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On the difference between consecutive squarefree integers

by

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1. Introduction. Let S_n be the nth squarefree integer. Results of the form $S_{n+1}-S_n = O(n^{\alpha+\varepsilon})$ have been obtained by Fogels [1] ($\alpha = 2/5$), Roth [6] ($\alpha = 3/13$), Richert [5] ($\alpha = 2/9$), Rankin [4] ($\alpha = 0.2219821\ldots$) and P. G. Schmidt ($\alpha = 0.2215834\ldots$). In this article, we prove $\alpha \leqslant 1057/4785 = 0.2208\ldots$

2. Notation. We use the letter D, with and without a subscript, to mean a domain which is bounded by $O(1)$ algebraic curves of bounded degree. $|D|$ is a number greater than or equal to the number of lattice points in D. We use the standard notation $e(w)$ to denote $e^{2\pi i w}$, $f(x) \ll g(x)$ denotes $f(x) = O(g(x))$, $f(x) \lll g(x)$ denotes $f(x) = O(x^{\varepsilon} g(x))$, $f(x) \approx g(x)$ denotes $f(x) = g(x)(1+o(1))$, and for a small positive number Δ, $f(x) \underset{\Delta}{\sim} g(x)$ means

$$\frac{\partial^i f}{\partial x^i} = \frac{\partial^i g}{\partial x^i} + O\left(\Delta \max\left|\frac{\partial^i g}{\partial x^i}\right|\right)$$

for all $x = (x_1, \ldots, x_n)$ and all $i = (i_1, \ldots, i_n)$ for which it makes sense.

Finally, $\psi(x) = x - [x] - 1/2$ and

$$H_j(f(x)) = \det\left[\frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}\right]$$

for $x = (x_1, \ldots, x_n)$ and $j \leqslant n$.

3. Reduction to exponential sums. Most of the arguments in this section are adapted from Richert [5]; we include them here for the sake of completeness.

To prove $S_{n+1} - S_n \lll n^{\alpha}$ it suffices to show that

$$\sum_{x-h<n\leqslant x} \mu^2(n) = 6h/\pi^2 + O(x^{\alpha+\varepsilon}) + o(h), \tag{1}$$

whenever $1 \leqslant h \ll x$. The left-hand side of (1) may be written as

$$\sum_{d\leqslant\sqrt{x}} \mu(d)\left([x/d^2] - [(x-h)/d^2]\right).$$

The above is equal to

$$mh\sum_{d\le t}\mu(d)d^{-2}+O(t)+O(S)=6h/\pi^2+O(ht^{-1}+t+S),$$

where $t=x^\alpha$ and

$$S=\sum_{t<d\le\sqrt{x}}([xd^{-2}]-[(x-h)d^{-2}])$$
$$=\sum_{t<d\le x^{1/3}}\left(\psi\left(\frac{x-h}{d^2}\right)-\psi\left(\frac{x}{d^2}\right)\right)+O(h/t)+\sum_{x^{1/3}<d\le\sqrt{x}}([xd^{-2}]-[(x-h)d^{-2}]).$$

The last sum equals

$$\sum_{x^{1/3}<d\le x^{1/2}}\ \sum_{x-h<nd^2\le x}1=\sum_{n\le x^{1/3}}\ \sum_{\sqrt{(x-h)/n}<d\le\sqrt{x/n}}1$$
$$=\sum_{n\le x^{1/3}}([\sqrt{x/n}]-[\sqrt{(x-h)/n}])-\sum_{n\le x^{1/3}}\ \sum_{\sqrt{(x-h)/n}<d\le x^{1/3}}1$$
$$\ll hx^{-1/3}+t+\sum_{t<n\le x^{1/3}}\left(\psi(\sqrt{x/n})-\psi\left(((x-h)/n)^{1/2}\right)\right)$$
$$+\sum_{(x-h)x^{-2/3}<n\le x^{1/3}}\left(1+x^{1/3}-((x-h)x^{-1/3})^{1/2}\right)$$
$$\ll t+hx^{-1/3}+\sum_{t<n\le x^{1/3}}\left(\psi(\sqrt{x/n})-\psi\left(((x-h)/n)^{1/2}\right)\right).$$

Denoting

$$S_1=\max_{t\le N\le x^{1/3}}\ \max_{x-h\le y\le x}\Big|\sum_{N\le n\le 2N}\psi(yn^{-2})\Big|,$$
$$S_2=\max_{t\le N\le x^{1/3}}\ \max_{x-h\le y\le x}\Big|\sum_{N\le n\le 2N}\psi(\sqrt{y/n})\Big|$$

and combining all of the above, we get

$$\sum_{x-h<n\le x}\mu^2(n)=6h/\pi^2+O(x^\alpha+hx^{-\alpha}+S_1+S_2).$$

To complete our proof, we need to show that

$$\Big|\sum_{N\le n\le 2N}\psi(yn^{-\beta})\Big|\ll x^\alpha \tag{2}$$

for $x^\alpha\le N\le x^{1/3}$, $\beta=2$ and $x-h\le y\le x$ (or $\beta=1/2$ and $\sqrt{(x-h)}\le y\le\sqrt{x}$).

The right-hand side of (2) can be converted into exponential sum by means of the following

LEMMA A. *Suppose $K>0$. There is a function $\psi^*(x)$ such that*

(i) $\psi^*(x)=\sum_{1\le k\le K}\gamma^*(k)e(kx)$;

(ii) $\gamma^*(k)\ll 1/k$ *and* $(\gamma^*(k))'\ll 1/k^2$;

(iii) $\psi^*(x)-\psi(x)\le(2K+2)^{-1}\sum_{k\le K}(1-k/K)e(kx)$.

The lemma is contained in Lemma 18 of [7] (in that paper, $\gamma^*(k)$ is given explicitly). If we set $\gamma(k)=\gamma^*(k)+(1-k/K)(K+2)^{-1}$, then we get

$$\psi(w)\le 1/(2K+2)+\sum_{1\le k\le K}\gamma(k)e(kw),$$

and so

$$\Big|\sum_{N<n\le 2N}\psi(yn^{-\beta})\Big|\le N/(2K+2)+\Big|\sum_{1\le k\le K}\gamma(k)\sum_{N<n\le 2N}e(kyn^{-\beta})\Big|.$$

By splitting the sum over k into subsums, we get

$$\Big|\sum_{N<n\le 2N}\psi(yn^{-\beta})\Big|\ll N/K+\max_{H\le K}\Big|\sum_{H\le k\le 2H}\gamma(k)\sum_{N<n\le 2N}e(kyn^{-\beta})\Big|.$$

Our next step is to remove the γ-factor by partial summation. Regard γ as a function of a real variable w. We shall write

$$V(H;w)=\sum_{H<k\le w}\ \sum_{N<n\le 2N}e(kyn^{-\beta}),$$

and when H is understood we shall simply write $V(w)$. With this convention, we get

$$\sum_{H<k\le 2H}\gamma(k)\sum_{N<n\le 2N}e(kyn^{-\beta})$$
$$=\int_H^{2H}\gamma(w)\,dV(w)=[\gamma(w)V(w)]_H^{2H}-\int_H^{2H}\gamma^*(w)V(w)\,dw$$
$$\ll(1/H)\max_{w\le 2H}|V(w)|.$$

The same bound holds for the corresponding sum over negative k, so

$$\Big|\sum_{N<n\le 2N}\psi(yn^{-\beta})\Big|\ll N/K+\max_{H\le K}\max_{H_1\le 2H}(1/H)|V(H,H_1)|.$$

We will complete the proof by obtaining an upper bound for $|V(H,H_1)|$.

4. Basic lemmas. We need 8 lemmas. Lemma 1 is Lemma 1 of [2]. Lemma 2 can be proved similarly to Lemma 1 of [3]; it will be used instead of Lemma 1 in some cases when Lemma 1 cannot be used or is not good enough. Lemma 3 is the generalization of Weyl–van der Corput's inequality (see, for example, Lemma 2 of [3]) and Lemma 4 is Lemma 3 of [3], Lemma 4 will be used in Lemmas 6, 7, 8 and Theorem 1 to choose parameters Q and Q_1.

LEMMA 1. *Let $f(x)$ be a real C^∞ function such that for all $x\in D\subseteq\{x: X_i\le x_i\le 2X_i\ (i=1,\dots,n)\}$ we have:*

(i) $|f^{i}(x)| \ll F X_1^{-i} \ldots X_n^{-i}$ *for some* F;

(ii) $|H_k(f(x))| \sim M_k^{-1}$ *for some* $k \in [1, n]$ *and some* $M_k \ll (X_1 \ldots X_k)^2 \times F^{1/3-k-\varepsilon_0}$;

(iii) *for all* $a_{i_l} \in [-C, C]$ *the functions* $\sum_{i} \prod_{l \le C} a_{i_l} f^{i_l}(x)$ *do not change sign, where* $i = (i_i, \ldots, i_c)$, $i_l = (i_{1l}, \ldots, i_{nl})$, $i_{jl} \le C$.

Then there exist some real numbers B_{ij} *such that for*

$$D_1 = \{(m_1, \ldots, m_k, x_{k+1}, \ldots, x_n):\ B_{1j} \le m_j \le B_{2j}\ (j = 1, \ldots, k);\ x \in D\}$$
$$\cap \{(m_1, \ldots, m_k, x_{k+1}, \ldots, x_n):\ f_{x_i}(x) = m_i,\ x \in D,\ B_{3j} \le x_j \le B_{4j}$$
$$(i = 1, \ldots, k;\ j = 1, \ldots, n)\}$$

we have

$$\Big|\sum_{x\in D} e(f(x))\Big| \ll \sqrt{M_k}\Big|\sum_{(m, x_{k+1}, \ldots, x_n)\in D_1} e\big(f_1(m, x_{k+1}, \ldots, x_n)\big)\Big|$$
$$+O\big(N^{1+\varepsilon}(F^{k-1} M_k (X_1 \ldots X_k)^{-2})^{1/2}\big)$$
$$\ll D/\sqrt{M_k} + N\big(M_k F^{k-1} (X_2 \ldots X_k)^{-2}\big)^{1/2}$$
$$+N\big((F^{k-1} M_k (X_1 \ldots X_k)^{-2}\big)^{1/2}\big),$$

where $N = X_1 \ldots X_n$, $f_1 = f(\varphi_1, \ldots, \varphi_k, x_{k+1}, \ldots, x_n) - m_1 x_1 - \ldots - m_k x_k$, $f_{x_i}(\varphi_1, \ldots, \varphi_k, x_{k+1}, \ldots, x_n) = m_i$ $(i = 1, \ldots, k)$, $f(x) \sim F$ *means that* $F \ll f \ll F$.

Lemma 2. *Let* $f(x_1, x_2)$ *be a real* C^∞ *function on* $D \subset \{(x_1, x_2):\ X_1 \le x_1 \le 2X_1,\ X_2 \le x_2 \le 2X_2\}$ *such that for large* X_1, X_2, M_1, M_2 *and* $N = X_1 X_2$ *we have*

(i) $|f_{x_1^2}| \sim M_1^{-1}$; $|f_{x_1 x_2}| \ll X_1/(X_2 M_1)$;

(ii) $|f_{x_1^k}| \ll U^k/(M_1 X^k)$, *where* $M_1 U^2 \log N \ll X_1^2$;

(iii) $|H_2(f(x))| \sim M_2^{-1}$.

Then

$$\Big|\sum_{x\in D} e(f(x))\Big| \ll |D|/\sqrt{M_2} + X_1\sqrt{M_2}/M_1 + (X_1 \log N)/M_1 + X_2\sqrt{M_1} + X_2 \log N.$$

Lemma 3. *Let* $f(x)$ *be a real function and let* D *be a subdomain of*

$$\{x:\ X_i \ll x_i \ll X_i \quad (i = 1, \ldots, n)\}.$$

Let $q_1, \ldots, q_k$ *be numbers such that*

$$q_1/X_1 = \ldots = q_k/X_k = Q/N; \quad q_1 \ldots q_k = Q \le |D|^2/N; \quad N = X_1 \ldots X_n;$$

$$f_1(x; h) = \int_0^1 \frac{\partial}{\partial t}(f(x+ht))\,dt; \quad D_1 = \{(x, h):\ h \ne 0,\ |h_j| \le q_j,\ x \in D,\ x+h \in D\};$$

$$S = N^{-1}\sum_{x\in D} e(f(x)), \quad S_1 = (NQ)^{-1}\sum_{(x,h)\in D_1} e(f(x)).$$

Then

$$|S| \ll Q^{-1/2} + (N^{-1}|D||S_1|)^{1/2}.$$

Lemma 4. *Let* M, N *be positive integers,* $u_m \ge 0$, $v_n \ge 0$, $A_m \ge 0$, $B_n \ge 0$. *Then for some* q

$$\sum_{1\le m\le M} A_m q^{u_m} + \sum_{1\le n\le N} B_n q^{v_n} \ll \sum_{1\le n\le N}\sum_{1\le m\le M} (A_m^{v_n} B_n^{u_m})^{1/(u_m+v_n)}.$$

Lemma 5. *Let* $f(x)$ *be a real* C^3 *function such that for* $X \le x \le X_1 \le 2X$ *we have*

$$|f''| \sim \lambda_2, \quad |f'''| \sim \lambda_3.$$

Then

$$S = \Big|\sum_{X\le x\le X_1} e(f)\Big| \ll (X_1 - X)/\sqrt{\lambda_2} + \min\{1/\sqrt{\lambda_2},\ 1/\lambda_3^{1/3}\}$$

and

$$|S| \ll (X_1 - X)\lambda_3^{1/6} + (X_1 - X)^{3/4}.$$

Lemma 5 is van der Corput's estimate.

Lemma 6. *Let* $f(x, y)$ *be a* C^∞ *function such that for* $(x, y) \in D \subset \{(x, y):\ X \le x \le 2X,\ Y \le y \le 2Y\}$ *and* $N = XY$ *we have*

(i) $|f_{x^i y^j}| \ll F X^{-i} Y^{-j}(U^{2j-3}+1) + F\Delta X^{-i} Y^{-j}(U^{2i+2j-3}+1)$,
$\Delta \ll 1$, $U \ge 1$, $\Delta U^3 \ll 1$;

(ii) $f_{x^i} = (F X^{-\alpha} Y^{-\beta} x^\alpha y^\beta)_{x^i} + O\big(F\Delta(U^{2i-3}+1)X^{-i}\big)$, *where*
$\alpha(\alpha-1)(\alpha-2)(\alpha-3)\beta(\beta-1)(\beta-2)(\alpha+\beta-2)(\alpha+\beta-3) \ne 0$;

(iii) $|H_2(f_x)| \sim F^2 a_0 X^{-4} Y^{-2}$, $|f_{x^3} f_{x^3 y} - f_{x^4} f_{x^2 y}| \gg F^2 X^{-6} Y^{-1}$, $|f_{x^i y}| \ll F U^2 X^{-i} Y^{-1}$.

Then

$$S = \Big|\sum_{(x,y)\in D} e(f(x, y))\Big|$$
$$\ll |D|\big((F^2 U^3 N^{-3})^{1/6}$$
$$+(X F^{-2} Y^{-3})^{1/22} + (F U^5 N^{-2} a_0^{-1})^{1/8} + (F^2 N X^{-8})^{1/6}\big)$$
$$+(|D|^5 N^2 Y/F)^{1/8} + (|D|\,Y)^{1/2} + (|D|^4 F^2 X^{-2} N^{-1})^{1/6}$$
$$+(|D|^8 F^2 N X^{-4} a_0^{-2})^{1/22} + (|D|^4 N^7 F^{-2})^{1/10}.$$

Proof. We apply Lemma 3 with $k = 2$ and obtain:

$$|S|^2 \ll |D|^2/Q + (|D|/Q)\sum_h\Big|\sum_{x,y} e(f_1(x, y))\Big|,$$

where

$$f_1(x, y) = h_1 f_x + h_2 f_y + f_2(x, y),$$
$$|(f_2(x, y))_{x^i y^j}| \ll \varrho^2 F U^{2j+1} X^{-i} Y^{-j}, \quad \varrho = |h_1|/X + |h_2|/Y,$$

Q is a parameter, to be chosen later to our advantage.

We divide D into subdomains of the type

$$D_a = \{(x, y)\colon\ a_1 \leqslant |h_1 f_{x^3} + h_2 f_{x^2 y}| \leqslant 2a_1,$$
$$a_2 \leqslant |H_2(h_1 f_x + h_2 f_y)| \leqslant 2a_2,\ (x+h_1,\ y+h_2) \in D\}$$

and assume that $a = (a_1, a_2)$ is such that

$$S^2 \ll |D|^2/Q + (|D|/Q) \sum_h \Big| \sum_{(x,y)\in D_a} e(f_1(x, y)) \Big|.$$

We consider the following cases:

1) $a_1 \ll (F^2 Q X^{-5-\varepsilon_0} Y^{-1})^{1/4} + (FYX^{-2}|D|^{-1})^{1/2}$, $\varepsilon_0 > 0$.

Using the condition (iii) of the lemma, one can verify that

$$\text{either} \quad a_1 \gg F\varrho X^{-2} \quad \text{or} \quad |h_1 f_{x^3} + h_2 f_{x^2 y}| \gg F\varrho X^{-3}.$$

Applying Lemma 4, we obtain:

$$S^2 \ll |D|^2/Q (|D|/Q) \sum_h \sum_y \big(\sqrt{a_1} \sum_x 1 + X(F\varrho)^{-1/3}\big)$$
$$\ll |D|^2/Q + |D|^2 (X^2 (a_1^3 N/Q)^{1/2}/F + (a_1 N/Q)^{1/2}/X) + |D| N (F^2 Q/N)^{-1/4}.$$

2) $a_1 \gg (F^2 Q X^{-5-\varepsilon_0} Y^{-1})^{1/4} + (FYX^{-2}/|D|)^{1/2}$,

$$a_2 \ll (F^6 Q a_0^2 X^{-4} |D|^{-4} N^{-5})^{1/4} + F^2 U^5 N^{-2} \sqrt{Q^3 N^{-3}}.$$

Since

$$(1/Q) \sum_h \sum_{(x,y)\in D_a} 1 \ll (|D|/X) \sqrt{N/Q} \big(1 + (a_2 X^4 Y^2 F^{-2}/a_0)^{1/2}\big),$$

we as in 1) use Lemma 4 and obtain

$$S^2 \ll |D|^2/Q + (|D|^2/X)(F^2 Q/N)^{1/4} (Y/(XQ))^{1/2} \big(1 + (a_2 X^4 Y^2 F^{-2}/a_0)^{1/2}\big)$$
$$+ |D| N (F^2 Q/N)^{-1/6}.$$

3) a_1 satisfies the same, a_2 satisfies the opposite inequality to as in 2). We use Lemma 2:

$$S^2 \ll |D|^2/Q + |D| (|D| \sqrt{a_2} + X a_1/\sqrt{a_2} + X a_1 \log N + Y/\sqrt[3]{a_1} + Y \log N).$$

Combining the obtained in 1)–3) estimates and using Lemma 4 to choose Q which minimizes the obtained expression, we obtain the claimed result.

LEMMA 7. *Let $f(x, y)$ be as in Lemma 6 and satisfy the condition*

$$f_{x^i y^j} = (F X^{-\alpha} Y^{-\beta} x^\alpha y^\beta)_{x^i y^j} + O\big(F\Delta (U^{2i+2j-3} + 1) X^{-i} Y^{-j}\big)$$

where

$$\alpha(\alpha-1)\beta(\beta-1)(\alpha+\beta-1)(\alpha+\beta-2) \neq 0.$$

Then

$$S = \Big| \sum_{(x,y)\in D} e(f(x, y)) \Big| \ll |D| (F^4 X^{-10} Y^{-3})^{1/20} + (|D|\, Y)^{1/2}$$
$$+ (|D|^4 X^8 Y^5 F^{-2})^{1/12} + (|D|^{11} F^2 X^{-5})^{1/12}$$
$$+ (X^6 Y^3 F^{-2})^{1/4} + (|D|^2 X^8 Y^3 F^{-2})^{1/8}.$$

Proof. We apply Lemma 3 with $k = 1$ and get:

$$S^2 \ll |D|^2/Q + (|D|/Q) \sum_{1\leqslant h\leqslant Q} \Big| \sum_{(x,y)\in D_1} e(f_1(x, y)) \Big|,$$

where

$$Q \leqslant |D|/Y, \quad D_1 = \{(x, y) \in D\colon (x+h, y) \in D\}, \quad f_1(x, y) = f(x+h, y) - f(x, y).$$

Now we apply Lemma 1 with $k = 1$ to the sum over x and Lemma 5 to the sum over y after that and obtain:

$$S^2 \ll |D|^2/Q + |D| (X^3/(FQ))^{1/2} (FQY^3/X)^{1/6} + |D|^2 (FQX^{-3})^{1/2} (FQ/(Y^3 X))^{1/6}$$
$$+ |D|\, Y^{3/4} (F^3/(FQ))^{1/2} + X^{1/4} |D|^{7/4} (FQX^{-3})^{1/2}.$$

Choosing Q to minimize the obtained above expression, we complete the proof of the lemma.

LEMMA 8. *Let $(\alpha, \beta, \gamma, \delta) = (1/7, 3/7, 4/7, 3/7)$ or $(2/8, 3/8, 5/8, 3/8)$. Let Δ be small, X_4 be large, $X_1 \geqslant X_2 \geqslant X_3 \geqslant X_4$, $X_1 X_2 X_3 X_4 = N$, $A X_1^\alpha X_2^\beta X_3^\gamma X_4^\delta = F$, and let $f(x) \underset{\Delta}{\sim} A x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta$, $D \subset \{x\colon X_i \leqslant x_i \leqslant 2X_i$ $(i = 1, 2, 3, 4)\}$. Then*

$$S = \Big| \sum_{x\in D} e(f(x)) \Big| \ll N \min \{(F^5 X_1^{-11} X_2^{-8} X_3^{-3})^{1/39} + \alpha_5^{1/4};$$
$$(F^{24} X_1^{-44} X_2^{-44} X_3^{-10} X_4^{-3})^{1/152} + \alpha_6^{1/4}\}$$
$$+ N\big((\alpha_0 + \alpha_1)^{1/2} + (\alpha_2 + \alpha_3 + \alpha_4)^{1/4}\big),$$

where $\alpha_0, \ldots, \alpha_6$ are defined in (3), (4), (6)–(10).

Proof. We apply Lemma 3 and obtain:

$$S^2 \ll N^2/Q + (N/Q) \sum_{h_1,h_2} \Big| \sum_{x\in D_h} e(g(x)) \Big|,$$

where

$$g(x) = f(x_1+h_1, x_2+h_2, x_3, x_4) - f(x),$$
$$D_h = \{x \in D\colon (x_1+h_1, x_2+h_2, x_3, x_4) \in D\}.$$

We assume that $h_1 h_2 \neq 0$ (otherwise the sum is easier to estimate) and divide D_h into subdomains of the type

$$D_a = \{x \in D_h\colon\ a_1 \leqslant X_1^2 X_2^2 F^{-2} \varrho_0^{-2} |H_2(h_1 f_{x_1} + h_2 f_{x_2})| \leqslant 2a_1,$$
$$a_2 \leqslant X_1^3 F^{-1} \varrho_0^{-1} |h_1 f_{x_1^4} + h_2 f_{x_1^3 x_2}| \leqslant 2a_2\},$$

where

$$\varrho_0 = |h_1|/X_1 + |h_2|/X_2.$$

We take one of them such that

$$S^2 \ll N^2/Q + (N/Q) \sum_h \Big| \sum_{x \in D_a} e(g(x)) \Big|.$$

If we denote $a_0 = \min\{a_1, a_2\}$, then one can show that

$$|D_a| \ll N a_0 + 1 \quad \text{and,} \quad \text{if } a_1 \ll 1, \quad \text{then } a_2 \gg 1,$$
$$|g_{x_1^3}| \sim F\varrho_0 X_1^{-3}, \quad |g_{x_1 x_2^2}| \sim F\varrho_0 X_1^{-1} X_2^{-2}, \quad |g_{x_1^k}| \ll F\varrho_0 X_1^{-k}.$$

If $a_1 \ll \Delta + (X_1 Q^7 F^{-2}/X_2)^{1/2}$, then we use Lemma 3 with $k = 1$ and van der Corput's estimate to get

$$S_1^2 = \Big| \sum_{x \in D_a} e(g(x)) \Big|^2 \ll |D_a|^2/Q_1 + (|D_a|/Q_1) \sum_{1 \leqslant h_3 \leqslant Q_1} \Big| \sum_x e(g_1(x)) \Big|,$$

where

$$g_1(x) = g(x_1 + h_3, x_2, x_3, x_4) - g(x),$$
$$|(g_1(x))_{x_1^2}| \sim h_3 |g_{x_1^3}|, \quad Q_1 \ll a_0^2 X_1 X_2,$$

so that (by van der Corput's estimate)

$$S_1/N \ll a_0^2/Q_1 + (a_0/(NQ_1)) \sum_{h_3} \big(|D_a| (h_3 F\varrho_0 X_1^{-3})^{1/2} + N(h_3 F\varrho_0 X_1^{-1})^{-1/2}\big)$$
$$\ll 1/(X_1 X_2) + a_0^{3/2}/\sqrt{X}_1 + a_0^2 (F\varrho_0 X_1^{-3})^{1/3} + (FX_2\varrho_0)^{-1/2}$$

and

(3) $$(S/N)^2 \ll (X_1 X_2)^{-1/2} + \Delta^{3/4} X_1^{-1/2} + (\Delta^{12} F^2 X_1^{-7} X_2^{-1})^{1/13}$$
$$+ (F^{10} X_1 X_2^7)^{-1/55} + (F^6 X_2^3 X_1^{-1})^{-1/29} + (FX_2)^{-1/4}$$
$$+ (X_1^{11} F^{-24} X_2^{-12})^{1/88} + (X_1^{21} F^{-48} X_2^{-25})^{1/178}$$
$$+ (\Delta^6 X_2^{-1} X_1^{-3})^{1/10} = \alpha_0.$$

Above we have used Lemma 4 to choose Q, Q_1 which minimize the estimate.

If

$$\Delta + (Q^7 X_1 F^{-2} X_2^{-1})^{1/2} \ll a_1 \ll (F^8 X_1^{-2} X_2^{-18} X_3^{-9} X_4^{-9})^{1/72},$$

then we use Lemma 2 and get

$$(S_1/N)^2 \ll a_0^2/Q_1 + (a_0/Q_1) \sum_{1 \leqslant h_3 \leqslant Q_1} \big(a_0 (a_1 F^2 \varrho_0^2 h_3^2 X_1^{-4} X_2^{-2})^{1/2}\big)$$
$$+ a_0^{-1/2}/X_1 + (Fh_3 \varrho_0/X_1)^{-1/2}.$$

Choosing Q_1 and Q to our advantage, we obtain

(4) $$(S/N)^2 \ll X_1^{-1/2} + (FX_2)^{-1/4} + (F^8 X_1^{-14} X_2^{-14} X_3^{-3} X_4^{-3})^{1/24}$$
$$+ (F^8 X_1^{-18} X_2^{-34} X_3^{-9} X_4^{-9})^{1/96} = \alpha_1.$$

If $a_1 \gg (F^8 X_1^{-2} X_2^{-18} X_3^{-9} X_4^{-9})^{1/72}$ then we apply Lemma 3 with $k = 2$ and obtain:

$$(S/N)^4 \ll Q^{-2} + |D_a| Q^{-3} N^{-2} \sum_{h_1, h_2} \sum_{h_3, h_4} \Big| \sum_{x \in D_{a,h}} e(\phi(x)) \Big|,$$

where

$$\phi(x) = g(x_1 + h_3, x_2 + h_4, x_3, x_4) - g(x), \quad |h_3| \leqslant q_3, \quad |h_4| \leqslant q_4,$$
$$q_3/X_1 = q_4/X_2 = Q/(X_1 X_2)^{1/2},$$
$$D_{a,h} = \{x \in D_a\colon (x_1 + h_3, x_2 + h_4, x_3, x_4) \in D_a\}.$$

Denoting

$$\phi_0(\theta) = \alpha(\alpha-1)\theta^2 + \alpha\beta\theta\sigma_1 + \beta(\beta-1)\sigma_2,$$
$$\sigma_1 = h_2/h_1 + h_4/h_3, \quad \sigma_2 = h_2 h_4/(h_1 h_3),$$
$$\phi_1(\theta) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)\theta^2 + \alpha(\alpha-1)(\alpha-2)\beta\theta\sigma_1 + \beta(\beta-1)(\beta-2)(\beta-3)\sigma_2,$$
$$\phi_2(\theta) = \alpha(\alpha-1)(\alpha-2)(\beta-1)\beta\theta^2 + \alpha\beta(\beta-1)(\beta-2)\theta\sigma_1 + \beta(\beta-1)(\beta-2)(\beta-3)\sigma_2,$$
$$\phi_3(\theta) = \alpha\beta(\alpha-1)(\alpha-2)\theta^2 + \alpha\beta(\alpha-1)(\beta-1)\theta\sigma_1 + \alpha\beta(\beta-1)(\beta-2)\sigma_2,$$
$$\phi_4 = \phi_1\phi_2 - \phi_3^2,$$
$$\phi_5(\theta) = \alpha(\alpha-1)(\alpha-2)\theta^2 + \alpha\beta(\alpha-1)\theta\sigma_1 + \beta(\beta-1)(\beta-2)\sigma_2,$$
$$\phi_6(\theta) = \alpha\beta(\alpha-1)\theta^2 + \alpha\beta(\beta-1)\theta\sigma_1 + \beta(\beta-1)(\beta-2)\sigma_2,$$
$$\phi_7(\theta) = (2\alpha + 2\beta + \gamma - 6)\phi_0\phi_1 + (\alpha + \beta + \gamma - 3)\phi_4,$$
$$\theta = x/y, \quad \varrho_1 = |h_3|/X_1 + |h_4|/X_2,$$

we subdivide $D_{a,h}$ into subdomains of the form

$$D_b = \{x \in D_{a,h}\colon |\phi_0(\theta)| \sim b_0, |\phi_1(\theta)| \sim b_1, |\phi_2(\theta)| \sim b_2, \ldots, |\phi_7(\theta)| \sim b_7\}.$$

We take a domain D_b such that

$$(S/N)^4 \ll Q^{-2} + |D_a| Q^{-3} N^{-2} \sum_h \Big| \sum_{x \in D_b} e(\phi(x)) \Big|.$$

One can verify that $b_1+b_4 \gg a_1^2$. If $b_4 \ll a_1^2$ and $b_4 \ll (F^2 Q^3 X_1^{-2} X_2^{-2})^{-1/4}$, then $b_1 \gg a_1^2$, and we apply van der Corput's estimate to get

$$(S/N)^4 \lll Q^{-2}+(F^2 Q^3 X_1^{-2} X_2^{-2})^{-1/4}+a_0 Q^{-3} N^{-1} \sum_h |D_b| (F^2 Q^3 X_1^{-6} X_2^{-2})^{1/4}.$$

Since for $b_0 = \min_j \{b_j\}$ we have $|D_b| \ll N\sqrt{b_0}$ and, if $b_2+b_3 \ll 1$ then $|D_b| \ll Nb_1$, we can use the above bounds for $|D_b|$ and, choosing Q appropriately, we get:

$$S/N \ll (F^2 X_1^{-10} X_2^{-2})^{1/38}+(X_1 X_2/F)^{1/5}.$$

If $b_4 \ll a_1^2$ and

(5) $$(F^2 Q^3 X_1^{-2} X_2^{-2})^{-1/4} \ll b_4 \ll (X_1^4 X_2^4 F^{-2} Q^{-7} a_0^{-2})^{1/2} + X_1 F_1^{-1}+(X_1 X_2 Q^{-2})^{-1/6}+\Delta^{1/3}+(X_1 X_2/F_1)^{-2/5}+F_1^{-2/5}$$

(where $F_1 = F\varrho_0\varrho_1$), then $b_1 \gg a_1^2$ and, using Lemma 2 and choosing an optimal Q, we get

(6) $$(S/N)^4 \lll (F^2 X_1^{-10} X_2^{-2})^{2/19}+(F^6 X_1^{-13} X_2^{-13})^{2/23}+X_2^{-1/4} (F^6 \Delta^2 X_1^{-12} X_2^{-12})^{2/21}+(X_1 X_2/F)^{4/5}+(F X_1^{-2} X_2^{-2})^{28/41} +(F^3 X_1^{-8} X_2^{-8})^{4/29} = \alpha_2.$$

Similarly as above, we obtain the same estimate if

$$\min\{b_1, b_2, b_3\} \ll X_1 X_2 F^{-1} Q^{-7/2}.$$

If $b_1 \ll X_1/F_1$ and $b_2 = o(1)$, then one can show that $b_4 \gg 1$ and, using Lemma 1 and choosing an appropriate Q, we obtain

(7) $$(S/N)^4 \lll 1/X_2+F^{-1/2} = \alpha_3.$$

If $b_1 \ll X_1/F_1$ and $b_2 \gg 1$ then $b_4 \gg a_1^2$ and we use Lemma 2 to get

(8) $$(S/N)^4 \lll (X_1 X_2^2)^{-1/4}+(F^2 X_1^{-4} X_2^{-6})^{2/11} = \alpha_4.$$

Now we assume that b_4 is larger than in (5),

$$b_1 \gg X_1/F_1, \quad \min_{i \leq 3} b_i \gg X_1 X_2 F^{-1} Q^{-7/2}.$$

Applying Lemma 1, we obtain

$$(S/N)^4 \lll Q^{-2}+|D_a| Q^{-3} N^{-2} \sum_h M_2^{-1/2} \Big| \sum_{m_1, m_2, x_3, x_4} e(\varphi(m_1, m_2, x_3, x_4))\Big| + R,$$

where

$$M_2^{-1} = F_1^2 b_4 X_1^{-2} X_2^{-2},$$

$$R = (F^2 Q^3 X_1^{-2} X_2^{-2})^{-1/4}+(F^2 Q X_1^{-4} X_2^{-4})^{1/2}+(F^2 Q^3 X_1^{-10} X_2^{-2})^{1/8},$$

$$\varphi = \phi(x_1(m), x_2(m), x_3, x_4)-x_1(m) m_1-x_2(m) m_2,$$

$$\phi_{x_i}(x_1(m), x_2(m), x_3, x_4) = m_i \sim b_5 F_1/X_i \quad (i = 1, 2),$$

$$\varphi_{x_3^i x_4^j} = A(m_1, m_2, h)(x_3^{\gamma_1} x_4^{\delta_1})_{x_3^i x_4^j}+O\big(F_1(\Delta+\varrho_0+\varrho_1) X_3^{-i} X_4^{-j}(1+b_1^{3-2i-2j})\big),$$

$$A(m_1, m_2, h) \sim b_0 F_1 X_3^{-\gamma_1} X_4^{-\delta_1}, \quad \gamma_1 = \gamma/(3-\alpha-\beta), \quad \delta_1 = 1/(3-\alpha-\beta),$$

$$|\varphi_{x_3^i m_2^j}| \ll X_2^j F_1^{1-j} X_3^{-i} b_4^{3-2i-2j}(\Delta+\varrho_0+\varrho_1)+(b_4^{3-2j}+1) X_2^j F_1^{1-j} X_3^{-i},$$

$$|\varphi_{x_3^3}\varphi_{x_3 m_2^2}-(\varphi_{x_3^2 m_2})^2| \sim b_7 X_2^3/(b_4 X_3^3 F_1).$$

Since for $0 \leq \alpha \leq 1$ we have

$$\sum_{m_1, x_4} \Big(\sum_{m_2, x_3} 1\Big)^\alpha \ll (F_1^2 b_1 |D_b| X_1^{-2} X_2^{-2})^\alpha (F_1 X_4/X_1)^{1-\alpha},$$

we can apply Lemma 6 to the sum over m_2 and x_3 or Lemma 7 to the sum over x_3, x_4 and, choosing Q to minimize the obtained expression, we complete the proof of the lemma:

(9) $$(S/N)^4 \lll X_1^{-2} X_2^{-2}+(F^5 X_1^{-11} X_2^{-8} X_3^{-3})^{4/39} +(F^{21} X_1^{-43} X_2^{-44} X_3^{-3})^{4/151}+(F^7 X_1^{-15} X_2^{-13} X_3^{-2})^{4/53} +(F^9 X_1^{-15} X_2^{-16} X_3^{-7})^{4/51}+(F^3 X_1^{-9} X_2^{-4} X_3^{-3})^{4/33} +(F X_1^{-3} X_2^{-2})^{4/11}+(F^{17} X_1^{-38} X_2^{-32} X_3^{-4})^{4/139} +(F^9 X_3 X_1^{-19} X_2^{-20})^{4/67} = (F^5 X_1^{-11} X_2^{-8} X_3^{-3})^{4/39}+\alpha_5$$

and

(10) $$(S/N)^4 \lll (F^{14} X_1^{-26} X_2^{-26} X_3^{-6}/X_4)^{2/45}+(F^{24} X_1^{-44} X_2^{-44} X_3^{-10} X_4^{-3})^{1/38} +(F^{10} X_1^{-22} X_2^{-22} X_4^{-3})^{2/39}+(F^8 X_1^{-16} X_2^{-16} X_3^{-2} X_4^{-3})^{1/14} +(X_1 X_2)^{-2}+(F^6 X_1^{-14} X_2^{-14} X_3^{-4} X_4^{-2})^{2/25} = (F^{24} X_1^{-44} X_2^{-44} X_3^{-10} X_4^{-3})^{1/38}+\alpha_6.$$

4. The main results.

THEOREM 1. *Let* $\alpha = 2$ *and* $X = x$ *or* $\alpha = 1/2$ *and* $X = x^{1/2}$, *and let* $f(k, n) = kXn^{-\alpha}$. *Let* N, X *be large positive numbers satisfying*

$$x^{1057/4785} \leq N \leq x^{1/3}, \quad K \leq N^{1+\varepsilon_0} x^{-1057/4785},$$

and let

$$S = \Big|\sum_{K \leq k \leq K_1 \leq 2K} \sum_{N \leq n \leq N_1 \leq 2N} e(f(k, n))\Big|.$$

Then

$$S \ll K \cdot x^{1057/4785}.$$

Proof. We apply Lemma 1 with $k=1$ and obtain:

$$S \ll NKF^{-1/2} + K + NF^{-1/2} \Big|\sum_k \sum_m e(g(m, k))\Big|,$$

where

$$g(m, k) = (\alpha+1)\alpha^{-\alpha/(\alpha+1)} X^{1/(\alpha+1)} k^{1/(\alpha+1)} m^{\alpha/(\alpha+1)},$$
$$m \sim M = KXN^{-\alpha-1} = F/N.$$

Now we apply Lemmas 3 and 1 with $k=1$ to the sum over m in the sum

$$S_1 = \Big|\sum_{k,m} e(g(m, k))\Big|$$

and obtain

$$S_1^2 \ll K^2 M^2/Q + (KM/Q) \max_{H \leqslant Q} \Big|\sum_{H \leqslant q \leqslant 2H} (M^3/(Fq))^{1/2} \sum_{k \sim K} \sum_{m \sim M} e(g_1(k, m_1, q))\Big| + K^2 M\big(1 + (M^3/(FQ))^{1/2}\big),$$

where

$$g_1(k, m_1, q) \underset{\Delta}{\sim} X_2 (km_1 q^{\alpha+1})^{1/(\alpha+2)},$$
$$X_2^{\alpha+2} \sim X, \quad \Delta = H/M, \quad g_1 \sim F_1 = FH/M, \quad m_1 \sim M_1 = FH/M^2,$$

Q is a parameter, to be chosen later.

We again apply Lemmas 3 and 1 with $k=1$ to the sum over m_1 in the last sum,

$$S_2 = \Big|\sum_{k,q} \sum_{m_1} e(g_1(k, m_1, q))\Big|.$$

We obtain:

$$S_2^2 \ll K^2 M_1^2 H^2/Q_1 + K^2 H^2 M_1 (M_1^3/(Q_1 F_1))^{1/2} + K^2 H^2 M_1 + KM_1 (H/Q_1)(M_1^3/(H_1 F_1))^{1/2} \Big|\sum_{q_1 \sim H_1} \sum_{k, m_2, q} e(g_2(k, m_2, q_1, q))\Big|,$$

where

$$g_2(k, m_2, q_1, q) \underset{\Delta_1}{\sim} X_3 (km_2^{\alpha+1} q_1^{\alpha+2} q^{\alpha+1})^{1/(2\alpha+3)}, \quad \Delta_1 = H/M + H_1/M_1,$$
$$X_3^{2\alpha+3} \sim X, \quad g_2 \sim F_2 = F_1 H_1/M_1, \quad m_2 \sim M_2 = F_2/M_1.$$

The function $g_2(k, m_2, q_1, q)$ satisfies the conditions of Lemma 8 with $k = x_1$, $m_2 = x_2$, $q_1 = x_3$, $q = x_4$ (note that $\alpha_0, \ldots, \alpha_6$ of Lemma 8 are small compared to the principal terms).

Using Lemma 8 (the first part of the minimum if $N \leqslant X^{1438/4785}$ and the second part otherwise) and choosing Q_1 and Q to minimize the obtained expressions, we complete the proof of the theorem.

THEOREM 2. *We have*

$$S_{n+1} - S_n = O(n^{1057/4785+\varepsilon}).$$

Proof. As shown in the introduction, we need to show that for

$$x^{1057/4785} \leqslant N \leqslant x^{1/3} \quad \text{and} \quad H \leqslant H_1 \leqslant 2H \ll Nx^{-1057/4785}$$

we have

$$\Big|\sum_{H \leqslant k \leqslant H_1} \sum_{N < n \leqslant 2N} e(kyn^{-\beta})\Big| \ll Hx^{1057/4785},$$

where $y \sim x$ and $\beta = 2$ or $y \sim x^{1/2}$ and $\beta = 1/2$. The needed estimate was proved in Theorem 1, which proves Theorem 2.

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