) as

(5.16) $$\theta = \theta(f) := \sup\{\beta:\ \varrho = \beta + i\gamma,\ Z(\varrho) = 0\},$$

in case $\theta > 1/2$ we have, by V and (5.15),

(5.17) $$\Big|r(x) + \sum_{\substack{\beta=\theta\\ |\gamma|<T}} \frac{x^{\varrho}}{\varrho}\Big| \leqslant \frac{x^{\theta}}{T^{1-\lambda(\theta-2\eta)}}$$

for a fixed but large T. Choosing ϱ_0 with minimal imaginary part as in (5.12) we are led to

(5.18) $$\Big|r(x) + \sum_{\substack{\varrho=\theta+i\\ 0<\gamma<T(\varepsilon,f)}} \Big(\frac{x^{\varrho}}{\varrho} + \frac{x^{\bar\varrho}}{\bar\varrho}\Big)\Big| < \frac{\varepsilon}{|\varrho_0|} x^{\theta} \quad (x > x_0(f, \varepsilon)),$$

if we suppose $Z(0) \neq 0$. Now (5.18) points out that to find an f with little oscillation is not easier in $\mathscr{Q}$ than in $\mathscr{P}$! Moreover, by (5.2), at least for large γ_0, this is equivalent to finding an almost periodic polynomial

(5.19) $$P(y) = 2\sum_{k=0}^{N} \frac{\sin(\lambda_k y)}{\lambda_k} \quad (y = \gamma_0 \log x)$$

where $\lambda_0 = 1$, $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_N$ are reals and $\|P\|_\infty$ is small. This leads to the extremal problem of Section 6, but before solving it, we summarize the present considerations as

THEOREM 2. *Let $0 < \varepsilon < 1$ be given; then we can find an $f \in \mathscr{P}$ such that $\varkappa(f) = 0$, i.e. $f(x) = r(x)$ and*

(5.20) $$|r(x)| \leqslant \Big(\frac{\pi}{2} + \varepsilon\Big) \frac{x^{\beta_0}}{|\varrho_0|} \quad (x \geqslant 1)$$

for some nonreal zero $\varrho_0 = \beta_0 + i\gamma_0$ of $Z(s)$.

Proof. Let some almost periodic polynomial of the type (5.19) be given with

(5.21) $$\|P\|_\infty \leqslant \tfrac{1}{2}\pi + \eta \quad (\eta := \varepsilon/10).$$

This will be justified by the next section; we suppose the existence here. Now take some B with $1/2 \leqslant B \leqslant 1$ and also some $\Gamma > 10/\eta$. Define

(5.22) $$\varrho_k = B + i\lambda_k \Gamma \quad (k = 0, 1, \ldots, N),$$

and take the continuous function

(5.23) $$f(x) = -\sum_{k=0}^{N} \Big\{\frac{x^{\varrho_k}}{\varrho_k} + \frac{1}{\varrho_k} + \frac{x^{\bar\varrho_k}}{\bar\varrho_k} + \frac{1}{\bar\varrho_k}\Big\} \quad (x > 1),$$

extending it with $f(1) = 0$. The corresponding $Z(s)$ is

(5.24) $$Z(s) = \prod_{k=0}^{N} \Big(1 - \frac{s}{\varrho_k}\Big)\Big(1 - \frac{s}{\bar\varrho_k}\Big),$$

so we have at once $f \in \mathscr{R}$ (and so obviously $f \in \mathscr{P}$) with

(5.25) $$K_1 = 2N+2,\quad K_2 = 0,\quad K_3 = 1,\quad K_4 = 2N+3,\quad K_5 = 1/\Gamma^{N+1}.$$

Let us denote

(5.26) $$\varphi_k = \arg(\varrho_k).$$

Then by (5.22) and $\lambda_k \geqslant \lambda_1$, $1/2 \leqslant B \leqslant 1$ and $\Gamma > 10/\eta > 100$,

(5.27) $$\tfrac{1}{2}\pi \geqslant \varphi_k \geqslant \varphi_0 = \arcsin\frac{\Gamma}{\sqrt{\Gamma^2 + B^2}} \geqslant \arcsin\Big(1 - \frac{1}{2\Gamma^2}\Big),$$

and for any $\omega \in \mathbf{R}$

(5.28) $$|\cos(\tfrac{1}{2}\pi + \omega) - \cos(\varphi_k + \omega)| \leqslant \tfrac{1}{2}\pi - \varphi_0 \leqslant 2/\Gamma.$$

Now compute with $\vartheta_k = \vartheta_k(f, x)$ where $|\vartheta_k| \leqslant 1$ that

(5.29) $$\begin{aligned} -\Big(\frac{x^{\varrho_k}}{\varrho_k} + \frac{x^{\bar\varrho_k}}{\bar\varrho_k}\Big) &= \frac{-2x^B}{|\varrho_k|} \frac{x^{i\gamma_k} e^{-i\varphi_k} + \bar{x}^{i\gamma_k} e^{i\varphi_k}}{2} \\ &= \frac{-2x^B}{|\varrho_k|} \cos(\varphi_k - \gamma_k \log x) \\ &= \frac{2x^B}{|\varrho_k|} \sin(\gamma_k \log x) + \frac{4x^B \vartheta_k}{|\varrho_k|\, \Gamma}, \end{aligned}$$

so from (5.19), (5.22) and (5.23) with some $\vartheta = \vartheta(f, x)$, $|\vartheta| \leqslant 1$,

(5.30) $$f(x) = \sum_{k=0}^{N} \frac{-2B}{|\varrho_k|^2} + \frac{2x^B}{|\varrho_0|} \sum_{k=0}^{N} \frac{\sin(\lambda_k(\gamma_0 \log x))}{\lambda_k} + \frac{4x^B}{|\varrho_0|}\Big(\frac{N+1}{\Gamma^2} + \frac{N+1}{\Gamma}\Big)\vartheta.$$

Now by (5.19) and (5.21) for any $x > 1$ we obtain

$$\Big|\frac{f(x)}{x^B/|\varrho_0|}\Big| \leqslant \|P\|_\infty + \frac{4(\Gamma+1)N+1)}{\Gamma^2} + \frac{2(N+1)}{\Gamma x^B} < \frac{\pi}{2} + \varepsilon$$

if we choose $\Gamma > (N+1)/\eta$. This proves (5.20) since $\varkappa(f) = 0$.

We note that similarly to the remarks following formula (5.14), this example can be modified to have $\varkappa(f) = 1$ and the zeros of Z satisfying the known properties of $\zeta(s)$. This really means that proving more than Theorem 1 implies that certain configurations of zeros — which fit all the informations we have up till now — are impossible.

6. The extremal problem. Let N be the set of nonnegative integers (so including 0). Define

$$C(n) = \inf\left\{\sup_y |P(y)|:\ P(y) = \sum_{k=1}^{n} \frac{a_k \sin(\lambda_k y)}{\lambda_k},\ a_k \in N,\ \lambda_k > 0,\ \lambda_1 = a_1 = 1\right\},$$

$$(6.1)\qquad C'(n) = \inf\left\{\sup_y |P(y)|:\ P(y) = \sum_{k=1}^{n} \frac{a_k \sin ky}{k},\ a_1 = 1,\ a_k \in N\right\},$$

$$C^*(n) = \inf\left\{\sup_y |P(y)|:\ P(y) = \sum_{k=0}^{n} \frac{a_{2k+1} \sin((2k+1)y)}{2k+1},\ a_1 = 1,\ a_k \in N\right\}.$$

Since $a_k = 0$ is possible, these sequences are nonincreasing, and obviously

$$(6.2)\qquad C(2n+1) \leqslant C'(2n+1) \leqslant C^*(n).$$

We are interested in the determination of

$$(6.3)\qquad C := \lim_{n\to\infty} C(n),\qquad C' := \lim_{n\to\infty} C'(n),\qquad C^* := \lim_{n\to\infty} C^*(n).$$

Now, for any almost periodic polynomial in the definition of $C(n)$ write

$$(6.4)\qquad \|P\|_\infty \geqslant \frac{\int_0^T P(y)\sin y\,dy}{\int_0^T |\sin y|\,dy} = \frac{\frac{1}{2}a_1 T + O_P(1)}{(2/\pi)T + O(1)} = \frac{\pi}{4} + O_P\left(\frac{1}{T}\right),$$

which shows together with (6.2), (6.3) that

$$(6.5)\qquad C^* \geqslant C' \geqslant C \geqslant \pi/4.$$

The trivial estimate for C^* from above would be

$$(6.6)\qquad C^* \leqslant \left\|\sum_{k=0}^{n} \frac{\sin((2k+1)y)}{2k+1}\right\|_\infty \to \frac{1}{2}\int_0^\pi \frac{\sin x}{x}\,dx = 0.93\ldots,$$

which is strictly between $\pi/4$ and 1. This would be sufficient for proving that Theorem 1 cannot be improved to $2-\varepsilon$ in place of $\frac{1}{2}\pi-\varepsilon$, but we want more, therefore we must avoid the Gibbs phenomenon for the partial sums of (5.4). This is the essence of the difficulty in the proof (carried out below) that

$$(6.7)\qquad \pi/4 = C^* = C' = C.$$

For this we take some ε with $0 < \varepsilon < 1/2$ and write for short

$$(6.8)\qquad K := 2k+1,\quad N := 2n+2,\quad M := 2m+2,\quad m := [\varepsilon n],$$

where k denotes any integer between 0 and n, and n satisfies

$$(6.9)\qquad n > 1/\varepsilon^{10}.$$

We write

$$\Omega = \{0, 1\}^{n-m},\qquad \omega = (\omega_{2m+3}, \ldots, \omega_K, \ldots, \omega_{2n+1}) \in \Omega$$

and

$$(6.10)\qquad F_\omega(u) := \sum_{k=0}^{n} \omega_{2k+1} \frac{\sin(2k+1)u}{2k+1}\qquad (\omega_1 = \ldots = \omega_{2m+1} = 1).$$

Taking the Fejér polynomials of $\frac{1}{4}\pi\,\mathrm{sgn}^*(u)$,

$$(6.11)\qquad \sigma_n(u) := \sum_{k=0}^{n}\left(1 - \frac{2k+1}{2n+2}\right)\frac{\sin(2k+1)u}{2k+1} = \sum_{K<N}\left(1-\frac{K}{N}\right)\frac{\sin Ku}{K},$$

we know by the positivity of the Cesàro-1 summation method that this Fejér mean satisfies

$$(6.12)\qquad |\sigma_n(u)| \leqslant \pi/4\qquad (u \in \mathbf{R}).$$

Now it will suffice to find some $\omega \in \Omega$ for which

$$(6.13)\qquad \|F_\omega(u) - \sigma_n(u)\|_\infty < 3\varepsilon.$$

Trivially by the definition of m

$$(6.14)\qquad \left|\sum_{k=0}^{m} \omega_{2k+1} \frac{\sin(2k+1)u}{2k+1} - \sum_{k=0}^{m}\left(1 - \frac{2k+1}{2n+2}\right)\frac{\sin(2k+1)u}{2k+1}\right| \leqslant \frac{m+1}{N} < \varepsilon.$$

Write

$$(6.15)\qquad G_\omega(u) = \sum_{M<K<N} \omega_K \frac{\sin Ku}{K}.$$

To find a good $\omega \in \Omega$, we introduce a probability measure in Ω and prove that the probability of the good ω's is positive. More precisely, take

$$P(\omega) := \prod_{M<K<N} P_K(\omega_K),\qquad P_K(1) = 1 - \frac{K}{N},\qquad P_K(0) = \frac{K}{N},$$

and observe that P_K depends solely on ω_K and that the random variables

$$X_K\colon \Omega \to \{0, 1\},\qquad X_K(\omega) := \omega_K$$

are totally independent. In this setting (6.15) is also a random variable on Ω, and its expectation is

$$E(u) = \int_\Omega G_\omega(u)\,dP(\omega) = \sum_{M<K<N}\left(1-\frac{K}{N}\right)\frac{\sin Ku}{K}.$$

Take

$$(6.16)\qquad u_l = \frac{\pi}{2}\frac{l}{L}\quad (l = 0, 1, \ldots, L),\qquad L = \frac{M^2}{2}\quad (> 10N);$$

then for any fixed u, Bernstein's inequality gives for any $\lambda > 0$

$$P(G_\omega(u_l)-E(u_l)\geqslant\varepsilon)\leqslant e^{-\lambda\varepsilon}E(e^{\lambda(G_\omega(u_l)-E(u_l))})$$

$$=e^{-\lambda\varepsilon}\prod_{M<K<N}E\exp\Big(\lambda\Big(\Big(X_K(\omega)-\Big(1-\frac{K}{N}\Big)\Big)\frac{\sin Ku_l}{K}\Big)\Big)$$

$$=e^{-\lambda\varepsilon}\prod_{M<K<N}\Big\{\Big(1-\frac{K}{N}\Big)\exp\Big(\frac{K}{N}\frac{\lambda\sin Ku_l}{K}\Big)+\frac{K}{N}\exp\Big(-\Big(1-\frac{K}{N}\Big)\frac{\lambda\sin Ku_l}{K}\Big)\Big\},$$

and similarly

$$P(G_\omega(u_l)-E(u_l)\leqslant-\varepsilon)$$

$$\leqslant e^{-\lambda\varepsilon}\prod_{M<K<N}\Big\{\Big(1-\frac{K}{N}\Big)\exp\Big(-\frac{K}{N}\frac{\lambda\sin Ku_l}{K}\Big)+\frac{K}{N}\exp\Big(\Big(1-\frac{K}{N}\Big)\frac{\lambda\sin Ku_l}{K}\Big)\Big\}.$$

Applying Proposition 5 with $\alpha=\dfrac{K}{N}$ and $z=\pm\dfrac{\lambda\sin Ku_l}{K}$ in these inequalities, we obtain

$$P(|G_\omega(u_l)-E(u_l)|\geqslant\varepsilon)\leqslant 2e^{-\lambda\varepsilon}\exp\Big(\sum_{M<K<N}\frac{\lambda^2\sin^2 Ku_l}{K^2}\Big)<2\exp\Big(\frac{\lambda^2}{2M}-\lambda\varepsilon\Big),$$

so trivially

(6.17) $$P(|G_\omega(u_l)-E(u_l)|\leqslant\varepsilon,\ l=0,1,\ldots,L)\geqslant 1-2(L+1)\exp\Big(\frac{\lambda^2}{2M}-\lambda\varepsilon\Big).$$

Now take $\lambda=\varepsilon M$ in (6.17), and call an ω "good" if

(6.18) $$|G_\omega(u_l)-E(u_l)|\leqslant\varepsilon\quad(l=0,1,\ldots,L).$$

Hence (6.17) combined with the choice of λ, L in (6.16), and with (6.8), (6.9) yields

$$P(\omega\text{ good})\geqslant 1-M^2\exp\Big(\frac{\lambda^2}{2M}-\varepsilon\lambda\Big)\geqslant 1-\exp\Big(2\log M-\frac{\varepsilon^2}{2}M\Big)>0.$$

This ensures the existence of at least one $\omega\in\Omega$ satisfying (6.18). Since G_ω and E are sine polynomials and with odd multiples only, for every $y\in\mathbf{R}$ there is some $u\in\mathbf{R}$ for which

(6.19) $$|D(u)|\leqslant\varepsilon;\quad D(u):=G_\omega(u)-E(u),\quad |u-y|\leqslant\pi/(4L).$$

Now we can apply the following well-known inequality of Bernstein:

(6.20) $$\|D'\|_\infty\leqslant N\|D\|_\infty.$$

Combining (6.20) and (6.19) we get for any $y\in\mathbf{R}$

$$|D(y)|\leqslant|D(u)|+\|D'\|_\infty|u-y|<\varepsilon+\frac{\|D\|_\infty}{10},$$

that is,

$$\|D\|_\infty\leqslant\varepsilon+\frac{\|D\|_\infty}{10},$$

so

$$\|D(u)\|_\infty=\|G_\omega(u)-E(u)\|_\infty<2\varepsilon.$$

Now this and (6.14) give (6.13), whence (6.7) is proved.

7. Large oscillation of the remainder in mean. As we mentioned in Section 1.3, $\mathscr{C}$ is defined very similarly to $\mathscr{R}$, but omitting the condition that f is real-valued, and simply taking complex-valued functions. Just as the theorem from Section 1.3, the following theorem is also true for the class $\mathscr{C}$.

THEOREM 3. *Let* $f\in\mathscr{C}$, $Z(\varrho_0)=0$, $\varrho_0=\beta_0+i\gamma_0$, $\beta_0=0.5$ *and*

(7.1) $$\log y\geqslant\max\{c_{11},|\varrho_0|,\log^2 K_1,K_2^4,\log K_4,K_3/K_5\}.$$

Then

(7.2) $$S(y)\geqslant M(y)\geqslant\frac{1}{y}\int_{y\exp(-7\sqrt{\log y})}^{y}r(x)\,dx>y^{\beta_0}\exp(-c_{12}K_4\sqrt{\log y}\log_2^2 y).$$

Here we have used the abbreviations

(7.3) $$S(y):=\max_{x\leqslant y}|r(x)|,\qquad M(y):=(1/y)\int_1^y|r(x)|\,dx$$

for the supremum and the mean value of the remainder.

8. Number-theoretic corollaries.

COROLLARY 1. *Let* $0<\varepsilon<1$ *and suppose that* $\zeta_K(\varrho_0)=0$. *Then for*

$$\log Y>\max\{c_7,|\varrho_0|,100/(\varepsilon^2\gamma_0^2),c_{13}n,\log|\Delta|\}$$

there exists an x *in the interval*

$$[Y,Y^{c_{14}(n\operatorname{Log}\gamma_0+\log|\Delta|)}],$$

for which

$$|\Delta_K(x)|>\Big(\frac{\pi}{2}-\varepsilon\Big)\frac{x^{\beta_0}}{|\varrho_0|}.$$

COROLLARY 2. *Let* $0<\varepsilon<1$ *and suppose* $\zeta(\varrho_0)=0$. *For any* Y *with*

$$\log Y>\max\{c_{15},|\varrho_0|,1/\varepsilon^2\}$$

there exists an x *in the interval*

$$[Y,Y^{c_{14}\log|\varrho_0|}]$$

for which

$$|\Delta(x)| > \left(\frac{\pi}{2}-\varepsilon\right)\frac{x^{\beta_0}}{|\varrho_0|}.$$

COROLLARY 3. *Let* $0<\varepsilon<1$ *be arbitrary and let* $\boldsymbol{K}$ *be any algebraic number field. Then for any*

$$Y > \max\{|\Delta|,\, c_{15}^n,\, \exp(1/\varepsilon^2)\}$$

we have at least one x *in the interval*

$$[Y,\, Y^{c_{16}(n+\log|\Delta|)}]$$

satisfying

$$|\Delta_{\boldsymbol{K}}(x)| > c_{17}\sqrt{x}.$$

COROLLARY 4. *Let* $0<\varepsilon<1$ *be arbitrary and* $Y>\max\{c_{18},\exp(1/\varepsilon^2)\}$. *Then in every interval of the type* $[Y, Y^{c_{19}}]$ *there exists an* x *for which*

$$|\Delta(x)| > \frac{(\pi/2)-\varepsilon}{|\varrho_0|}\sqrt{x}$$

where ϱ_0 *is the first zeta-zero. In particular, taking* $\varepsilon=0.1$ *we get*

$$|\Delta(x)| > 0.1\sqrt{x}.$$

COROLLARY 5. *Let* $\zeta_{\boldsymbol{K}}(\varrho_0)=0$ *and suppose that*

$$\log y \geqslant \max\{|\varrho_0|,\, c_{20}^n,\, |\Delta|\}.$$

Then we have

$$\max_{1\leqslant x\leqslant y}|\Delta(x)| \geqslant \frac{1}{y}\int_{y\exp(-7\sqrt{\log y})}^{y}|\Delta(x)|\,dx > y^{\beta_0}\exp(-c_{21}n\sqrt{\log y}\log_2^2 y).$$

COROLLARY 6 (Pintz [9]). *Let* $\zeta(\varrho_0)=0$ *and take* $y\geqslant\max(e^{|\varrho_0|}, c_{22})$. *Then we have*

$$\max_{1\leqslant x\leqslant y}|\Delta(x)| \geqslant \frac{1}{y}\int_{y\exp(-7\sqrt{\log y})}^{y}|\Delta(x)|\,dx > y^{\beta_0}\exp(-c_{21}\sqrt{\log y}\log_2^2 y).$$

The proofs rely on the explicit calculations of Lemmas 1, 2, 3 and 4. Similarly to Corollaries 3 and 4, unconditional estimates are valid for the mean of the remainders — here we omit them.

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