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Transformations that preserve uniform distribution

by

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Dedicated to Professor Pál Erdős
on the occasion of his 75th birthday

The purpose of this paper is to describe some properties of functions that preserve uniformly distributed sequences of real numbers. Here we say that a map T of the unit interval $I = \langle 0, 1\rangle$ to itself is a *uniform distribution preserving* (*u.d.p.*) *transformation* if $\{T(x_n)\}_{n=1}^{\infty}$ is a uniformly distributed sequence (u.d.) sequence in I for every u.d. sequence $\{x_n\}_{n=1}^{\infty} \subset I$.

In the course of our discussion we shall see that the study of u.d.p. transformations leads to the opposite question to that investigated in the ergodic theory. Namely, given a measure μ (in our case this will be the Jordan measure), describe properties of transformations with respect to which μ is invariant. Perhaps our results may motivate other directions in the theory of dynamical systems, besides the study of properties of sets of points with periodical, recurrent, dense, etc. orbits to study, for instance, sets of points which orbits are uniformly distributed or to investigate sequences of integrals of iterations of transformations. Another direction is the study of the orbit behaviour of concrete points. It will be worth to answer these questions at least for piecewise linear transformations.

1. General criteria. From the well-known integral criterion ([4], p. 2) for u.d. sequences the following necessary and sufficient condition for a map of I to be a u.d.p. transformation results immediately.

THEOREM 1. *A map T: $I \to I$ is a u.d.p. transformation if and only if for every Riemann-integrable function g: $I \to R$ the composition $g \circ T$ is also Riemann-integrable and*

$$\int_0^1 g(x)\,dx = \int_0^1 g\big(T(x)\big)\,dx. \tag{1}$$

Proof. Suppose that T is a u.d.p. transformation and $\{x_n\}_{n=1}^{\infty}$ is a u.d.

sequence. Then $\{T(x_n)\}_{n=1}^{\infty}$ is also a u.d. sequence and consequently

$$\lim_{n\to\infty}\frac{g(x_1)+\ldots+g(x_n)}{n}=\int_0^1 g(x)\,dx=\lim_{n\to\infty}\frac{g(T(x_1))+\ldots+g(T(x_n))}{n}$$

for every Riemann-integrable function g on I.

The existence of the limes on the right-hand side implies that $g\circ T$ is Riemann-integrable. In the opposite case there exists ([1] or [2]) a u.d. sequence $\{t_n\}_{n=1}^{\infty}\subset I$ for which the sequence

$$\left\{\frac{g\circ T(t_1)+\ldots+g\circ T(t_n)}{n}\right\}_{n=1}^{\infty}$$

does not have a finite limit. The Riemann-integrability of $g\circ T$ implies

$$\lim_{n\to\infty}\frac{g(T(x_1))+\ldots+g(T(x_n))}{n}=\int_0^1 g(T(x))\,dx$$

and the necessary condition follows.

For the proof of the sufficient condition suppose on the contrary that T is not a u.d.p. transformation. Then there exists a u.d. sequence $\{x_n\}_{n=1}^{\infty}$ for which the sequence $\{T(x_n)\}_{n=1}^{\infty}$ is not u.d. This means that there exist a Riemann-integrable function $h\colon I\to R$ for which the sequence

$$\left\{\frac{h\circ T(x_1)+\ldots+h\circ T(x_n)}{n}\right\}_{n=1}^{\infty} \tag{2}$$

does not converge to $\int_0^1 h(x)\,dx$. We can suppose that $h\circ T$ is Riemann-integrable. Then (2) necessarily converges to $\int_0^1 h\circ T(x)\,dx$ which contradicts (1) and the theorem is proved.

Theorem 1 implies that every u.d.p. transformation is Riemann-integrable. The function

$$f(x)=\begin{cases}x & \text{if } x \text{ is a rational number},\\ 0 & \text{otherwise}\end{cases}$$

shows that there are functions not integrable in the Riemann sense which are not u.d.p. transformations though they transform infinitely many u.d. sequences into u.d. ones.

By Theorem 1 every composition $g\circ T$ of a u.d.p. transformation and a Riemann-integrable function $g\colon I\to I$ is again Riemann-integrable. Note that this is a restriction, because every Lebesgue-measurable function is expressible as a composition of two Riemann-integrable functions ([6], [7]).

The well-known approximation technique enables us to replace the Riemann-integrable functions in the integrable criterion for the u.d. sequences used above by continuous functions. More generally, instead of the system of the continuous functions one can take any system of functions which linear hull is dense in the system of the all continuous functions. The systems of functions $\{x^n\}_{n=1}^{\infty}$ and $\{e^{2\pi inx}\}_{n=-\infty}^{\infty}$ lead to the following result (the verification of the composition property in Theorem 1 is trivial):

THEOREM 2. *A Riemann-integrable function $T\colon I\to I$ is a u.d.p. transformation if and only if one of the following condition is satisfied:*

(a) $\int_0^1 g(x)\,dx=\int_0^1 g(T(x))\,dx$ *for every continuous function* $g\colon I\to R$,

(b) $\int_0^1 T^n(x)\,dx=1/(n+1)$ *for every* $n=1,2,\ldots$,

(c) $\int_0^1 e^{2\pi inT(x)}\,dx=0$ *for every* $n=\pm1,\pm2,\ldots$

The next theorem shows that for a Riemann-integrable function of I to itself the question whether it is a u.d.p. transformation can be decided using only one suitable sequence. For the formulation of the result we shall need the following notion:

Let $\{N_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. A sequence $\{x_n\}_{n=1}^{\infty}\subset I$ is called *$\{N_k\}$-uniformly distributed* if

$$\lim_{k\to\infty}\frac{A(\langle 0,x\rangle,N_k,\{x_n\})}{N_k}=x \quad\text{for}\quad 0\leqslant x\leqslant 1,$$

where $A(E,N,\{x_n\})$ denotes the number of terms x_n, $1\leqslant n\leqslant N$ for which $x_n\in E$. Such sequences are usually called *almost u.d. sequences* ([4], p. 53). The reason for our terminology is that we shall need to stress the rôle of the sequence $\{N_k\}$ in this section.

THEOREM 3. *Let $T\colon I\to I$ be Riemann-integrable. Then T is a u.d.p. transformation if and only if there exists an increasing sequence of positive integers $\{N_k\}_{k=1}^{\infty}$ and an $\{N_k\}$-uniformly distributed sequence $\{x_n\}_{n=1}^{\infty}\subset I$ for which the sequence $\{T(x_n)\}_{n=1}^{\infty}$ is also $\{N_k\}$-uniformly distributed.*

Proof. The necessity is obvious. For the sufficiency suppose that both $\{x_n\}_{n=1}^{\infty}$ and $\{T(x_n)\}_{n=1}^{\infty}$ are $\{N_k\}$-uniformly distributed. Then we have

$$\lim_{k\to\infty}\frac{T^m(x_1)+T^m(x_2)+\ldots+T^m(x_{N_k})}{N_k}=\int_0^1 T^m(x)\,dx \quad\text{for } m=1,2,\ldots$$

However the left-hand side can be in turn written in the form

$$\lim_{k\to\infty}\frac{(T(x_1))^m+(T(x_2))^m+\ldots+(T(x_{N_k}))^m}{N_k}=\int_0^1 x^m\,dx=1/(m+1)$$

and Theorem 2(b) finishes the proof.

The affinity between the Riemann integrability and the Jordan measurability leads to the next result.

THEOREM 4. *A map $T\colon I \to I$ is a u.d.p. transformation if and only if*
(a) *T is measurable in the Jordan sense,*
(b) *$|T^{-1}(I_1)| = |I_1|$ for every interval $I_1 \subset I$ ($|E|$ denotes the Jordan or Lebesgue measure of E).*

Proof. First of all we have

$$A(I_1, N, \{T(x_n)\}) = A(T^{-1}(I_1), N, \{x_n\}) \tag{3}$$

for every sequence $\{x_n\}_{n=1}^{\infty} \subset I$.

Suppose (a) and (b) are true. Then (a) implies that $T^{-1}(I_1)$ is measurable in the Jordan sense. Thus for every u.d. sequence $\{x_n\}_{n=1}^{\infty}$ we have

$$\lim_{N\to\infty} \frac{A(T^{-1}(I_1), N, \{x_n\})}{N} = |T^{-1}(I_1)|.$$

Then (3) and (b) imply that

$$\lim_{N\to\infty} \frac{A(I_1, N, \{T(x_n)\})}{N} = |I_1|$$

which means that $\{T(x_n)\}_{n=1}^{\infty}$ is a u.d. sequence.

Conversely, let T be a u.d.p. transformation and I_1 a subinterval of I. If $T^{-1}(I_1)$ is Jordan measurable then (3) implies (b). Suppose therefore that $T^{-1}(I_1)$ is not Jordan measurable. Then the indicator χ of $T^{-1}(I_1)$ is not Riemann integrable and thus there exists [2] a u.d. sequence, say, $\{x_n\}_{n=1}^{\infty}$ for which the sequence

$$\left\{\frac{1}{N}\sum_{n\leqslant N} \chi(x_n)\right\}_{N=1}^{\infty}$$

does not have a finite limit. On the other hand,

$$\sum_{n\leqslant N} \chi(x_n) = A(T^{-1}(I_1), N, \{x_n\})$$

and (3) leads to a contradiction that T is a u.d.p. transformation.

2. Miscellanea. In this section we shall present several simple properties of u.d.p. transformations which may be of some interest. The proof of the first proposition is straightforward.

PROPOSITION 1. *Let T and G be u.d.p. transformations and α a real number. Then $T\circ G$, $1-T$ and the fractional part $\{T+\alpha\}$ are also u.d.p. transformations.*

PROPOSITION 2. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of u.d.p. transformations uniformly converging to H. Then H is a u.d.p. transformation.*

Proof. If $T_n \rightrightarrows H$ then also $T_n^k \rightrightarrows H^k$ for every exponent $k = 1, 2, \ldots$, which implies that

$$\lim_{n\to\infty} \int_0^1 T_n^k(x)\,dx = \int_0^1 H^k(x)\,dx.$$

Owing to Theorem 2(b) every integral on the left-hand side is equal to $1/(k+1)$. Consequently the same is true for the right-hand side and the same theorem finishes the proof.

PROPOSITION 3. *Let $T\colon I \to I$ be a u.d.p. transformation and let at least one of the following conditions be satisfied:*
(a) *T is monotone,*
(b) *T has the derivative at each point of I,*
(c) *T has the Darboux property (i.e. every interval is mapped onto an interval) and T is injective,*
(d) *T is continuous and either $T(x) \leqslant x$ for each $x \in I$ or $T(x) \geqslant 1-x$ for each $x \in I$.*

Then either $T(x) = x$ for each $x \in I$ or $T(x) = 1-x$ for each $x \in I$.

Proof. (a) If T is a u.d.p. transformation and monotone then T is necessarily continuous (no jumps are possible). This means that for each subinterval I_1 of I the set $I_2 = T^{-1}(I_1)$ is also an interval. Then $I_1 = T(I_2)$ and by Theorem 4 $|T^{-1}(I_1)| = |I_1|$. This implies that $|I_2| = |T(I_2)|$, i.e.

$$\frac{|T(I_2)|}{|I_2|} = 1.$$

This gives in turn that the derivative of T in every point of I equals 1 or -1 and the conclusion follows.

(b) We reduce this case to (a) showing that the derivative T' of T has no sign changes. The opposite case would lead to a point $x \in I$ with $T'(x) = 0$ and this in turn to a sequence $\{I_n\}_{n=1}^{\infty}$ of subintervals of I such that

$$0 < |I_n|, \quad \lim_{n\to\infty} |I_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{|T(I_n)|}{|I_n|} = 0.$$

But then

$$|T^{-1}(T(I_n))| \geqslant |I_n| > \frac{1}{\varepsilon}|T(I_n)|$$

for $n > n_0(\varepsilon)$ and an arbitrarily small $\varepsilon > 0$. The contradiction with Theorem 4 finishes the proof of (b).

(c) If T is injective and I_1 any subinterval of I then

$$\frac{A(I_1, N, \{x_n\})}{N} = \frac{A(T(I_1), N, \{T(x_n)\})}{N}.$$

If T possesses the Darboux property then $T(I_1)$ is a subinterval of I and for a u.d. sequence $\{x_n\}_{n=1}^{\infty}$ the right-hand side converges to $|T(I_1)|$ and the left-hand side to $|I_1|$. Thus $|I_1| = |T(I_1)|$ for every subinterval $I_1 \subset I$. This implies that T is continuous. But a continuous and injective transformation is obviously monotone and the proof returns again to (a).

(d) In analogy with the definition of the u.d. sequences in I we say that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a subinterval $I_1 \subset I$ is uniformly distributed in I_1 if

$$\lim_{N\to\infty} \frac{A(I_2, N, \{x_n\})}{N} = \frac{|I_2|}{|I_1|} \quad \text{for every interval } I_2 \subset I_1.$$

Given an $y \in I$, denote by I_y the interval $\langle 0, y \rangle$.

Suppose now that $T(x) \leqslant x$ for each $x \in I$. Then the contraction T/I_y is a transformation of I_y to I_y preserving the u.d. sequences in the just mentioned sense in the interval I_y. We can prove in a manner analogous to the proof of Theorem 1 that

$$\int_0^y g(T(x))\,dx = \int_0^y g(x)\,dx$$

for a u.d.p. transformation T/I_y on interval I_y and for every Riemann-integrable function $g\colon I_y \to R$. For $g(x) = x$ this implies that

$$\int_0^y T(x)\,dx = \int_0^y x\,dx.$$

The differentiation at the continuity point y of T yields that $T(y) = y$. This proves the case $T(x) \leqslant x$. The case $T(x) \geqslant 1-x$ can be proved along similar lines.

PROPOSITION 4. *If T is a u.d.p. transformation then every its iteration $T^{(n)}$ is a u.d.p. transformation and*

$$\int_0^1 T^{(n)}(x)\,dx = 1/2 \quad \textit{for each } n = 1, 2, \ldots$$

The proof follows immediately from (1) by induction on n beginning with $g(x) = x$.

As an immediate consequence of Theorem 3 we have the following result.

PROPOSITION 5. *Let $T\colon I \to I$ be Riemann-integrable. Then T is a u.d.p. transformation if one of the following conditions is satisfied:*

(a) *There exists an almost u.d. sequence $\{x_n\}_{n=1}^{\infty}$ in I for which the sequence $\{T(x_n)-x_n\}_{n=1}^{\infty}$ converges to a finite limit.*

(b) *There exists at least one $x \in I$ for which the orbit*

$$x,\ T(x),\ T(T(x)),\ \ldots$$

is an almost u.d. sequence in I.(1)

(1) We are indebted to M. Paštéka for calling our attention to this result.

The following observation forms the background for the next proposition:

If a u.d.p. transformation $T\colon I \to I$ is differentiable at a point $t \in I$, and if T is continuous in a neighbourhood of this point then $|T'(t)| \geqslant 1$.

The opposite inequality would namely imply the existence of a closed neighbourhood, say, U of t on which T can be supposed to be continuous and where

$$|T(x)-T(t)| < |x-t| \quad \text{for all } x \in U.$$

Let $x_1 \neq x_2$ be elements of U such that

$$T(x_1) = \max_{y\in U} T(y) \quad \text{and} \quad T(x_2) = \min_{y\in U} T(y).$$

If t lies between x_1 and x_2 then

$$|T(x_1)-T(x_2)| < |x_1-x_2|.$$

But this yields a contradiction with Theorem 4 for I_1 defined by endpoints $T(x_1)$ and $T(x_2)$ because then

$$|T^{-1}(I_1)| \geqslant |x_1-x_2| > |T(x_1)-T(x_2)| = |I_1|.$$

A similar contradiction can be obtained for t's outside the closed interval determined by x_1 and x_2.

We have more generally:

PROPOSITION 6. *Let $T\colon I \to I$ be piecewise differentiable. Then T is a u.d.p. transformation if and only if*

$$\sum_{x\in T^{-1}(y)} \frac{1}{|T'(x)|} = 1 \tag{4}$$

for all but a finite number of points $y \in I$.

Proof. Suppose that T is piecewise differentiable. If T is a u.d.p. transformation or if (4) is true then T is a piecewise continuous strictly monotone and surjective map. To such a map one can always find two systems of disjoint open intervals (similarly as for piecewise linear maps in the next section)

$$\{J_j = (y_{j-1}, y_j)\colon j = 1, 2, \ldots, n\},$$
$$\{I_{j,i}\colon i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, n\}$$

with

$$T^{-1}(J_j) = \bigcup_{i=1}^{n_j} I_{j,i}, \quad T(I_{j,i}) = J_j$$

and such that the contraction $T/I_{j,i}$ is continuous, strictly monotone and

$$\sum_{j=1}^{n} |J_j| = 1, \quad \sum_{j=1}^{n}\sum_{i=1}^{n_j} |I_{j,i}| = 1$$

for all $i = 1, 2, \ldots, n_j$, $j = 1, 2, \ldots, n$.

Further let $G_{j,i}\colon J_j \to I_{j,i}$ be the inverse map to $T/I_{j,i}$ for $i = 1, \ldots, n_j$, $j = 1, \ldots, n$. According to Theorem 4 transformation T is a u.d.p. one if and only if we have

$$(5)\qquad y - y_{j-1} = \sum_{i=1}^{n_j} |G_{j,i}(y) - G_{j,i}(y_{j-1})|$$

for every $y \in J_j$ and every $j = 1, \ldots, n$. Then the differentiation of (5) gives

$$(6)\qquad 1 = \sum_{i=1}^{n_j} |G'_{j,i}(y)|$$

what is equivalent to (4).

Now suppose that (4), and consequently also (6), is true. Then ([8], p. 199)

$$(7)\qquad y - y_{j-1} = \int_{y_{j-1}}^{y} 1 \cdot dy = \sum_{i=1}^{n_j} \int_{y_{j-1}}^{y} |G'_{j,i}(y)|\, dy \leqslant \sum_{i=1}^{n_j} |G_{j,i}(y) - G_{j,i}(y_{j-1})|.$$

This implies that

$$|J_k| \leqslant \sum_{i=1}^{n_k} |I_{k,i}| \quad \text{for all } k = 1, \ldots, n.$$

On the other hand if the inequality in (7) would be strict for some $y \in J_j$ then we obtain

$$|J_j| < \sum_{i=1}^{n_j} |I_{j,i}|$$

for this j. But this together contradicts the fact that the open intervals J_j for $j = 1, \ldots, n$ form a disjoint decomposition of I. Thus (4) implies (5) and the theorem is proved.

Proposition 6 can be employed for construction of nontrivial u.d.p. transformation in the following manner. On some subintervals of I we can choose T arbitrarily but with sufficiently large derivatives in magnitude. On the remaining subintervals of I we complete T in such a way that (4) or (5) is satisfied. For instance, let $I_1 = \langle 0, 1/2 \rangle$, $I_2 = \langle 1/2, 1 \rangle$. Let T/I_1 be strictly increasing, T/I_2 strictly decreasing with $T(0) = 0$, $T(1/2) = 1$, $T(1) = 0$ and let G_1, G_2 be the corresponding inverse mappings. According to (5) T is a u.d.p. transformation if and only if

$$y - 0 = |G_1(y) - G_1(0)| + |G_2(y) - G_2(0)| = G_1(y) - G_2(y) + 1.$$

Now if $G_1(y)$ will be increasing with $0 < G'_1(y) < 1$ for $y \in (0, 1)$ then

$$G_2(y) = G_1(y) - y + 1$$

is the required complement. Thus for instance, for $G_1(y) = y^2/2$ we obtain $G_2(y) = y^2/2 - y + 1$ and therefore T given by

$$T(x)/I_1 = \sqrt{2x}, \qquad T(x)/I_2 = 1 - \sqrt{2x-1}$$

is a u.d.p. transformation.

3. Piecewise linear transformation. A little calculation using simple geometrical devices shows by means of Theorem 4 that the "saw-functions" on I with the height of the all teeth equal to 1 are u.d.p. transformations. Motivated by this observation we shall characterize in this section the u.d.p. transformations which are piecewise linear (p.l.). Then we show that the condition of Proposition 4 is not sufficient even for the p.l. transformations. Finally, we find a necessary and sufficient condition for a p.l. transformation T with the property that the orbit x, $T(x)$, $T^{(2)}(x)$, ... is u.d. in I for some (and consequently for almost all) $x \in I$.

Let T be a surjective p.l. transformation of I onto I. Let

$$0 = y_0 < y_1 < y_2 < \ldots < y_n = 1$$

be the sequence of ordinates of the ends of the line segments of the graph of T in the unit square $I \times I$. Let

$$J_j = (y_{j-1}, y_j), \quad j = 1, 2, \ldots, n$$

be the corresponding system of open disjoint subintervals of I. Then for every $j = 1, 2, \ldots, n$ the set $T^{-1}(J_j)$ can be written in the form

$$T^{-1}(J_j) = \bigcup_{i=1}^{n_j} I_{j,i},$$

where $I_{j,i}$, $i = 1, \ldots, n_j$ are disjoint open intervals such that

$$T(I_{j,i}) = J_j \quad \text{and} \quad T/I_{j,i} \text{ is linear for every } i = 1, 2, \ldots, n_j.$$

For the sake of brevity we shall say that the system of intervals

$$(8.1)\qquad \{J_j\colon 1 \leqslant j \leqslant n\}$$

form the ordinate decomposition of I and that the system of intervals

$$(8.2)\qquad \{I_{j,i}\colon 1 \leqslant i \leqslant n_j,\ 1 \leqslant j \leqslant n\}$$

form the abscissa decomposition of I with respect to T.

The connection between both decompositions of I with respect to a p.l. and u.d.p. transformation is given in the next result.

Proposition 7. *A p.l. map T of I onto I is a u.d.p. transformation if and only if*

$$|J_j| = |I_{j,1}| + |I_{j,2}| + \ldots + |I_{j,n}|$$

for every $j = 1, 2, \ldots, n$.

For the proof use Theorem 4 with the fact that given an interval $I^* \subset J_j$ we have

$$|T^{-1}(I^*)| = |I^*| \cdot \frac{|I_{j,1}| + \ldots + |I_{j,n}|}{|J_j|},$$

provided T is p.l.

Now it is not difficult to give a general rule for the construction of the all p.l. and u.d.p. transformations, provided the ordinate or the abscissa decomposition of I is given. For instance, given an arbitrary decomposition of I into disjoint open intervals, grouping them arbitrarily into n groups we obtain the initial abscissa decomposition (8.2). After dividing I into open disjoint subintervals $\{J_j\}_{j=1}^n$ with

$$|J_j| = \sum_{i=1}^{n_j} |I_{j,i}|, \quad j = 1, \ldots, n$$

we can construct T as a map which graph in $I \times I$ consists of arbitrarily chosen diagonals of rectangles $I_{j,i} \times J_j$ (endpoints of diagonals can be assigned arbitrarily). The above mentioned saw-functions or the u.d.p. transformation $\{kx\}$ with integral k correspond to the case $n = 1$. (This construction can be generalized to a certain extent if in every rectangle $I_{j,i} \times J_j$ we choose a map which transforms u.d. sequences in $I_{j,i}$ into u.d. sequences in J_j.)

We now turn to the iterations of p.l. transformations T of I onto I. Let

$$J_j^{(2)} \quad \text{and} \quad I_{j,i}^{(2)} \quad \text{for} \quad i = 1, \ldots, n_j^{(2)},\ j = 1, \ldots, n^{(2)}$$

be the ordinate and abscissa decomposition of I with respect to the second iteration $T^{(2)}$ of T. Then

$$\{J_j^{(2)}:\ 1 \leqslant j \leqslant n^{(2)}\}$$

is formed by the minimal (with respect to the set inclusion) nonzero intersections of intervals from the system

$$\{T(J_s \cap I_{j,i}):\ J_s \cap I_{j,i} \neq \emptyset,\ 1 \leqslant s, j \leqslant n,\ 1 \leqslant i \leqslant n_j\}$$

and

$$\{I_{j,i}^{(2)}:\ 1 \leqslant j \leqslant n^{(2)},\ 1 \leqslant i \leqslant n_j^{(2)}\} = \{T^{-2}(J_j^{(2)}):\ 1 \leqslant j \leqslant n^{(2)}\}.$$

Generally, for the kth iteration $T^{(k)}$ we have

$$\{J_j^{(k)}\} = \{\text{minimal} \cap T(J_s^{(k-1)} \cap I_{j,i}) \neq \emptyset\} \tag{8.3}$$
$$= \{\text{minimal} \cap T^{(e)}(J_s^{(f)} \cap I_{j,i}^{(e)}) \neq \emptyset\}$$

and

$$\{I_{j,i}^{(k)}\} = \{T^{-k}(J_s^{(k)})\} \tag{8.4}$$

with $e+f = k$.

One of the simplest examples of p.l. transformations with respect to the iterations we obtain when the ordinate decomposition of I is the same for the all iterations of T. We shall call such p.l. transformations of I onto I *simple*.

PROPOSITION 8. *Let T be a p.l. transformation of I onto I. Then T is simple if and only if one of the following equivalent conditions is satisfied for its ordinate and abscissa decomposition* (8.1–2) *of I*

(a) *the intersection $J_s \cap I_{j,i}$ equals either $I_{j,i}$ or it is empty for all $1 \leqslant s, j \leqslant n$ and $1 \leqslant i \leqslant n_j$;*

(b) $|J_j| = \sum_{I_{1,i} \subset J_j} |I_{1,i}| + \sum_{I_{2,i} \subset J_j} |I_{2,i}| + \ldots + \sum_{I_{n,i} \subset J_j} |I_{n,i}|$ *for every $j = 1, 2, \ldots, n$.*

Given a p.l. transformation T, let (8.1) and (8.2) be the ordinate and abscissa decomposition of I with respect to T. To formulate our next results assign to T the following n-dimensional vectors and $n \times n$ matrices (vectors will always denote column vectors and the row vectors we shall write as their transposes):

$\boldsymbol{a}' = (a_1, a_2, \ldots, a_n)$ where $a_j = |J_j|$,
$\boldsymbol{b}' = (b_1, b_2, \ldots, b_n)$ where $b_j = |J_j|(|J_1| + |J_2| + \ldots + \frac{1}{2}|J_j|)$,
$\boldsymbol{c}' = (c_1, c_2, \ldots, c_n)$ where $c_j = (\sum_{i=1}^{n_j} |I_{j,i}|)/|J_j|$,
$\boldsymbol{1}' = (1, 1, \ldots, 1)$,
$\boldsymbol{A} = (a_{js})$ where $a_{js} = (\sum_{I_{j,i} \subset J_s} |I_{j,i}|)/|J_j|$,
$\boldsymbol{B} = (b_{js})$ where $b_{jj} = 1/2$, $b_{js} = 1$ if $j < s$ and $b_{js} = 0$ if $j > s$,
$\boldsymbol{Q} = (p_{js})$ where $p_{js} = (\sum_{I_{j,i} \subset J_s} |I_{j,i}|)/|J_s|$ (note that p_{js} is the conditional probability that $T(x) \in J_j$ for $x \in J_s$),
$\operatorname{diag} \boldsymbol{a} = \operatorname{diag}(|J_1|, |J_2|, \ldots, |J_n|)$,
$\operatorname{diag}^{-1} \boldsymbol{a} = \operatorname{diag}(|J_1|^{-1}, |J_2|^{-1}, \ldots, |J_n|^{-1})$.

As usual, $\operatorname{diag} \boldsymbol{a}$ denotes the diagonal matrix whose only nonzero elements are elements of vector $\boldsymbol{a}$ on the leading diagonal.

In the analogous way we can define the corresponding vectors and matrices, say, $\boldsymbol{a}^{(k)}$, etc. for the kth iteration $T^{(k)}$ of T.

We can immediately characterize some properties of a transformation T in terms of matrices A and Q as follows.

PROPOSITION 9. (a) *A p.l. map $T: I \to I$ is simple if and only if $\boldsymbol{1}'\boldsymbol{Q} = \boldsymbol{1}'$, i.e. if Q is a Markov matrix.*

(b) *A simple p.l. map $T: I \to I$ is a u.d.p. transformation if and only if $\boldsymbol{A} \cdot \boldsymbol{1} = \boldsymbol{1}$, i.e. if A is a stochastic matrix.*

(c) $\boldsymbol{Q} = \operatorname{diag} \boldsymbol{a} \cdot \boldsymbol{A} \cdot \operatorname{diag}^{-1} \boldsymbol{a}$, $\boldsymbol{b}' = \boldsymbol{a}' \cdot \boldsymbol{B} \cdot \operatorname{diag} \boldsymbol{a}$, $\boldsymbol{c} = \boldsymbol{A} \cdot \boldsymbol{1}$, $\operatorname{diag} \boldsymbol{a} \cdot \boldsymbol{1} = \boldsymbol{a}$.

PROPOSITION 10. *If a p.l. map $T: I \to I$ is simple then $\boldsymbol{A}^{(k)} = \boldsymbol{A}^k$, $\boldsymbol{Q}^{(k)} = \boldsymbol{Q}^k$ and*

$$\int_0^1 T^{(k)}(x)\,dx = \boldsymbol{a}'\boldsymbol{B}\boldsymbol{Q}^k\boldsymbol{a} \quad \textit{for every } k = 1, 2, \ldots$$

Proof. To establish the first part of the proposition note that

$$a_{js}^{(k)} = \Big(\sum_{I_{j,i}^{(k)} \subset J_s} |I_{j,i}^{(k)}|\Big)/|J_j| = \Big(\sum_{i=1}^{n} \sum_{I_{j,t}^{(k-1)} \subset J_i} \sum_{I_{i,r} \subset J_s} (|I_{j,t}^{(k-1)}| \cdot |I_{i,r}|/|J_i|)\Big)/|J_j| = \sum_{i=1}^{n} a_{ji}^{(k-1)} \cdot a_{is},$$

i.e. $\boldsymbol{A}^{(k)} = \boldsymbol{A}^{(k-1)} \cdot \boldsymbol{A}$. We can similarly prove that $\boldsymbol{Q}^{(k)} = \boldsymbol{Q}^k$.

For the proof of the expression of the integral note first that a similar reasoning as above leads to the relation $\boldsymbol{c}^{(k)} = \boldsymbol{A}\boldsymbol{c}^{(k-1)}$.

The direct computation of the area under the graph of T gives that

$$\int_0^1 T(x)\,dx = \boldsymbol{b}'\boldsymbol{c}.$$

If T is simple then we similarly obtain

$$\int_0^1 T^{(k)}(x)\,dx = \boldsymbol{b}'\boldsymbol{c}^{(k)} = \boldsymbol{b}'\boldsymbol{A}^{k-1}\boldsymbol{c} = \boldsymbol{b}'\boldsymbol{A}^k\boldsymbol{1}.$$

Proposition 9 (c) finishes the proof of the theorem.

More can be proved in general. Namely, given a simple p.l. map $T: I \to I$, we have

$$\int_0^1 g(T^{(k)}(x))\,dx = \Big(\int_{J_1} g(x)\,dx, \int_{J_2} g(x)\,dx, \ldots, \int_{J_n} g(x)\,dx\Big)\boldsymbol{c}^{(k)} = \Big(\int_{J_1} g(x)\,dx, \int_{J_2} g(x)\,dx, \ldots, \int_{J_n} g(x)\,dx\Big)\operatorname{diag}^{-1}\boldsymbol{a}\boldsymbol{Q}^k\boldsymbol{a}$$

for every Riemann-integrable function $g: I \to I$. This relation yields another proof of Theorem 1 for simple p.l. transformations T because $\int_0^1 g(T(x))\,dx = \int_0^1 g(x)\,dx$ is true for all these g if and only if $\boldsymbol{c} = \boldsymbol{1}$.

Note that also the expression for the integral in the proof of Proposition 10 implies another proof of Proposition 4 for simple p.l. and u.d.p. transformations $T: I \to I$. For if $\boldsymbol{A}$ is a stochastic matrix then $\boldsymbol{b}'\boldsymbol{A}^k\boldsymbol{1} = \boldsymbol{b}'\boldsymbol{1} = 1/2$.

We now show that the conditions of Proposition 4 are not sufficient for a map $T: I \to I$ to be a u.d.p. transformation. The point of departure for the remainder of this section are some basic results of the theory of Markov matrices which enables us to determine under which conditions the sequences

$$\Big\{\int_0^1 T^{(k)}(x)\,dx\Big\}_{k=1}^{\infty}, \quad \Big\{\sum_{I_{j,i}^{(k)} \subset J_s} |I_{j,i}^{(k)}|\Big\}_{k=1}^{\infty}$$

converge. It is known [5] that if $\boldsymbol{Q}$ is a Markov matrix then either $\lim_{k\to\infty} \boldsymbol{Q}^k = \boldsymbol{Q}^\infty$ or there exists a positive integer h such that $\lim_{k\to\infty} \boldsymbol{Q}^{hk} = (\boldsymbol{Q}^h)^\infty$. In either case the sequence of the averages

$$\Big\{\frac{1}{N}\sum_{k=0}^{N-1} \boldsymbol{Q}^k\Big\}_{N=1}^{\infty}$$

converges. In particular, if $\boldsymbol{Q}$ is an irreducible and primitive Markov matrix then

$$\lim_{k\to\infty} \boldsymbol{Q}^k = \boldsymbol{Q}^\infty$$

where $\boldsymbol{Q}' = \boldsymbol{p}\boldsymbol{1}'$, $\boldsymbol{Q}\boldsymbol{p} = \boldsymbol{p}$, $\boldsymbol{p}'\boldsymbol{1} = 1$ and $p_i > 0$ for $i = 1, \ldots, n$.

Suppose $n \geq 3$ is given. Let $\boldsymbol{p}$ be one of the solutions of the system

$$\boldsymbol{a}'\boldsymbol{B}\boldsymbol{x} = 1/2, \quad \boldsymbol{1}'\boldsymbol{x} = 1, \quad x_i \geq 0. \tag{9}$$

The set of these solutions is infinite and $\boldsymbol{a}$ is one of them. Then for the matrix $\boldsymbol{Q} = \boldsymbol{p}\boldsymbol{1}'$ we have

$$\boldsymbol{a}'\boldsymbol{B}\boldsymbol{Q}^k\boldsymbol{a} = 1/2.$$

Proposition 10 implies that a simple p.l. transformation $T: I \to I$ corresponding to the just chosen $\boldsymbol{Q}$ fullfils Proposition 4 for all n. However, if $\boldsymbol{p} \neq \boldsymbol{a}$ then this T is not a u.d.p. transformation for the corresponding matrix $\boldsymbol{A}$ is not stochastic what can be readily verified.

To be concrete, let $n = 3$ and $J_j = ((j-1)/3, j/3)$ for $j = 1, 2, 3$. Then $\boldsymbol{a}' = (1/3, 1/3, 1/3)$ and one can take $\boldsymbol{p}' = (1/4, 1/2, 1/4)$. Let further $n_1 = n_2 = n_3 = 3$. Then the intervals

$$\begin{aligned} &I_{1,1} = (0, 1/12), && I_{2,1} = (1/12, 1/4), && I_{3,1} = (1/4, 1/3),\\ &I_{3,2} = (1/3, 5/12), && I_{2,2} = (5/12, 7/12), && I_{1,2} = (7/12, 8/12),\\ &I_{1,3} = (8/12, 9/12), && I_{2,3} = (9/12, 11/12), && I_{3,3} = (11/12, 1) \end{aligned}$$

form the abscissa decomposition of a (unique) continuous p.l. function T for which

$$\int_0^1 T^{(k)}(x)\,dx = 1/2 \quad \text{for all } k = 1, 2, \ldots \tag{10}$$

but

$$|T^{-1}(J_2)| \neq |J_2|$$

and so T is not a u.d.p. transformation.

Note that in the case $n = 2$ the vector $\boldsymbol{a}$ is the only solution of (9). Therefore if T is p.l. and simple with $n = 2$ then T is a u.d.p. transformation if and only if (10) is true.

We conclude this section with a characterization of simple p.l. transformations T with a u.d. orbit

$$x,\ T(x),\ T^{(2)}(x),\ \ldots \tag{11}$$

for at least one $x \in I$.

THEOREM 5. *Let T be a simple p.l. transformation of I onto I with the ordinate and abcissa decomposition* (8.1–2) *and A the matrix assigned to T above. Then there exists an $x \in I$ for which the orbit* (11) *is u.d. in I if and only if*

(a) *A is a stochastic and irreducible matrix,*

(b) $\{J_j\colon 1 \leqslant j \leqslant n\} \neq \{I_{j,i}\colon 1 \leqslant i \leqslant n_j,\ 1 \leqslant j \leqslant n\}$.

Moreover, if (11) *is u.d. for one $x \in I$ then this is true for almost all $x \in I$.*

Proof. If (11) is a u.d. sequence for a simple p.l. map T of I onto I then Proposition 5 (b) implies that T is a u.d.p. transformation. Proposition 9 (b) yields in turn that A is consequently a stochastic matrix.

Using the known reformulation of the notion of irreducibility of a matrix ([5], p. 281) in term of directed graphs it is enough to show for the irreducibility of A that for every couple of intervals J_i, J_j there exists a positive integer k and a $t \in J_i$ with $T^{(k)}(t) \in J_j$. But this can readily be seen for the orbit (11) is dense in I and therefore there exist positive integers s, k with $T^{(s)}(x) \in J_i$ and $T^{(k)}(x) \in J_j$.

To see (b) note that the equality between the ordinate and abscissa decomposition implies that the ordinate decomposition splits into cycles of the form

$$J_{j_1} \xrightarrow{T} J_{j_2} \xrightarrow{T} \ldots \xrightarrow{T} J_{j_s} = J_{j_1} \tag{12}$$

with the property that the contraction T/J_{j_i} is linear and $T(J_{j_i}) = J_{j_{i+1}}$ for $i = 1, \ldots, s-1$. This in turn implies the existence of a positive integer k such that $T^{(k)}(x) = x$ for all interior points x of intervals (8.1). This contradiction with the density of orbit (11) finishes the proof of the necessity.

Conversely, suppose that (a) and (b) are satisfied. First of all we show that

$$\lim_{k \to \infty} \max |I_{j,i}^{(k)}| = 0. \tag{13}$$

A look on the graph of $T^{(k)}$ shows that

$$|I_{j,i}^{(k)}| = |I_{j,i_1}| \cdot \frac{|I_{s,w}^{(k-1)}|}{|J_s|}$$

where $I_{j,i_1} \subset J_s$. Thus we can write

$$\frac{|I_{j,i}^{(k)}|}{|J_j|} = \frac{|I_{j,i_1}|}{|J_j|} \cdot \frac{|I_{s,i_2}|}{|J_s|} \cdot \frac{|I_{t,i_3}|}{|J_t|} \ldots$$

where $I_{j,i_1} \subset J_s$, $I_{s,i_2} \subset J_t$, ... and the product on the right-hand side contains exactly k factors of the indicated form. Since A is a stochastic, none of these factors exceeds 1. Then the relation (13) will follow if we prove that the number

of consecutive factors equal to 1 cannot be greater than n. To see this note that the equality $|I_{p,q}| = |J_p|$ implies $n_p = 1$ and this undoubtedly the existence of a cycle (12) if the number of consecutive factors equal to 1 is greater than n. The existence of such cycle exhausting all the system (8.1) leads to a contradiction with the assumption (b). A shorter cycle contradicts the irreducibility of A.

Secondly we show that

$$\frac{|A \cap I_1|}{|I_1|} = \frac{|A \cap T(I_1)|}{|T(I_1)|} \tag{14}$$

provided $A \subset I$ is a Lebesgue-measurable T-invariant set (i.e. $T^{-1}(A) = A$) and I_1 a subinterval of I on which T is linear. To prove this suppose that the endpoints of I_1 are α, β and that $T/I_1 = ax+b$. Then with χ the indicator of A we have

$$|A \cap I_1| = \int_{\alpha}^{\beta} \chi(ax+b)\,dx = \frac{1}{|a|} \int_{a\alpha+b}^{a\beta+b} \chi(t)\,dt = \frac{1}{|a|} |A \cap T(I_1)|$$

and (14) follows.

We now derive from (14) that if $A \subset I$ is a Lebesgue-measurable T-invariant set with a non-zero measure, then A is of the full measure. This would imply that T is ergodic (more precisely the ergodicity is equivalent to (a) and (b)) and this in turn implies that (11) is u.d. for almost all $x \in I$.

Suppose therefore that $|A| > 0$. It is known that for a measurable set almost all its points are density points [8]. This means that one can find a point $x \in I$ not an endpoint of the intervals (8.1–2) such that to every $\varepsilon > 0$ the relation (13) implies the existence of an interval $I_{j,i}^{(k)}$ containing x and satisfying

$$\frac{|A \cap I_{j,i}^{(k)}|}{|I_{j,i}^{(k)}|} > 1 - \varepsilon.$$

Since $T^{(k)}(I_{j,i}^{(k)}) = J_j$ and $T^{(k)}/I_{j,i}^{(k)}$ is linear, this together with (14) gives that also

$$\frac{|A \cap J_j|}{|J_j|} > 1 - \varepsilon.$$

To this J_j there exists $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$ such that $\lim_{\varepsilon \to 0} \varepsilon_1 = 0$ and

$$\frac{|A \cap I_{s,i}|}{|I_{s,i}|} > 1 - \varepsilon_1 \quad \text{for every } I_{s,i} \subset J_j.$$

This together with (14) imply that

$$\frac{|A \cap J_s|}{|J_s|} > 1 - \varepsilon_1$$

for every J_s connected with J_j through a directed edge in the directed graph of matrix A. Similarly there exists an $\varepsilon_2 > 0$ such that $\lim_{\varepsilon\to 0} \varepsilon_2 = 0$ and

$$\frac{|A \cap J_m|}{|J_m|} > 1 - \varepsilon_2$$

for every J_m which is in the above mentioned graph connected with J_j through a directed path of length 2, etc. After a finite number of steps we find, say, an ε_s with $\lim_{\varepsilon\to 0} \varepsilon_s = 0$ and

$$\frac{|A \cap J_k|}{|J_k|} > 1 - \varepsilon_s$$

for every $k = 1, 2, \ldots, n$. Consequently $|A| = 1$ and the proof of theorem is finished.

Note that the conditions (a), (b) are satisfied for instance if A is a stochastic irreducible and regular matrix.

Let $T\colon I \to I$ be a simple p.l. transformation. Let X denote the set of all those points of I which are not endpoints of any of the intervals in (8.3–4) for all k. On X we can define a function which assigns to every $x \in X$ a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers defined through

(15) $$T^{(i)}(x) \in J_{k_i} \quad \text{for every } i = 1, 2, \ldots$$

One sees immediately that the image of X under this function is the set of such sequences $\{k_i\}_{i=1}^{\infty}$ for which the sets

(16) $$\{s\colon s = 1, \ldots, n_k \text{ with } I_{k_{i+1},s} \subset J_{k_i}\}$$

are non-empty for every $i = 1, 2, \ldots$ (i.e. $p_{k_{i+1}k_i} > 0$).

The system of all such sequences $\{k_i\}_{i=1}^{\infty}$ together with the shift transformation

$$\{k_i\}_{i=1}^{\infty} \to \{k_i\}_{i=2}^{\infty}$$

is the subject of the theory of symbolic dynamical systems. Motivated by the standpoint of this theory, we shall characterize the behaviour of orbits of elements of X under the simple p.l. transformations which satisfy the following additional conditions:

(i) the cardinality of the set $\{i\colon I_{j,i} \subset J_s,\ i = 1, 2, \ldots, n_j\}$ is at most 1 for every $j, s = 1, 2, \ldots, n$,

(ii) $\lim_{k\to r} \max |I_{j,i}^{(k)}| = 0$.

For the sake of simplicity we shall call simple p.l. transformations of I to itself satisfying these two conditions (i), (ii) *strictly simple*. Plainly, if T is strictly simple then the function defined in (15) is injective.

Given a strictly simple transformation, we can reindex the intervals $I_{j,i}$ in such a way that

$$I_{j,i} \subset J_i \quad \text{and} \quad T(I_{j,i}) = J_j.$$

The same can be done for intervals $I_{j,i}^{(k)}$ corresponding to the kth iteration $T^{(k)}$ of T. More precisely, we can assume that $I^{(k)}_{j,j_2,\ldots,j_{k+1}}, I^{(k)}_{t,j_2,\ldots,j_{k+1}}, I^{(k)}_{u,j_2,\ldots,j_{k+1}}, \ldots$ are the dividing subintervals of $I^{(k-1)}_{j_2,j_3,\ldots,j_{k+1}}$, $T^{(k)}/I^{(k)}_{j_1,j_2,\ldots,j_{k+1}}$ is linear and $T^{(k)}(I^{(k)}_{j_1,j_2,\ldots,j_{k+1}}) = J_{j_1}$, $T^{(k-1)}(I^{(k)}_{j_1,j_2,\ldots,j_{k+1}}) = I_{j_1,j_2}$, etc.

Using this notation we have

(17) $$\begin{aligned} x \in I^{(k)}_{j_1,j_2,\ldots,j_{k+1}} &\Leftrightarrow \big(T^{(k)}(x) \in J_{j_1} \text{ and } x \in I^{(k-1)}_{j_2,j_3,\ldots,j_{k+1}}\big)\\ &\Leftrightarrow \big(T^{(k)}(x) \in J_{j_1} \text{ and } T^{(k-1)}(x) \in J_{j_2} \text{ and} \ldots \text{and } x \in J_{j_{k+1}}\big). \end{aligned}$$

Therefore

(18) $$\frac{|I^{(k)}_{j_1,j_2,\ldots,j_{k+1}}|}{|J_{j_1}|} = \frac{|I_{j_1,j_2}|}{|J_{j_1}|} \cdot \frac{|I^{(k-1)}_{j_2,j_3,\ldots,j_{k+1}}|}{|J_{j_2}|} = \frac{|I_{j_1,j_2}|}{|J_{j_1}|} \cdot \frac{|I_{j_2,j_3}|}{|J_{j_2}|} \cdots \frac{|I_{j_k,j_{k+1}}|}{|J_{j_k}|}.$$

We shall apply the foregoing to explicit determination of points of X in terms of $I_{j,i}$'s and the sequence $\{k_i\}_{i=1}^{\infty}$. If $T\colon I \to I$ is a strictly simple p.l. transformation and if (16) holds then

$$x = \sum_{i=1}^{k_1-1} |J_i| + \sum\nolimits_{1,u} |I_{u,k_1}| + \sum\nolimits_{2,v}^{(2)} |I^{(2)}_{v,k_2,k_1}| + \sum\nolimits_{3,z}^{(3)} |I^{(3)}_{z,k_3,k_2,k_1}| + \ldots$$

for every $x \in X$ where $\{k_i\}_{i=1}^{\infty}$ is determined by (15) and $\sum_1$ denotes the sum over those intervals I_{u,k_1} which lie on the left of I_{k_2,k_1}, $\sum_2^{(2)}$ over those $I^{(2)}_{v,k_2,k_1}$ which lie on the left of $I^{(2)}_{k_3,k_2,k_1}$, etc. Then (18) gives

(19) $$\begin{aligned} x = {}& |J_1| + |J_2| + \ldots + |J_{k_1-1}| + \sum\nolimits_1 |I_{u,k_1}|\\ &+ \frac{|I_{k_2,k_1}|}{|J_{k_2}|} \cdot \sum\nolimits_2 |I_{t,k_2}| + \frac{|I_{k_2,k_1}|}{|J_{k_2}|} \cdot \frac{|I_{k_3,k_2}|}{|J_{k_3}|} \cdot \sum\nolimits_3 |I_{s,k_3}| + \ldots \end{aligned}$$

Here in the summation $\sum_2$ the intervals run over those I_{t,k_2} which are on the left of I_{k_3,k_2} if the slope of T in $I_{k_2,k_1} \times J_{k_2}$ is positive, otherwise the intervals are taken from the right of I_{k_3,k_2}. Similarly, in $\sum_3$ the intervals I_{s,k_3} are taken from the left of I_{k_4,k_3} if the product of slopes in $I_{k_2,k_1} \times J_{k_2}$ and $I_{k_3,k_2} \times J_{k_3}$ is positive and from the left otherwise, etc.

We can now state the following two results:

(a) An $x \in X$ is a fixed point of a strictly simple p.l. transformation T if and only if the sequence $\{k_i\}_{i=1}^{\infty}$ defined through (15) is constant. If, moreover, all the slopes of T are positive then the fixed points of T belonging to X are given through the formula

$$|J_1|+|J_2|+\ldots+|J_{k-1}|+\frac{|J_k|}{|J_k|-|I_{k,k}|}\cdot\sum\nolimits_1 |I_{u,k}|$$

with $k=1,2,\ldots,n$ satisfying the condition $p_{kk}>0$.

(b) An $x\in X$ is a periodical point of a strictly simple p.l. transformation if and only if the sequence $\{k_i\}_{i=1}^{\infty}$ is periodical. If, moreover, all the slopes of T are positive then the periodical points from X of order s are of the form

$$|J_1|+|J_2|+\ldots+|J_{k_1-1}|$$
$$+\frac{|J_{k_1}|\cdot|J_{k_2}|\ldots|J_{k_s}|}{|J_{k_1}|\cdot|J_{k_2}|\ldots|J_{k_s}|-|I_{k_2,k_1}|\cdot|I_{k_3,k_2}|\ldots|I_{k_s,k_{s-1}}|\cdot|I_{k_1,k_s}|}$$
$$\times\Big(\sum\nolimits_1 |I_{u,k_1}|+\frac{|I_{k_2,k_1}|}{|J_{k_2}|}\cdot\sum\nolimits_2 |I_{t,k_2}|+\frac{|I_{k_2,k_1}|}{|J_{k_2}|}\cdot\frac{|I_{k_3,k_2}|}{|J_{k_3}|}\cdot\sum\nolimits_3 |I_{z,k_3}|+\ldots\Big),$$

where the sum in the parentheses has s terms and the s-tuple $k_1, k_2, \ldots, k_s$ satisfies the conditions $p_{k_{i+1},k_i}>0$ for $i=1,2,\ldots,s-1$.

The characterization of $x\in X$ with u.d. orbit is more complex. To do this we shall denote by $d(A)$ the asymptotic density of the set A.

Proposition 11. *Let $T\colon I\to I$ be a strictly simple p.l. transformation. The orbit x, $T(x)$, $T^{(2)}(x)$, … of an element $x\in X$ is u.d. in I if and only if for every s and every $(s+1)$-tuple $j_1, j_2, \ldots, j_{s+1}$ of positive integers satisfying $p_{j_i,j_{i+1}}>0$ for $i=1,2,\ldots,s$ we have*

(20) $$d(\{n\colon k_{n+s}=j_1,\ k_{n+s-1}=j_2,\ \ldots,\ k_{n-1}=j_{s+1}\}) = |J_{j_1}|\cdot\frac{|I_{j_1,j_2}|}{|J_{j_1}|}\cdot\frac{|I_{j_2,j_3}|}{|J_{j_2}|}\cdots\frac{|I_{j_s,j_{s+1}}|}{|J_{j_s}|}.$$

Proof. Since $|I^{(s)}_{j_1,j_2,\ldots,j_{s+1}}|\to 0$ for $s\to\infty$, the standard approximation argument shows that the uniform distribution of our orbit is equivalent with the equality

$$d(\{n\colon T^{(n)}(x)\in I^{(s)}_{j_1,j_2,\ldots,j_{s+1}}\}) = |I^{(s)}_{j_1,j_2,\ldots,j_{s+1}}|$$

for every sufficiently large positive integer s and every $(s+1)$-tuple $j_1, j_2, \ldots, j_{s+1}$ with $p_{j_i,j_{i+1}}>0$. But then using (17) and (18) the assertion of proposition follows immediately.

Note that in the case of simple p.l. transformations the intervals J_j are uniquely determined by lengths of $I_{j,i}$'s, so that in the foregoing formulae the lengths of J_j's can be eliminated.

We close this section with the following application. Let $T\colon I\to I$ be a strictly simple p.l. transformation for which

(i) $n_j=n$ for every $j=1,2,\ldots,n$, $|I_{j,i}|=1/n^2$ for every $i=1,2,\ldots,n$ (consequently $|J_j|=1/n$ for every j),

(ii) the graph of T in every rectangle $I_{j,i}\times J_j$ has a positive slope.

No assumptions are made about the location of intervals $I_{1,i}, I_{2,i}, \ldots, I_{n,i}$ within J_i which can be arbitrary for every $i=1,2,\ldots,n$. However, let $n_{j,i}$ denote the number of them which lie on the left of $I_{j,i}$. Then owing to (19) every $x\in X$ with $\{k_i\}_{i=1}^{\infty}$ defined in (15) has the following representation in the scale of n

$$x=0.(k_1-1)\,n_{k_2,k_1}\,n_{k_3,k_2}\,n_{k_4,k_3}\ldots$$

According to (20) the number x has a u.d. orbit if and only if the number

$$\alpha=0.(k_1-1)(k_2-1)(k_3-1)\ldots$$

is normal in the scale of n.

One interesting prototype of a transformation satisfying (i) and (ii) we obtain when the intervals $I_{1,i}, I_{2,i}, \ldots, I_{n,i}$ are ordered from the left to the right. Then namely,

$$T(x)=\{nx\} \quad\text{and}\quad x=\alpha.$$

4. Two topological properties. In this section we endow the set of the all transformations $T\colon I\times I$ with the supremum metric

$$d(T,G)=\sup|T(x)-G(x)|.$$

From the topological point of view it is known that every u.d. sequence in I is dense in I. In the next lines we shall construct a map of I to itself which preserve the all dense sequences in I but which is not a u.d.p. transformation. To do this we shall modify slightly a function of [9]. Let $\{(r_n,s_n)\}_{n=1}^{\infty}$ be a one-to-one sequence of the all distinct couples of rational numbers from I. Define

$$G(x)=\begin{cases} x & \text{if } x=0 \text{ or } x=1,\\ r_n & \text{if } x\in\langle 1/(n+1),1/n) \text{ and } x \text{ is rational},\\ s_n & \text{if } x\in\langle 1/(n+1),1/n) \text{ and } x \text{ is irrational}.\end{cases}$$

It can be shown (see [9] for details) that G transforms every dense sequence of I into a dense sequence whereas G is discontinuous at every point of I and therefore not Riemann-integrable.

Theorem 6. *The system of the all u.d.p. transformations is nowhere dense in the space M of the all maps of I to itself.*

Proof. Let $F\in M$ and $x_0\in I$, $F(x_0)\neq 1$. Let $K(F,\varepsilon)$ be an open ball in M

with $0 < \varepsilon < 1 - F(x_0)$. Define the map $G\colon I \to I$ as follows:

$$G(x) = \begin{cases} F(x_0) + \varepsilon/3 & \text{if } |F(x) - F(x_0)| < \varepsilon/3, \\ F(x) & \text{otherwise.} \end{cases}$$

Plainly,

$$|G(x) - F(x_0)| \geqslant \varepsilon/3 \quad \text{and} \quad |G(x) - F(x)| < 2\varepsilon/3$$

for every $x \in I$.

We claim that the open ball $K(G, \varepsilon/9)$ does not contain a u.d.p. transformation. Let $H \in K(G, \varepsilon/9)$ and

$$I_1 = \big(F(x_0) - \varepsilon/9,\ F(x_0) + \varepsilon/9\big).$$

Then $H^{-1}(I_1)$ is an empty set which is impossible for u.d.p. transformations according to Theorem 4. The proof of theorem is thus finished.

Note that by Theorem 4 the u.d.p. transformations F share the following property: for every open subinterval $I_1 \subset I$ the condition $F^{-1}(I_1) \neq \emptyset$ implies that the interior $\mathrm{Int}\big(F^{-1}(I_1)\big)$ is also non-empty (i.e. that the u.d.p. transformations are somewhat continuous [3]).

THEOREM 7. *The system of the all u.d.p. transformations is a perfect set in M, the space of the all maps of I to itself.*

Proof. We have to verify two conditions:

(a) The system of the all u.d.p. transformations is closed in M. But this the content of Proposition 2.

(b) It is an everywhere dense set in M. Given a u.d.p. transformation F, we have to construct a sequence $\{F_n\}_{n=1}^{\infty}$, $F_n \neq F$ for all n, of u.d.p. transformations converging to F in M. But this can be easily done. Let $x_0 \in I$ be such that $F(x_0) \neq 1$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad F(x_0) + \varepsilon_n \leqslant 1 \quad \text{for all } n.$$

Then define

$$F_n(x) = \begin{cases} F(x) & \text{if } x \in I \text{ and } x \neq x_0, \\ F(x_0) + \varepsilon_n & \text{if } x = x_0 \end{cases}$$

and the required conclusion follows immediately.

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