Transformations that preserve uniform distribution

by

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Dedicated to Professor Pál Erdős on the occasion of his 75th birthday

The purpose of this paper is to describe some properties of functions that preserve uniformly distributed sequences of real numbers. Here we say that a map $T$ of the unit interval $I = (0, 1)$ to itself is a uniform distribution preserving (u.d.p.) transformation if \{\{T(x)\}_{x=1}^{\infty}\} is a uniformly distributed sequence (u.d.) sequence in $I$ for every u.d. sequence \{x_{n}\}_{n=1}^{\infty} \subseteq I$.

In the course of our discussion we shall see that the study of u.d.p. transformations leads to the opposite question to that investigated in the ergodic theory. Namely, given a measure $\mu$ (in our case this will be the Jordan measure), describe properties of transformations with respect to which $\mu$ is invariant. Perhaps our results may motivate other directions in the theory of dynamical systems, besides the study of properties of sets of points with periodical, recurrent, dense, etc. orbits to study, for instance, sets of points which orbits are uniformly distributed or to investigate sequences of integrals of iterations of transformations. Another direction is the study of the orbit behaviour of concrete points. It will be worth to answer these questions at least for piecewise linear transformations.

1. General criteria. From the well-known integral criterion ([4], p. 2) for u.d. sequences the following necessary and sufficient condition for a map of $I$ to be a u.d.p. transformation results immediately.

**Theorem 1.** A map $T: I \to I$ is a u.d.p. transformation if and only if for every Riemann-integrable function $g: I \to \mathbb{R}$ the composition $g \circ T$ is also Riemann-integrable and

$$
\int_0^1 g(x) \, dx = \int_0^1 g(T(x)) \, dx.
$$

**Proof.** Suppose that $T$ is a u.d.p. transformation and \{x_{n}\}_{n=1}^{\infty} is a u.d.
sequence. Then $\{T(x_n)\}_{n=1}^{\infty}$ is also a u.d. sequence and consequently
\[
\lim_{n \to \infty} \frac{g(x_1) + \ldots + g(x_n)}{n} = \frac{1}{0} \int g(x) \, dx = \lim_{n \to \infty} \frac{g(T(x_1)) + \ldots + g(T(x_n))}{n}
\]
for every Riemann-integrable function $g$ on $I$.

The existence of the limit on the right-hand side implies that $g \circ T$ is Riemann-integrable. In the opposite case there exists $([1]$ or $[2])$ a u.d. sequence $\{x_n\}_{n=1}^{\infty} \subset I$ for which the sequence
\[
\left\{ g \circ T(x_1) + \ldots + g \circ T(x_n) \right\}_{n=1}^{\infty}
\]
does not have a finite limit. The Riemann-integrability of $g \circ T$ implies
\[
\lim_{n \to \infty} \frac{g(T(x_1)) + \ldots + g(T(x_n))}{n} = \int_0^1 g(T(x)) \, dx
\]
and the necessary condition follows.

For the proof of the sufficient condition suppose on the contrary that $T$ is not a u.d.p. transformation. Then there exists a u.d. sequence $\{x_n\}_{n=1}^{\infty}$ for which the sequence $\{T(x_n)\}_{n=1}^{\infty}$ is not u.d. This means that there exist a Riemann-integrable function $h: I \to R$ for which the sequence
\begin{equation}
\left\{ h \circ T(x_1) + \ldots + h \circ T(x_n) \right\}_{n=1}^{\infty}
\end{equation}
does not converge to $\int_0^1 h(x) \, dx$. We can suppose that $h \circ T$ is Riemann-integrable. Then (2) necessarily converges to $\int_0^1 h \circ T(x) \, dx$ which contradicts (1) and the theorem is proved.

Theorem 1 implies that every u.d.p. transformation is Riemann-integrable. The function
\[
f(x) = \begin{cases} x & \text{if } x \text{ is a rational number}, \\
0 & \text{otherwise}
\end{cases}
\]
shows that there are functions not integrable in the Riemann sense which are not u.d.p. transformations though they transform infinitely many u.d. sequences into u.d. ones.

By Theorem 1 every composition $g \circ T$ of a u.d.p. transformation and a Riemann-integrable function $g: I \to I$ is again Riemann-integrable. Note that this is a restriction, because every Lebesgue-measurable function is expressible as a composition of two Riemann-integrable functions ([6], [7]).

The well-known approximation technique enables us to replace the Riemann-integrable functions in the integrable criterion for the u.d. sequences used above by continuous functions. More generally, instead of the system of the continuous functions one can take any system of functions which linear hull is dense in the system of all continuous functions. The systems of functions $\{x^n\}_{n=1}^{\infty}$ and $\{x^{2n}\}_{n=1}^{\infty}$ lead to the following result (the verification of the composition property in Theorem 1 is trivial):

**Theorem 2.** A Riemann-integrable function $T: I \to I$ is a u.d.p. transformation if and only if one of the following conditions is satisfied:

1. $\int_0^1 g(x) \, dx = \int_0^1 g(T(x)) \, dx$ for every continuous function $g: I \to R$,
2. $\int_0^1 T^n(x) \, dx = 1/(n+1)$ for every $n = 1, 2, \ldots$,
3. $\int_0^1 x^{2n} \, dx = 0$ for every $n = \pm 1, \pm 2, \ldots$

The next theorem shows that for a Riemann-integrable function $I$ to itself the question whether it is a u.d.p. transformation can be decided using only one suitable sequence. For the formulation of the result we shall need the following notion:

Let $\{N_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. A sequence $\{x_n\}_{n=1}^{\infty} \subset I$ is called $\{N_k\}$-uniformly distributed if
\[
\lim_{k \to \infty} \frac{A(\{0, x\}, N_k; \{x_n\})}{N_k} = x \quad \text{for } 0 \leq x \leq 1,
\]
where $A(E, N, \{x_n\})$ denotes the number of terms $x_n$, $1 \leq n \leq N$ for which $x_n \in E$. Such sequences are usually called almost u.d. sequences ([4], p. 53). The reason for our terminology is that we shall need to stress the role of the sequence $\{N_k\}$ in this section.

**Theorem 3.** Let $T: I \to I$ be Riemann-integrable. Then $T$ is a u.d.p. transformation if and only if there exists an increasing sequence of positive integers $\{N_k\}_{k=1}^{\infty}$ and an $\{N_k\}$-uniformly distributed sequence $\{x_n\}_{n=1}^{\infty} \subset I$ for which the sequence $\{T(x_n)\}_{n=1}^{\infty}$ is also $\{N_k\}$-uniformly distributed.

Proof. The necessity is obvious. For the sufficiency suppose that both $\{x_n\}_{n=1}^{\infty}$ and $\{T(x_n)\}_{n=1}^{\infty}$ are $\{N_k\}$-uniformly distributed. Then we have
\[
\lim_{k \to \infty} \frac{T^n(x_1) + T^n(x_2) + \ldots + T^n(x_N)}{N_k} = \frac{1}{0} \int T^n(x) \, dx \quad \text{for } m = 1, 2, \ldots
\]
However the left-hand side can be in turn written in the form
\[
\lim_{k \to \infty} \frac{(T(x_1))^m + (T(x_2))^m + \ldots + (T(x_N))^m}{N_k} = \frac{1}{0} \int x^m \, dx = 1/(m+1)
\]
and Theorem 2(b) finishes the proof.
The affinity between the Riemann integrability and the Jordan measurability leads to the next result.

**Theorem 4.** A map \( T : I \to I \) is a u.d.p. transformation if and only if 
(a) \( T \) is measurable in the Jordan sense, 
(b) \( |T^{-1}(I)| = |I| \) for every interval \( I \subset I \) \(||E|\) denotes the Jordan or Lebesgue measure of \( E \).

**Proof.** First of all we have 

\[
A(I_1, \{x_n\}) = A(T^{-1}(I_1), N, \{x_n\})
\]

for every sequence \( \{x_n\}_{n=1}^{\infty} \subset I \).

Suppose (a) and (b) are true. Then (a) implies that \( T^{-1}(I_1) \) is measurable in the Jordan sense. Thus for every u.d. sequence \( \{x_n\}_{n=1}^{\infty} \) we have 

\[
\lim_{N \to \infty} \frac{A(T^{-1}(I_1), N, \{x_n\})}{N} = |T^{-1}(I_1)|.
\]

Then (3) and (b) imply that 

\[
\lim_{N \to \infty} \frac{A(I_1, \{x_n\})}{N} = |I_1|
\]

which means that \( \{T(x_n)\}_{n=1}^{\infty} \) is a u.d. sequence.

Conversely, let \( T \) be a u.d.p. transformation and \( I_1 \) a subinterval of \( I \). If \( T^{-1}(I_1) \) is Jordan measurable then (3) implies (b). Suppose therefore that \( T^{-1}(I_1) \) is not Jordan measurable. Then the indicator \( \chi \) of \( T^{-1}(I_1) \) is not Riemann integrable and thus there exists \( \{x_n\}_{n=1}^{\infty} \) a u.d. sequence, say, \( \{x_n\}_{n=1}^{\infty} \) for which the sequence 

\[
\left\{ \frac{1}{N} \sum_{n \leq N} \chi(x_n) \right\}_{N=1}^{\infty}
\]

does not have a finite limit. On the other hand,

\[
\sum_{n \leq N} \chi(x_n) = A(T^{-1}(I_1), N, \{x_n\})
\]

and (3) leads to a contradiction that \( T \) is a u.d.p. transformation.

**2. Miscellanea.** In this section we shall present several simple properties of u.d.p. transformations which may be of some interest. The proof of the first proposition is straightforward.

**Proposition 1.** Let \( T \) and \( G \) be u.d.p. transformations and \( x \) a real number. Then \( T \circ G, 1 - T \) and the fractional part \( \{T+x\} \) are also u.d.p. transformations.

**Proposition 2.** Let \( \{T_n\}_{n=1}^{\infty} \) be a sequence of u.d.p. transformations uniformly converging to \( H \). Then \( H \) is a u.d.p. transformation.

**Proof.** If \( T_n \to H \) then also \( T_k \to H \) for every exponent \( k = 1, 2, \ldots \), which implies that

\[
\lim_{n \to \infty} \frac{1}{N} \int T_n(x) \, dx = \frac{1}{N} \int H(x) \, dx.
\]

Owing to Theorem 2(b) every integral on the left-hand side is equal to \( 1/(k+1) \). Consequently the same is true for the right-hand side and the same theorem finishes the proof.

**Proposition 3.** Let \( T : I \to I \) be a u.d.p. transformation and let at least one of the following conditions be satisfied:

(a) \( T \) is monotone,

(b) \( T \) has the derivative at each point of \( I \),

(c) \( T \) has the Darboux property (i.e. every interval is mapped onto an interval) and \( T \) is injective,

(d) \( T \) is continuous and either \( T(x) \leq x \) for each \( x \in I \) or \( T(x) \geq 1 - x \) for each \( x \in I \).

Then either \( T(x) = x \) for each \( x \in I \) or \( T(x) = 1 - x \) for each \( x \in I \).

**Proof.** (a) If \( T \) is a u.d.p. transformation and monotone then \( T \) is necessarily continuous (no jumps are possible). This means that for each subinterval \( I_1 \) of \( I \) the set \( I_2 = T^{-1}(I_1) \) is also an interval. Then \( T_1 = T(I_1) \) and by Theorem 4 \(|T^{-1}(I_1)| = |I_1| \). This implies that \( |T_2| = |T(I_2)| \), i.e.

\[
\left| \frac{T(I_2)}{|I_2|} \right| = 1.
\]

This gives in turn that the derivative of \( T \) in every point of \( I \) equals 1 or -1 and the conclusion follows.

(b) We reduce this case to (a) showing that the derivative \( T' \) of \( T \) has no sign changes. The opposite case would lead to a point \( x \in I \) with \( T'(x) = 0 \) and this in turn to a sequence \( \{I_n\}_{n=1}^{\infty} \) of subintervals of \( I \) such that

\[
0 < |I_n| \quad \lim_{n \to \infty} \frac{|I_n|}{|T(I_n)|} = 1.
\]

But then

\[
|T^{-1}(T(I_n))| \geq \frac{|I_n|}{1 + \frac{1}{n}} = |I_n|.
\]

for \( n > n_0(e) \) and an arbitrarily small \( e > 0 \). The contradiction with Theorem 4 finishes the proof of (b).

(c) If \( T \) is injective and \( I_1 \) any subinterval of \( I \) then

\[
A(I_1, \{x_n\}) = A(T(I_1), N, \{T(x_n)\})
\]

If \( T \) possesses the Darboux property then \( T(I_1) \) is a subinterval of \( I \) and for a
The following observation forms the background for the next proposition: If a u.d.p. transformation $T: I 	o I$ is differentiable at a point $t \in I$, and if $T$ is continuous in a neighbourhood of this point then $|T'(t)| \geq 1$.

The opposite inequality would merely imply the existence of a closed neighbourhood, say, $U$ of $t$ on which $T$ can be supposed to be continuous and where

$$|T(x) - T(t)| < |x - t|$$

for all $x \in U$.

Let $x_1 \neq x_2$ be elements of $U$ such that $T(x_1) = \max_{y \in U} T(y)$ and $T(x_2) = \min_{y \in U} T(y)$. If $t$ lies between $x_1$ and $x_2$ then

$$|T(x_1) - T(x_2)| < |x_1 - x_2|.$$ 

But this yields a contradiction with Theorem 4 for $I_1$ defined by endpoints $T(x_1)$ and $T(x_2)$ because then

$$|T^{-1}(I_1)| \geq |x_1 - x_2| > |T(x_1) - T(x_2)| = |I_1|.$$ 

A similar contradiction can be obtained for $t$'s outside the closed interval determined by $x_1$ and $x_2$.

We have more generally:

**Proposition 6.** Let $T: I \to I$ be piecewise differentiable. Then $T$ is a u.d.p. transformation if and only if

$$\sum_{x \in T^{-1}(y)} \frac{1}{|T'(x)|} = 1$$

for all but a finite number of points $y \in I$.

**Proof.** Suppose that $T$ is piecewise differentiable. If $T$ is a u.d.p. transformation or if (4) is true then $T$ is a piecewise continuous strictly monotone and surjective map. To such a map one can always find two systems of disjoint open intervals (similarly as for piecewise linear maps in the next section)

$$\{ J_1 = (y_{j-1}, y_j) : j = 1, 2, \ldots, n_j \},$$

$$\{ I_{j,d} : i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, n \}$$

with

$$T^{-1}(J_j) = \bigcup_{i=1}^{n_j} I_{j,d}, \quad T(I_{j,d}) = J_j$$

and such that the contraction $T/I_{j,d}$ is continuous, strictly monotone and

$$\sum_{j=1}^{n} |J_j| = 1, \quad \sum_{j=1}^{n} \sum_{i=1}^{n_j} |I_{j,d}| = 1$$

for all $i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, n$. 

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\(^{(*)}\) We are indebted to M. Paštěka for calling our attention to this result.
Further let $G_{J_j} : J_j \to I_{J_t}$ be the inverse map to $T/I_{J_t}$ for $i = 1, \ldots, n_j, j = 1, \ldots, n$. According to Theorem 4 transformation $T$ is a u.d.p. one if and only if we have

$$y - y_{j-1} = \sum_{i=1}^{n_j} |G_{J_j}(y) - G_{J_j}(y_{j-1})|$$

for every $y \in J_j$ and every $j = 1, \ldots, n$. Then the differentiation of (5) gives

$$1 = \sum_{i=1}^{n_j} |G_{J_j}(y)|$$

what is equivalent to (4).

Now suppose that (4), and consequently also (6), is true. Then ([8], p. 199)

$$y - y_{j-1} = \int_{y_{j-1}}^{y} 1 \cdot dy = \sum_{i=1}^{n_j} \int_{y_{j-1}}^{y} |G_{J_j}(y)| dy \leq \sum_{i=1}^{n_j} |G_{J_j}(y) - G_{J_j}(y_{j-1})|.$$  

This implies that

$$|J_j| \leq \sum_{i=1}^{n_j} |I_{J_i}|$$

for all $k = 1, \ldots, n$.

On the other hand if the inequality in (7) would be strict for some $y \in J_j$ then we obtain

$$|J_j| < \sum_{i=1}^{n_j} |I_{J_i}|$$

for this $j$. But this together contradicts the fact that the open intervals $J_j$ for $j = 1, \ldots, n$ form a disjoint decomposition of $I$. Thus (4) implies (5) and the theorem is proved.

Proposition 6 can be employed for construction of nontrivial u.d.p. transformation in the following manner. On some subintervals of $I$ we can choose $T$ arbitrarily but with sufficiently large derivatives in magnitude. On the remaining subintervals of $I$ we complete $T$ in such a way that (4) or (5) is satisfied. For instance, let $I_1 = \langle 0, 1/2 \rangle$, $I_2 = \langle 1/2, 1 \rangle$. Let $T|I_1$ be strictly increasing, $T|I_2$ strictly decreasing with $T(0) = 0$, $T(1/2) = 1$, $T(1) = 0$ and let $G_1, G_2$ be the corresponding inverse mappings. According to (5) $T$ is a u.d.p. transformation if and only if

$$y - 0 = |G_1(y) - G_1(0)| + |G_2(y) - G_2(0)| = G_1(y) - G_2(y) + 1.$$

Now if $G_1(y)$ will be increasing with $0 < G_1(y) < 1$ for $y \in (0, 1)$ then

$$G_2(y) = G_1(y) + y - 1$$

is the required complement. Thus for instance, for $G_1(y) = y^2/2$ we obtain $G_2(y) = y^2/2 - y + 1$ and therefore $T$ given by

$$T(x)/I_1 = \sqrt{2x}, \ T(x)/I_2 = 1 - \sqrt{2x - 1}$$

is a u.d.p. transformation.

3. Piecewise linear transformation. A little calculation using simple geometrical devices shows by means of Theorem 4 that the "saw-functions" on $I$ with the height of the all teeth equal to 1 are u.d.p. transformations. Motivated by this observation we shall characterize in this section the u.d.p. transformations which are piecewise linear (p.l.). Then we show that the condition of Proposition 4 is not sufficient even for the p.l. transformations. Finally, we find a necessary and sufficient condition for a p.l. transformation $T$ with the property that the orbit $x, T(x), T^2(x), \ldots$ is u.d. in $I$ for some (and consequently for almost all) $x \in I$.

Let $T$ be a surjective p.l. transformation of $I$ onto $I$. Let

$$0 = y_0 < y_1 < y_2 < \ldots < y_n = 1$$

be the sequence of ordinates of the ends of the line segments of the graph of $T$ in the unit square $I \times I$. Let

$$J_j = (y_{j-1}, y_j), \ j = 1, 2, \ldots, n$$

be the corresponding system of open disjoint subintervals of $I$. Then for every $j = 1, 2, \ldots, n$ the set $T^{-1}(J_j)$ can be written in the form

$$T^{-1}(J_j) = \bigcup_{i=1}^{n_j} I_{J_i},$$

where $I_{J_i}, i = 1, \ldots, n_j$ are disjoint open intervals such that

$$T(I_{J_i}) = J_j \quad \text{and} \quad T/I_{J_i} \text{ is linear for every } i = 1, 2, \ldots, n_j.$$
For the proof use Theorem 4 with the fact that given an interval \( I^* \subset J_j \) we have
\[
\| T^{-1}(I^*) \| = \frac{|I^*|}{|J_j|},
\]
provided \( T \) is p.l.

Now it is not difficult to give a general rule for the construction of the all p.l. and u.d.p. transformations, provided the ordinate or the abscissa decomposition of \( I \) is given. For instance, given an arbitrary decomposition of \( I \) into disjoint open intervals, grouping them arbitrarily into \( n \) groups we obtain the initial abscissa decomposition (8.2). After dividing \( I \) into open disjoint subintervals \( \{ J_{j_i} \}_{i=1}^n \) with
\[
|J_{j_i}| = \sum_{i=1}^n |I_{j_i}|, \quad j = 1, \ldots, n
\]
we can construct \( T \) as a map which graph in \( I \times I \) consists of arbitrarily chosen diagonals of rectangles \( I_{j_i} \times J_{j_i} \) (endpoints of diagonals can be assigned arbitrarily). The above mentioned saw-functions or the u.d.p. transformation \( \{ k \} \) with integral \( k \) correspond to the case \( n = 1 \). (This construction can be generalized to a certain extent if in every rectangle \( I_{j_i} \times J_{j_i} \) we choose a map which transforms u.d. sequences in \( I_{j_i} \) into u.d. sequences in \( J_{j_i} \).)

We now turn to the iterations of p.l. transformations of \( I \) onto \( I \). Let
\[
J_{j_i}^{(2)} \quad \text{and} \quad I_{j_i}^{(2)} \quad \text{for} \quad i = 1, \ldots, n^{(2)}, j = 1, \ldots, n^{(2)}
\]
be the ordinate and abscissa decomposition of \( I \) with respect to the second iteration \( T^{(2)} \) of \( T \). Then
\[
\{ J_{j_i}^{(2)} : 1 \leq j \leq n^{(2)} \}
\]
is formed by the minimal (with respect to the set inclusion) nonzero intersections of intervals from the system
\[
\{ T(J_{j_i} \cap I_{j_i}) : J_{j_i} \cap I_{j_i} \neq \emptyset, 1 \leq s, j \leq n, 1 \leq i \leq n_j \}
\]
and
\[
\{ I_{j_i}^{(2)} : 1 \leq j \leq n^{(2)}, 1 \leq i \leq n^{(2)} \} = \{ T^{-2}(J_{j_i}^{(2)}) : 1 \leq j \leq n^{(2)} \}.
\]
Generally, for the \( k \)th iteration \( T^{(k)} \) we have
\[
\{ J_{j_i}^{(k)} \} = \{ \text{minimal } \cap T(J_{j_i}^{(k-1)} \cap I_{j_i}) \neq \emptyset \}
\]
and
\[
\{ I_{j_i}^{(k)} \} = \{ T^{-k}(J_{j_i}^{(k)}) \}
\]
with \( e + f = k \).

One of the simplest examples of p.l. transformations with respect to the iterations we obtain when the ordinate decomposition of \( I \) is the same for the all iterations of \( T \). We shall call such p.l. transformations of \( I \) onto \( I \) simple.

**Proposition 8.** Let \( T \) be a p.l. transformation of \( I \) onto \( I \). Then \( T \) is simple if and only if one of the following equivalent conditions is satisfied for its ordinate and abscissa decomposition (8.1–2) of \( I \)

(a) the intersection \( J_{j_i} \cap I_{j_i} \) equals either \( I_{j_i} \) or it is empty for all \( 1 \leq s, j \leq n \) and \( 1 \leq i \leq n_j \);

(b) \( |J_{j_i}| = \sum_{l_{j_i}=j} |I_{j_i}| + \sum_{l_{j_i}=j} |I_{j_i}| + \ldots + \sum_{l_{j_i}=j} |I_{j_i}| \) for every \( j = 1, 2, \ldots, n \).

Given a p.l. transformation \( T \), let (8.1) and (8.2) be the ordinate and abscissa decomposition of \( I \) with respect to \( T \). To formulate our next results assign to \( T \) the following \( n \)-dimensional vectors and \( n \times n \) matrices (vectors will always denote column vectors and the row vectors we shall write as their transposes):

\[
a' = (a_1, a_2, \ldots, a_n) \quad \text{where} \quad a_j = |J_{j_i}|,
\]

\[
b' = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_j = |J_{j_i}|(|J_{j_i}| + |J_{j_i}| + \ldots + \frac{1}{2}|J_{j_i}|),
\]

\[
c' = (c_1, c_2, \ldots, c_n) \quad \text{where} \quad c_j = (\sum_{i=1}^{n_j} |I_{j_i}|/|J_{j_i}|),
\]

\[
\gamma' = (1, 1, \ldots, 1),
\]

\[
\lambda = (a_j) \quad \text{where} \quad \lambda_j = (\sum_{l_{j_i}=j} |I_{j_i}|/|J_{j_i}|),
\]

\[
\beta = (b_j) \quad \text{where} \quad b_j = \frac{1}{2}, \quad b_{j+1} = 1 \quad \text{if} \quad j < s \quad \text{and} \quad b_{j+1} = 0 \quad \text{if} \quad j > s,
\]

\[
\varphi = (p_j) \quad \text{where} \quad p_j = (\sum_{l_{j_i}=j} |I_{j_i}|/|J_{j_i}|) \quad (\text{note that} \quad p_j \quad \text{is the conditional probability that} \quad T(x) \in J_{j_i} \quad \text{for} \quad x \in I_{j_i}),
\]

\[
\text{diag} a = \text{diag}(|J_{j_i}|, |J_{j_i}|, \ldots, |J_{j_i}|),
\]

\[
\text{diag}^{-1} a = \text{diag}(|J_{j_i}^{-1}|, |J_{j_i}^{-1}|, \ldots, |J_{j_i}^{-1}|).
\]

As usual, \( \text{diag} a \) denotes the diagonal matrix whose only nonzero elements are elements of vector \( a \) on the leading diagonal.

In the analogous way we can define the corresponding vectors and matrices, say, \( a^{(k)} \), etc. for the \( k \)th iteration \( T^{(k)} \) of \( T \).

We can immediately characterize some properties of a transformation \( T \) in terms of matrices \( A \) and \( Q \) as follows.

**Proposition 9.** (a) A p.l. map \( T : I \to I \) is simple if and only if \( T Q = T \), i.e. if \( Q \) is a Markov matrix.

(b) A simple p.l. map \( T : I \to I \) is a u.d.p. transformation if and only if \( A \cdot 1 = 1 \), i.e. if \( A \) is a stochastic matrix.

(c) \( Q = \text{diag} a \cdot A \cdot \text{diag}^{-1} a \), \( b' = a' \cdot B' \cdot \text{diag} a \), \( c = A \cdot 1 \), \( \text{diag} a \cdot 1 = a \).

**Proposition 10.** If a p.l. map \( T : I \to I \) is simple then \( a^{(k)} = A^k, \quad Q^{(k)} = Q^k \) and

\[
\int_0^1 T^{(k)}(x) \, dx = a' B Q^k a \quad \text{for every} \quad k = 1, 2, \ldots
\]
Proof. To establish the first part of the proposition note that
\[
a^{(k)}_{ij} = \left( \sum_{j_{i-1} \subseteq J_k} |J_k|/|J_i| \right)^k
\]
\[
= \left( \sum_{i=1}^{n} \sum_{j_{i-1} \subseteq J_k} \frac{|J_k|}{|J_i|} \right)^k
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} a^{(k-1)}_{ij} \cdot a_{ij},
\]
i.e. \( A^{(k)} = A^{(k-1)} \cdot A \). We can similarly prove that \( Q^{(k)} = Q^{k} \).

For the proof of the expression of the integral note first that a similar reasoning as above leads to the relation \( a^{(k)} = A e^{(k-1)} \).

The direct computation of the area under the graph of \( T \) gives that
\[
\frac{1}{0} T(x) \, dx = b' c.
\]
If \( T \) is simple then we similarly obtain
\[
\frac{1}{0} T^{(k)}(x) \, dx = b' c^{(k)} = b' A^{k-1} c = b' A^k 1.
\]
Proposition 9 (c) finishes the proof of the theorem.

More can be proved in general. Namely, given a simple p.l. map \( T: I \rightarrow I \), we have
\[
\frac{1}{0} \int g(T^{(k)}(x)) \, dx = \left( \int g(x) \, dx \right) \left( \int g(x) \, dx, \ldots, \int g(x) \, dx \right) e^{(k)}
\]
\[
= \left( \int g(x) \, dx, \ldots, \int g(x) \, dx, \ldots, \int g(x) \, dx \right) \text{diag}^{-1} aQ a
\]
for every Riemann-integrable function \( g: I \rightarrow I \). This relation yields another proof of Theorem 1 for simple p.l. transformations \( T \) because \( \frac{1}{0} \int g(T(x)) \, dx \)
\[
= \frac{1}{0} g(x) \, dx \text{ is true for all these } g \text{ if and only if } c = 1.
\]

Note that also the expression for the integral in the proof of Proposition 10 implies another proof of Proposition 4 for simple p.l. and u.d.p. transformations \( T: I \rightarrow I \). If \( A \) is a stochastic matrix then \( b' A^k 1 = b' 1 = 1/2 \).

We now show that the conditions of Proposition 4 are not sufficient for a map \( T: I \rightarrow I \) to be a u.d.p. transformation. The point of departure for the remainder of this section are some basic results of the theory of Markov matrices which enables us to determine under which conditions the sequences
\[
\left\{ \frac{1}{0} T^{(k)}(x) \, dx \right\}_{k=1}^{\infty}, \quad \left\{ \sum_{i=1}^{n} |J_k|/|J_i| \right\}_{k=1}^{\infty}
\]
converge. It is known [5] that if \( Q \) is a Markov matrix then either \( \lim_{k \rightarrow \infty} Q^k = Q^{\infty} \) or there exists a positive integer \( k \) such that \( Q^k = (Q^{\infty})^k \). In either case the sequence of the averages
\[
\left\{ \frac{1}{N} \sum_{k=0}^{N-1} Q^k \right\}_{k=1}^{\infty}
\]
converges. In particular, if \( Q \) is an irreducible and primitive Markov matrix then
\[
\lim_{k \rightarrow \infty} Q^k = Q^{\infty}
\]
where \( Q' = p1' \), \( Qp = p \), \( p' = 1 \) and \( p_i > 0 \) for \( i = 1, \ldots, n \).

Suppose \( n \geq 3 \) is given. Let \( p \) be one of the solutions of the system
\[
a'Bx = 1/2, \quad 1'x = 1, \quad x_j \geq 0.
\]
The set of these solutions is infinite and \( a \) is one of them. Then for the matrix \( Q = p1' \) we have
\[
a'BQ a = 1/2.
\]

Proposition 10 implies that a simple p.l. transformation \( T: I \rightarrow I \) corresponding to the just chosen \( Q \) fulfills Proposition 4 for all \( n \). However, if \( p \neq a \) then this \( T \) is not a u.d.p. transformation for the corresponding matrix \( A \) is not stochastic which can be readily verified.

To be concrete, let \( n = 2 \) and \( J_j = ([j-1]/3, j/3) \) for \( j = 1, 2, 3 \) Then \( a' = (1/3, 1/3, 1/3) \) and one can take \( p' = (1/4, 1/2, 1/4) \). Let further \( n_1 = n_2 = n_3 = 3 \). Then the intervals
\[
I_{1,1} = (0, 1/12), \quad I_{1,2} = (1/12, 1/4), \quad I_{1,3} = (1/4, 1/3),
\]
\[
I_{2,1} = (1/3, 5/12), \quad I_{2,2} = (5/12, 7/12), \quad I_{2,3} = (7/12, 8/12),
\]
\[
I_{3,1} = (8/12, 9/12), \quad I_{3,2} = (9/12, 11/12), \quad I_{3,3} = (11/12, 1)
\]
form the abscissa decomposition of a (unique) continuous p.l. function \( T \) for which
\[
\frac{1}{0} T^{(k)}(x) \, dx = 1/2 \quad \text{for all } k = 1, 2, \ldots
\]
but
\[
|T^{-1}(J_2)| \neq |J_2|
\]
and so \( T \) is not a u.d.p. transformation.

Note that in the case \( n = 2 \) the vector \( a \) is the only solution of (9). Therefore if \( T \) is p.l. and simple with \( n = 2 \) then \( T \) is a u.d.p. transformation if and only if (10) is true.

We conclude this section with a characterization of simple p.l. transfor-
mations $T$ with a u.d. orbit

$$(11) \quad x, \ T(x), \ T^{(2)}(x), \ldots$$

for at least one $x \in I$.

**Theorem 5.** Let $T$ be a simple p.l. transformation of $I$ onto $I$ with the ordinate and abcissa decomposition (8.1–2) and $A$ the matrix assigned to $T$ above. Then there exists an $x \in I$ for which the orbit (11) is u.d. in $I$ if and only if

(a) $A$ is a stochastic and irreducible matrix,

(b) $\{J_j : 1 \leq j \leq n\} \neq \{I_{j,i} : 1 \leq i \leq n_j, 1 \leq j \leq n\}$.

Moreover, if (11) is u.d. for one $x \in I$ then this is true for almost all $x \in I$.

**Proof.** If (11) is a u.d. sequence for a simple p.l. map $T$ of $I$ onto $I$ then Proposition 5 (b) implies that $T$ is a u.d.p. transformation. Proposition 9 (b) yields in turn that $A$ is consequently a stochastic matrix.

Using the known reformulation of the notion of irreducibility of a matrix ([5], p. 281) in terms of directed graphs it is enough to show for the irreducibility of $A$ that for every couple of intervals $J_s, J_j$ there exists a positive integer $k$ and a $t \in J_t$ with $T^{(k)}(t) \in J_j$.

To see (b) the equality between the ordinate and abcissa decomposition implies that the ordinate decomposition splits into cycles of the form

$$(12) \quad J_{j_1} \twoheadrightarrow J_{j_2} \twoheadrightarrow \ldots \twoheadrightarrow J_{j_s} = J_j,$$

with the property that the contraction $T/J_{j_i}$ is linear and $T(J_{j_i}) = J_{j_{i+1}}$, for $i = 1, \ldots, s - 1$. This in turn implies the existence of a positive integer $k$ such that $T^{(k)}(x) = x$ for all interior points $x$ of intervals (8.1). This contradiction with the density of orbit (11) finishes the proof of the necessity.

Conversely, suppose that (a) and (b) are satisfied. First of all we show that

$$(13) \quad \lim_{k \to \infty} \max_{k \in I} |I_{j_i}^{(k)}| = 0.$$ A look on the graph of $T^{(k)}$ shows that

$$|I_{j_i}^{(k)}| = |I_{j_i}^{(k-1)}| \frac{|I_{j_i}^{(k-1)}|}{|J_{j_i}|},$$

where $I_{j_i}^{(k)} \subset J_{j_i}$. Thus we can write

$$|I_{j_i}^{(k)}| = |I_{j_i}| \frac{|I_{j_i}^{(k-1)}|}{|J_{j_i}|} \frac{|I_{j_i}^{(k-2)}|}{|J_{j_i}|} \ldots$$

where $I_{j_i}^{(k)} \subset J_{j_i} \subset I_{j_i} \subset J_{j_i} \ldots$ and the product on the right-hand side contains exactly $k$ factors of the indicated form. Since $A$ is a stochastic, none of these factors exceeds 1. Then the relation (13) will follow if we prove that the number of consecutive factors equal to 1 cannot be greater than $n$. To see this note that the inequality $|I_{j_i}^{(k)}| = |J_{j_i}|$ implies $n_j = 1$ and this undoubtedly the existence of a cycle (12) if the number of consecutive factors equal to 1 is greater than $n$. The existence of such cycle exhausting all the system (8.1) leads to a contradiction with the assumption (b). A shorter cycle contradicts the irreducibility of $A$.

Secondly we show that

$$(14) \quad \frac{|A \cap J_{j_i}|}{|J_{j_i}|} = \frac{|A \cap T(I_{j_i})|}{|T(I_{j_i})|},$$

provided $A \subset I$ is a Lebesgue-measurable $T$-invariant set (i.e. $T^{-1}(A) = A$) and $I_{j_i}$ a subinterval of $I$ on which $T$ is linear. To prove this suppose that the endpoints of $I_{j_i}$ are $\alpha, \beta$ and that $T/I_{j_i} = ax + b$. Then with $\chi$ the indicator of $A$ we have

$$|A \cap I_{j_i}| = \int \chi(ax + b) \, dx = \frac{1}{a} \int \chi(t) \, dt = \frac{1}{a} |A \cap T(I_{j_i})|$$

and (14) follows.

We now derive from (14) that if $A \subset I$ is a Lebesgue-measurable $T$-invariant set with a non-zero measure, then $A$ is of the full measure. This would imply that $T$ is ergodic (more precisely the ergodicity is equivalent to (a) and (b)) and this in turn implies that (11) is u.d. for almost all $x \in I$.

Suppose therefore that $|A| > 0$. It is known that for a measurable set almost all its points are density points [8]. This means that one can find a point $x \in I$ not an endpoint of the intervals (8.1) such that to every $\varepsilon > 0$ the relation (13) implies the existence of an interval $I_{j_i}^{(k)}$ containing $x$ and satisfying

$$\frac{|A \cap I_{j_i}^{(k)}|}{|I_{j_i}^{(k)}|} > 1 - \varepsilon.$$
such a way that

$$I_{j,i} \subset J_i \quad \text{and} \quad T(I_{j,i}) = J_i.$$  

The same can be done for intervals $I_{j,i}^{(k)}$ corresponding to the $k$th iteration $T^{(k)}$ of $T$. More precisely, we can assume that $I_{j,i}^{(k)} \subset J_i$, $I_{j,i}^{(k)} \subset J_i$, $I_{j,i}^{(k)} \subset J_i$, ..., are the dividing subintervals of $I_{j,i}^{(k)} \subset J_i$, $T^{(k)} I_{j,i}^{(k)} \subset J_i$, $T^{(k-1)}(I_{j,i}^{(k-1)}) = I_{j,i}^{(k-1)}$, $T^{(k-1)}(I_{j,i}^{(k-1)}) = I_{j,i}^{(k-1)}$, etc.

Using this notation we have

\begin{align*}
17 & \quad x \in I_{j,i}^{(k)} \quad \Leftrightarrow \quad T^{(k)}(x) \in J_i \quad \text{and} \quad x \in I_{j,i}^{(k-1)} \\
& \quad \Leftrightarrow \quad T^{(k)}(x) \in J_i \quad \text{and} \quad T^{(k-1)}(x) \in J_i \quad \text{and} \quad \ldots \quad x \in J_i. \\
\end{align*}

Therefore

\begin{align*}
18 & \quad \frac{|I_{j,i}^{(k)}|}{|I_{j,i}|} = \frac{|I_{j,i}^{(k-1)}|}{|I_{j,i}|} \frac{|I_{j,i}^{(k-2)}|}{|I_{j,i}^{(k-1)}|} \ldots \frac{|I_{j,i}^{(1)}|}{|I_{j,i}^{(0)}|} = \frac{|I_{j,i}|}{|I_{j,i}|} \frac{|I_{j,i}|}{|I_{j,i}|} \ldots \frac{|I_{j,i}|}{|I_{j,i}|} \\
& \quad \text{We shall apply the foregoing to explicit determination of points of $X$ in terms of $I_{j,i}$'s and the sequence $\{k_i\}_{i=1}^n$, If $T: I \to I$ is a strictly simple p.l. transformation and if (16) holds then} \\
& \quad x = \sum_{i=1}^{k-1} \left| J_i \right| + \sum_{i=1}^{n} \left| I_{s,i} \right| + \sum_{i=2}^{n} \left| J_{s,i+1} \right| + \sum_{i=3}^{n} \left| I_{s,i+2} \right| + \ldots \\
& \quad \text{for every } x \notin X \text{ where } \{k_i\}_{i=1}^n \text{ is determined by (15) and } \sum \text{ denotes the sum over those intervals } I_{s,i} \text{ which lie on the left of } J_{s,i+1}, \sum \text{ over those } I_{s,i+2}, \sum \text{ on the left of } I_{s,i+2}, \text{ etc.} \quad \text{Then (18) gives} \\
19 & \quad x = |J_1| + |J_2| + \ldots + |J_{k-1}| + \sum_{i=1}^n |I_{s,i}| + \sum_{i=2}^n |I_{s,i}| |I_{s,i+1}| + \sum_{i=3}^n |I_{s,i+2}| + \ldots \\
& \quad \text{Here in the summation } \sum \text{ the intervals run over those } I_{s,i} \text{ which are on the left of } I_{k,s}, \text{ if the slope of } T \text{ in } I_{k,s} \times J_s \text{ is positive, otherwise the intervals are taken from the right of } I_{k,s}. \text{ Similarly, in } \sum \text{ the intervals } I_{s,i} \text{ are taken from the left of } I_{s,i}, \text{ if the product of slopes in } I_{k,s} \times J_s \text{ and } I_{k,s} \times J_{s+1} \text{ is positive and from the left otherwise, etc.} \\
& \quad \text{We can now state the following two results:} \\
& \quad (a) \text{ An } x \in X \text{ is a fixed point of a strictly simple p.l. transformation } T \text{ if and only if the sequence } \{k_i\}_{i=1}^n \text{ defined through (15) is constant. If, moreover, all} \

the slopes of $T$ are positive then the fixed points of $T$ belonging to $X$ are given through the formula

$$|J_1| + |J_2| + \ldots + |J_{k-1}| + \frac{|J_k|}{|J_1||J_2| \cdots |J_{k-1}|} \sum_{k} |I_{a,k}|$$

with $k = 1, 2, \ldots, n$ satisfying the condition $p_{ab} > 0$.

(b) An $x \in X$ is a periodical point of a strictly simple p.l. transformation if and only if the sequence $\{k_i\}_{i=1}^\infty$ is periodical. If, moreover, all the slopes of $T$ are positive then the periodical points from $X$ of order $s$ are of the form

$$|J_1| + |J_2| + \ldots + |J_{s-1}| + \frac{|J_{s-1}|}{|J_1||J_2| \cdots |J_{s-2}|} \sum_{k} |I_{a,k}|$$

where the sum in the parentheses has $s$ terms and the $s$-tuple $k_1, k_2, \ldots, k_s$ satisfies the conditions $p_{k_i, k_{i+1}} > 0$ for $i = 1, 2, \ldots, s-1$.

The characterization of $x \in X$ with u.d. orbit is more complex. To do this we shall denote by $d(A)$ the asymptotic density of the set $A$.

**Proposition 11.** Let $T : I \to I$ be a strictly simple p.l. transformation. The orbit $x, T(x), T^2(x), \ldots$ of an element $x \in X$ is u.d. in $I$ if and only if for every $s$ and every $(s+1)$-tuple $j_1, j_2, \ldots, j_{s+1}$ of positive integers satisfying $p_{j_i, j_{i+1}} > 0$ for $i = 1, 2, \ldots, s$ we have

$$d(\{(n; k_{n+s} = j_1, k_{n+s+1} = j_2, \ldots, k_{n+s-1} = j_{s+1})\})$$

where $s$ tends to $\infty$, the standard approximation argument shows that the uniform distribution of our orbit is equivalent with the equality

$$d(\{n; T^{(s)}(x) \in I_{j_1, j_2, \ldots, j_{s+1}}\}) = |I_{j_1, j_2, \ldots, j_{s+1}}|$$

for every sufficiently large positive integer $s$ and every $(s+1)$-tuple $j_1, j_2, \ldots, j_{s+1}$ with $p_{j_i, j_{i+1}} > 0$. Then using (17) and (18) the assertion of proposition follows immediately.

**Theorem 6.** The system of all u.d.p. transformations is nowhere dense in the space $M$ of all maps of $I$ to itself.

**Proof.** Let $F \in M$ and $x_0 \in I$, $F(x_0) \neq 1$. Let $K(F, \varepsilon)$ be an open ball in $M$. Then for every $y \in K(F, \varepsilon)$ we have $F(y) < 1 - \varepsilon$ and $x_0 \notin F([x_0]) = F(I)$.
with \( 0 < \varepsilon < 1 - F(x_0) \). Define the map \( G : I \rightarrow I \) as follows:

\[
G(x) = \begin{cases} 
F(x_0) + \varepsilon/3 & \text{if } |F(x) - F(x_0)| < \varepsilon/3, \\
F(x) & \text{otherwise.}
\end{cases}
\]

Plainly,

\[ |G(x) - F(x_0)| \geq \varepsilon/3 \quad \text{and} \quad |G(x) - F(x)| < 2\varepsilon/3 \]

for every \( x \in I \).

We claim that the open ball \( K(G, \varepsilon/9) \) does not contain a u.d.p. transformation. Let \( H \in K(G, \varepsilon/9) \) and

\[ I_1 = (F(x_0) - \varepsilon/9, F(x_0) + \varepsilon/9). \]

Then \( H^{-1}(I_1) \) is an empty set which is impossible for u.d.p. transformations according to Theorem 4. The proof of theorem is thus finished.

Note that by Theorem 4 the u.d.p. transformations \( F \) share the following property: for every open subinterval \( I_1 \subset I \) the condition \( F^{-1}(I_1) \neq \emptyset \) implies that the interior \( \operatorname{Int}(F^{-1}(I_1)) \) is also non-empty (i.e. that the u.d.p. transformations are somewhat continuous [3]).

**Theorem 7.** The system of the all u.d.p. transformations is a perfect set in \( M \), the space of the all maps of \( I \) to itself.

**Proof.** We have to verify two conditions:
(a) The system of the all u.d.p. transformations is closed in \( M \). But this the content of Proposition 2.
(b) It is an everywhere dense set in \( M \). Given a u.d.p. transformation \( F \), we have to construct a sequence \( \{F_n\}_{n=1}^{\infty} \), \( F_n \neq F \) for all \( n \), of u.d.p. transformations converging to \( F \) in \( M \). But this can be easily done. Let \( x_0 \in I \) be such that \( F(x_0) \neq 1 \).

Let \( \{\varepsilon_n\}_{n=1}^{\infty} \) be a sequence of positive real numbers with

\[
\lim_{n \to \infty} \varepsilon_n = 0 \quad \text{and} \quad F(x_0) + \varepsilon_n \leq 1 \quad \text{for all } n.
\]

Then define

\[
F_n(x) = \begin{cases} 
F(x) & \text{if } x \in I \text{ and } x \neq x_0, \\
F(x_0) + \varepsilon_n & \text{if } x = x_0
\end{cases}
\]

and the required conclusion follows immediately.

**References**


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