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Elementary methods in the theory of L-functions
IX. Density theorems

by

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To Professor Paul Erdős
on the occasion of his 75th birthday

1. In the last paper of this series [11] we used elementary methods and some complex function theory to prove some well-known theorems (that of Landau, Page, Siegel, Tatuzawa) on real zeros of real L-functions. The purpose of this paper is to give simple (we may call quasi-elementary) proofs for various density theorems including the deep density theorems of Halász and Turán.

In the following let $\varrho = \beta + i\gamma$ denote zeros of Dirichlet's L-function,

$$N(\sigma, T, \chi) = \sum_{\substack{L(\varrho,\chi)=0 \\ \beta \geqslant \sigma, |\gamma| \leqslant T}} 1 \tag{1.1}$$

and let $\sum\limits_{\chi(q)}^{*}$ denote summation over all primitive characters mod q.

Our first result will be

THEOREM 1. *For $\sigma \geqslant 3/4$, $T \geqslant 2$ we have*

$$\sum_{q \leqslant Q} \sum_{\chi(q)}^{*} N(\sigma, T, \chi) \ll Q^{16(1-\sigma)} T^{1600(1-\sigma)^{3/2}} \log^{19}(QT). \tag{1.2}$$

This was proved (in a slightly weaker form) by Halász and Turán [5] for $Q = 1$, but their method can be generalized for L-functions too [6]. Their result was proved later in a simpler way by Bombieri [1] and Montgomery [8], Corollary 12.5.

Supposing the Generalized Lindelöf Hypothesis (GLH)

$$L(\sigma + it, q, \chi) \ll_{\varepsilon} \big(q(|t|+2)\big)^{\varepsilon} \quad \text{if} \quad \sigma \geqslant 1/2, \quad |\sigma + it - 1| \geqslant 1/4 \tag{1.3}$$

for all characters of arbitrary modulus and arbitrary $\varepsilon > 0$ we obtain

THEOREM 2. *Assume GLH. Then for $\sigma > 3/4$, $T \geqslant 2$ and any $\varepsilon > 0$ we have*

$$\sum_{q \leqslant Q} \sum_{\chi(q)}^{*} N(\sigma, T, \chi) \ll_{\sigma,\varepsilon} (QT)^{\varepsilon}. \tag{1.4}$$

This might be expressed in the following more precise form.

THEOREM 2′. *If* (1.3) *holds with* ε^2 *in place of* ε *and for all* $q \leqslant Q^2$, *then*

$$\sum_{q \leqslant Q} \sum_{\chi(q)}^{*} N(3/4+\varepsilon,\, T,\, \chi) \ll_{\varepsilon} (QT)^{\varepsilon}. \tag{1.5}$$

This was also shown (again in a slightly weaker form) by Halász and Turán [6] (for every Q). A simpler proof (for $Q = 1$) was given by Montgomery [8], Theorem 12.3.

Finallly we give a short proof for the following

THEOREM 3. *For* $\sigma > 1/2$, $T \geqslant 2$ *we have*

$$\sum_{q \leqslant Q} \sum_{\chi(q)}^{*} N(\sigma,\, T,\, \chi) \ll (Q^2 T)^{4\sigma(1-\sigma)} \log^7 (QT). \tag{1.6}$$

Our method is similar to that of Montgomery [8] and Karatsuba [7] but it reminds also for the method used in the works [9] and [10] of the author. Naturally the notion of a zero of $\zeta(s)$ in the critical strip already assumes complex function theory, so we cannot hope for completely elementary proofs. We use the inequality

$$N(\sigma,\, T+1,\, \chi) - N(\sigma,\, T,\, \chi) \ll \log q(T+2) \qquad (\sigma \geqslant 1/2) \tag{1.7}$$

which follows by Jensen's inequality. (It would be interesting to show (1.7) in an elementary way.) Accordingly we shall show Theorems 3′ and 4′ in place of Theorem 3 and 4 (for Theorems 3′, 4 and 4′ see Section 2).

But our crucial Theorem 4′ really follows in a simple elementary way from Lemmas A, B and 1 (see Section 3). Lemma 1 naturally uses the notion of a complex function but can be proved in a simple way without using contour integration. For the sake of completeness we present proofs for both Lemmas A and B. Theorem 3′ follows from the large sieve Lemma C and the standard Lemmas 1 and 2. Here again Lemma 2 can be proved in a relatively simple way without contour integration; further it can be avoided if we are contented with an estimate of the form $(Q^2 T)^{4(1-\sigma)}$ in place of $(Q^2 T)^{4\sigma(1-\sigma)}$.

In order to derive Theorem 1 from Theorem 4 we need only trigonometric sums estimate, e.g. an estimate of type ($2 \leqslant N < N' \leqslant 2N$, $|t| \geqslant \sqrt{N}$, $0 < w \leqslant 1$)

$$\sum_{N \leqslant n < N'} (n+w)^{-it} \ll N^{1-c(\log N/\log|t|)^2} \tag{1.8}$$

which follows by Vinogradov's method (cf. [4], Theorem 10.3).

In order to obtain explicit constants and for the sake of simplicity we used the bound (2.4) of Richert [12].

Finally to deduce Theorem 2 from Theorem 4 we use a standard contour integration technique. This might be avoided if we replace the Generalized Lindelöf Hypothesis by the following equivalent form of it (cf. (2.3)):

$$\sum_{n=1}^{N} \frac{\chi(n)}{n^{1/2+it}} \ll_{\varepsilon} \big(q(|t|+2)\big)^{\varepsilon} + \frac{E(\chi)\sqrt{N}}{|t|+1} \qquad \text{for any } \varepsilon > 0. \tag{1.9}$$

2. Theorems 1 and 2 will follow from

THEOREM 4. *Let us suppose that with a real* $\alpha \geqslant 0$ *and* $M = M(\alpha, Q, T) \geqslant 1$ *we have the estimate*

$$\sum_{n=1}^{N} \frac{\chi(n)}{n^{it}} \ll_{\alpha} N^{\alpha} M + \frac{E(\chi) N}{|t|+1} \tag{2.1}$$

for every $|t| \leqslant T$ *and every character* $\chi \bmod q \leqslant Q^2$ *where* $E(\chi) = 1$ *if* χ *is principal and* $E(\chi) = 0$ *if* χ *is non-principal. Then for* $\sigma > (1+\alpha)/2$, $T \geqslant 2$ *we have*

$$\sum_{q \leqslant Q} \sum_{\chi(q)}^{*} N(\sigma,\, T,\, \chi) \ll_{\sigma,\alpha} M^{4(1-\sigma)/(2\sigma-1-\alpha)} \big(\log(MQT)\big)^{20(1-\sigma)/(2\sigma-1-\alpha)+7}. \tag{2.2}$$

Remark. By Abel's inequality we see that (2.1) implies the same estimate if we replace the exponent it by $\delta + it$, $\delta = \delta(\chi) \geqslant 0$.

Let us denote by $N'(\sigma, T, \chi)$ the maximal number R of zeros $\{\varrho_\nu\}_{\nu=1}^{R}$ of $L(s, \chi)$ with the property $\beta_\nu \geqslant \sigma$, $|\gamma_\nu| \leqslant T$ ($\nu = 1, 2, \ldots, R$) and with $|\gamma_\nu - \gamma_\mu| \geqslant 1$ if $1 \leqslant \nu < \mu \leqslant R$. Then, by Jensen's inequality, Theorem 4 is an easy consequence of

THEOREM 4′. *Theorem* 4 *is valid if* $N(\sigma, T, \chi)$ *is replaced by* $N'(\sigma, T, \chi)$ *and* 7 *by* 6 *in the exponent of* $\log(MQT)$.

Analogously, instead of Theorem 3 it is sufficient to prove

THEOREM 3′. *Theorem* 3 *is valid if* $N(\sigma, T, \chi)$ *is replaced by* $N'(\sigma, T, \chi)$ *and* $\log^7(QT)$ *by* $\log^6(QT)$.

Now we shall show that Theorem 4 implies Theorems 1 and 2. First we remark that by Abel's inequality (2.1) will follow if we can show that

$$\sum_{n=1}^{N} \frac{\chi(n)}{n^{\alpha+it}} \ll_{\alpha} M + \frac{E(\chi) N^{1-\alpha}}{|t|+1}. \tag{2.3}$$

This is trivially true if $|t| \leqslant 2$, independently of M. On the other hand for $|t| \geqslant 2$ we have for any $w \in (0, 1]$, and $K > 0$

$$\sum_{m=1}^{K} (m+w)^{-\alpha-it} \leqslant C_1 |t|^{100(1-\alpha)^{3/2}} \log^{2/3} |t| \tag{2.4}$$

from the work of Richert [12]. By (2.4) we have

$$(2.5)\qquad \Big|\sum_{n=1}^{N}\frac{\chi(n)}{n^{\alpha+it}}\Big| \leqslant \sum_{n\leqslant q}\frac{1}{n^{\alpha}}+\frac{1}{q^{\alpha}}\sum_{l=1}^{q}\Big|\sum_{m=1}^{[(N-l)/q]}\frac{1}{(m+l/q)^{\alpha+it}}\Big|$$

$$\ll_{\alpha} q^{1-\alpha}+q^{1-\alpha}|t|^{100(1-\alpha)^{3/2}}\log^{2/3}|t|$$

$$\ll_{\alpha} Q^{2(1-\alpha)}\,T^{100(1-\alpha)^{3/2}}\log^{2/3}T \stackrel{\text{def}}{=} M(\alpha, Q, T)$$

if $q \leqslant Q^2$, $|t| \leqslant T$. Choosing $\alpha = 4\sigma-3$ we obtain Theorem 1 since

$$1-\alpha = 4(1-\sigma) \quad\text{and}\quad 2\sigma-1-\alpha = 2(1-\sigma).$$

In order to prove Theorem 2′ let $\sigma = 3/4+\varepsilon$, $\alpha = 1/2+2\varepsilon^2$ with an $\varepsilon \leqslant 1/4$. Then $4(1-\sigma) \leqslant 1$, $2\sigma-\alpha-1 = 2\varepsilon-2\varepsilon^2 \geqslant 3\varepsilon/2$. Using Perron's formula with $K = (NT)^{5/4}Q^{2\varepsilon^2}$ we obtain for any χ mod $q \leqslant Q^2$ and $|t| \leqslant T$:

$$(2.6)\qquad \sum_{n=1}^{N}\frac{\chi(n)}{n^{it}} = \frac{1}{2\pi i}\int_{9/8-iK}^{9/8+iK} L(it+w, \chi)\frac{N^w}{w}\,dw+O(1).$$

Transforming the path of integration to $\sigma = 1/2$ we get

$$(2.7)\qquad \sum_{n=1}^{N}\frac{\chi(n)}{n^{it}} = \frac{E(\chi)\varphi(q)N^{1-it}}{q(1-it)}+\frac{1}{2\pi i}\int_{1/2-iK}^{1/2+iK} L(it+w, \chi)\frac{N^w}{w}\,dw+O(1).$$

Here the integral is

$$(2.8)\qquad \ll_{\varepsilon} N^{1/2}(qK)^{\varepsilon^2}\log K \ll_{\varepsilon} N^{1/2+2\varepsilon^2}(Q^2T)^{4\varepsilon^2/3} \stackrel{\text{def}}{=} N^{\alpha}M.$$

Therefore we have by Theorem 4

$$(2.9)\qquad \sum_{q\leqslant Q}\sideset{}{^*}\sum_{\chi(q)} N(\sigma, T, \chi) \ll_{\varepsilon} (Q^2T)^{\frac{4\varepsilon^2/3}{3\varepsilon/2}}\big(\log(QT)\big)^{\frac{5}{3\varepsilon/2}+7} \ll_{\varepsilon} (Q^2T)^{\varepsilon}.$$

3. We shall need the following lemmas.

LEMMA A (Bombieri [2]). *If $\xi, \varphi_1, \ldots, \varphi_R$ are elements of an inner product space over the field of complex numbers, then*

$$(3.1)\qquad \sum_{\nu=1}^{R}|(\xi, \varphi_\nu)|^2 \leqslant \|\xi\|^2 \max_{1\leqslant\nu\leqslant R}\sum_{\mu=1}^{R}|(\varphi_\nu, \varphi_\mu)|.$$

LEMMA B. *Let $S(s, \chi) = \sum_{n=1}^{N} a_n\chi(n)n^{-s}$ and let $\mathscr{A}$ be a finite set of pairs $(s, \chi) = (\sigma+it, \chi)$ where χ is mod $q \leqslant Q$ primitive, $|t| \leqslant T$ $(T \geqslant 2)$, $\sigma \geqslant \sigma_0 \geqslant 0$ for all pairs and $|t-t'| \geqslant 1$ for any distinct pairs $(\sigma+it, \chi)$ and $(\sigma'+it', \chi)$ in $\mathscr{A}$. Supposing* (2.1) *we have*

$$(3.2)\qquad \sum_{(s,\chi)\in\mathscr{A}}|S(s, \chi)|^2 \ll (N\log T+|\mathscr{A}|N^{\alpha}M)\sum_{n=1}^{N}\frac{|a_n|^2}{n^{2\sigma_0}}.$$

LEMMA C. *Let us suppose the conditions of Lemma* B, *with the exception of* (2.1). *Then we have*

$$(3.3)\qquad \sum_{(s,\chi)\in\mathscr{A}}|S(s, \chi)|^2 \ll (N+Q^2T)\log N\sum_{n=1}^{N}\frac{|a_n|^2}{n^{2\sigma_0}}\Big(1+\log\frac{\log 2N}{\log 2n}\Big).$$

LEMMA 1. *For $\sigma \geqslant 1/2$, $x \geqslant |t|$ we have*

$$(3.4)\qquad \sum_{n\leqslant x}\frac{1}{n^s} = \zeta(s)+\frac{x^{1-s}}{1-s}+O(x^{-\sigma}).$$

LEMMA 2. *If χ is non-principal* mod q, $\sigma \geqslant 1/2$, $x \geqslant q(|t|+1)$ *then*

$$(3.5)\qquad \sum_{n\leqslant x}\frac{\chi(n)}{n^s} = L(s, \chi)+O(qx^{-\sigma}).$$

For Lemmas 1 and 2 see Theorem 4.11 of [13] and Lemma 2 in § 26 of [3], resp. Lemma B is very similar to Theorem 8.4 of [8], whilst Lemma C is Theorem 7.5 of [8]. We prove here Lemmas A and B needed in the proofs of Theorems 1 and 2.

Proof of Lemma A. For arbitrary complex numbers c_ν

$$(3.6)\qquad \Big|\sum_{\nu=1}^{R}c_\nu(\xi, \varphi_\nu)\Big|^2 = \Big|\Big(\xi, \sum_{\nu=1}^{R}\overline{c_\nu}\varphi_\nu\Big)\Big|^2 \leqslant \|\xi\|^2\Big\|\sum_{\nu=1}^{R}\overline{c_\nu}\varphi_\nu\Big\|^2$$

$$= \|\xi\|^2\sum_{\nu=1}^{R}\sum_{\mu=1}^{R}\overline{c_\nu}c_\mu(\varphi_\nu, \varphi_\mu)$$

$$\leqslant \|\xi\|^2\sum_{\nu=1}^{R}|c_\nu|^2\sum_{\mu=1}^{R}|(\varphi_\nu, \varphi_\mu)|$$

$$\leqslant \|\xi\|^2\sum_{\nu=1}^{R}|c_\nu|^2\max_{1\leqslant\nu\leqslant R}\sum_{\mu=1}^{R}|(\varphi_\nu, \varphi_\mu)|.$$

Taking $c_\nu = \overline{(\xi, \varphi_\nu)}$ we obtain

$$(3.7)\qquad \Big(\sum_{\nu=1}^{R}|(\xi, \varphi_\nu)|^2\Big)^2 \leqslant \|\xi\|^2\sum_{\nu=1}^{R}|(\xi, \varphi_\nu)|^2\max_{1\leqslant\nu\leqslant R}\sum_{\mu=1}^{R}|(\varphi_\nu, \varphi_\mu)|. \;\blacksquare$$

Proof of Lemma B. Let

$$\xi = \{a_n n^{-\sigma_0}\}_1^N, \qquad \varphi_{s,\chi} = \{\overline{\chi}(n)n^{-(\sigma-\sigma_0)+it}\}_1^N$$

Then

$$(\xi, \varphi_{s,\chi}) = S(s, \chi), \qquad \|\xi\|^2 = \sum_{n=1}^{N}|a_n|^2 n^{-2\sigma_0}$$

and

$$\sum_{s',\chi'\in\mathscr{A}} |(\varphi_{s_1,\chi_1}, \varphi_{s',\chi'})| = \sum_{s',\chi'\in\mathscr{A}} \left|\sum_{n=1}^{N} \frac{\bar{\chi}_1(n)\chi'(n)}{n^{\sigma_1-\sigma_0+\sigma'-\sigma_0+i(t'-t_1)}}\right|$$
$$\ll |\mathscr{A}|\, N^{\alpha} M + \sum_{(s',\chi_1)\in\mathscr{A}} \left|\sum_{n=1}^{N} \frac{|\chi_1(n)|^2}{n^{\sigma_1-\sigma_0+\sigma'-\sigma_0+i(t'-t_1)}}\right|$$
$$\ll |\mathscr{A}|\, N^{\alpha} M + N\log T. \ \blacksquare$$

4. For the proof of Theorem 4′ let

(4.1) $$\eta = 2\sigma-1-\alpha \ (>0), \quad X = C_{\alpha,\sigma}(M\log^5(MT))^{1/\eta}, \quad \mathscr{L} = \log XT$$

with a sufficiently large constant $C_{\alpha,\sigma}$. The constants implied by the $\ll$ sign in this section may depend on α and σ but not on $C_{\alpha,\sigma}$. Further we define for a zero ϱ of $L(s,\chi)$ with $\beta \geqslant \sigma$

(4.2) $$a_n = \sum_{\substack{d\mid N\\ d\leqslant X}} \mu(d), \qquad H = H(X,\varrho) = \sum_{n\leqslant X^2} \frac{\chi(n)a_n}{n^{\varrho}} = 1 + \sum_{X<n\leqslant X^2} \frac{\chi(n)a_n}{n^{\varrho}}.$$

So we have for any non-principal χ

(4.3) $$|H| = \left|\sum_{d\leqslant X} \frac{\chi(d)\mu(d)}{d^{\varrho}} \sum_{m\leqslant X^2/d} \frac{\chi(m)}{m^{\varrho}}\right| \leqslant \sum_{d\leqslant X} \frac{1}{d^{\beta}} \left|\sum_{m>X^2/d} \frac{\chi(m)}{m^{\varrho}}\right|$$
$$\ll X^{1-\sigma} \int_X^{\infty} \frac{u^{\alpha}M}{u^{1+\sigma}}\,du \ll X^{1-\sigma} M X^{\alpha-\delta} < 1/2$$

if $C_{\alpha,\sigma}$ is large enough.

For $q=1$ ($\chi=\chi_0$) and $|\varrho| > X^{3-3\sigma}\mathscr{L}^2$ we have by Lemma 1

(4.4) $$\sum_{m\leqslant Z} \frac{1}{m^{\varrho}} \ll \frac{Z^{1-\beta}}{|\varrho|} \quad \left(< \frac{1}{X^{1-\sigma}\mathscr{L}^2}\right) \quad \text{if} \quad |\varrho| \leqslant Z \quad \left(= \frac{X^2}{d}\right).$$

If $X^2/d < |\varrho|$, using (4.4) we obtain from (2.1) (where the second term may be deleted if $N \leqslant |t|+1$) similarly to (4.3)

(4.5) $$\sum_{m\leqslant X^2/d} \frac{1}{m^{\varrho}} = \sum_{m\leqslant|\varrho|} \frac{1}{m^{\varrho}} - \sum_{X^2/d<m\leqslant|\varrho|} \frac{1}{m^{\varrho}} \ll \frac{1}{X^{(3/2)(1-\sigma)}\mathscr{L}} + \frac{M}{X^{\sigma-\alpha}}.$$

So we obtain for $\chi=\chi_0$, $\varrho\in\mathscr{A}'$

(4.6) $$H = H(X,\varrho) \ll X^{1-\sigma}\left\{\frac{1}{X^{1-\sigma}\mathscr{L}} + \frac{M}{X^{\sigma-\alpha}}\right\} < \frac{1}{2}$$

where $\mathscr{A}' = \mathscr{A}\setminus\{(\varrho,\chi_0)\in\mathscr{A};\, |\varrho| \leqslant X^{3-3\sigma}\mathscr{L}^2\}$.

By (4.2) and Cauchy's inequality there exist $N\in[X, X^2]$ and $N'\in[N, 3N]$ such that

(4.7) $$\sum_{(\varrho,\chi)\in\mathscr{A}'} \left|\sum_{n=N}^{N'} \frac{\chi(n)a_n}{n^{\varrho}}\right|^2 > \frac{|\mathscr{A}'|}{4\mathscr{L}^2}.$$

Applying Lemma B to the above partial sums we obtain

(4.8) $$\frac{|\mathscr{A}'|}{4\mathscr{L}^2} \ll (N\mathscr{L} + |\mathscr{A}'|\, N^{\alpha} M)\,\mathscr{L}^3 N^{1-2\sigma}$$
$$\ll \mathscr{L}^4 X^{4-4\sigma} + |\mathscr{A}'|\,\mathscr{L}^3 M X^{-\eta}.$$

But if $C_{\alpha,\sigma}$ in (4.1) is large enough then the second term on the right-hand side can be deleted and we obtain

(4.9) $$|\mathscr{A}'| \ll \mathscr{L}^6 X^{4-4\sigma} \Rightarrow |\mathscr{A}| \ll \mathscr{L}^6 X^{4-4\sigma}. \ \blacksquare$$

5. In order to prove Theorem 3 let

(5.1) $$D = Q^2 T, \qquad X = D^{2\sigma-1}, \qquad Y = CD^{2\sigma}, \qquad \mathscr{L} = \log D$$

where C is a sufficiently large absolute constant. The constants implied by the $\ll$ signs are absolute and independent of C. Let us define a_n as in (4.2) and let

(5.2) $$H = H(X,Y,\varrho) = \sum_{n\leqslant Y} \frac{\chi(n)a_n}{n^{\varrho}} = 1 + \sum_{X<n\leqslant Y} \frac{\chi(n)a_n}{n^{\varrho}}.$$

Since $Y/X = CD > QT$ we obtain by Lemma 2 for $\chi\neq\chi_0$

(5.3) $$|H| \leqslant \sum_{d\leqslant X} \frac{1}{d^{\beta}} \left|\sum_{m>Y/d} \frac{\chi(m)}{m^{\varrho}}\right| \ll \sum_{d\leqslant X} \frac{1}{d^{\beta}}\cdot\frac{q}{(Y/d)^{\beta}}$$
$$\ll qXY^{-\sigma} \ll C^{-1/2} D^{1/2+2\sigma-1-2\sigma^2} \ll C^{-1/2} < 1/2.$$

In case of $q=1$, $\chi=\chi_0$ we have here by $Y/d > T$ from (4.4)

(5.4) $$|H(X,Y,\varrho)| \ll \sum_{d\leqslant X} \frac{1}{d^{\beta}}\cdot\frac{Y^{1-\beta}}{d^{1-\beta}|\varrho|} \ll \frac{\mathscr{L}Y^{1-\sigma}}{|\varrho|} < \frac{1}{2}$$

if $\varrho\notin\mathscr{A}' \stackrel{\text{def}}{=} \mathscr{A}\setminus\{(\varrho,\chi_0)\in\mathscr{A};\, |\varrho| \leqslant \mathscr{L}^2 Y^{1-\sigma}\}$. Finally, similarly to (4.7)–(4.8) we get with an $N\in[X, Y]$ from Lemma C

(5.5) $$\frac{|\mathscr{A}'|}{\mathscr{L}^2} \ll \mathscr{L}(D+N)\,\mathscr{L}^3 N^{1-2\sigma} \ll \mathscr{L}^4 (DX^{1-2\sigma} + Y^{2-2\sigma}),$$

(5.6) $$|\mathscr{A}'| \ll \mathscr{L}^6 D^{4\sigma(1-\sigma)} \Rightarrow |\mathscr{A}| \ll \mathscr{L}^6 D^{4\sigma(1-\sigma)}. \ \blacksquare$$

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Nombres hautement composés

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A Paul Erdős
pour fêter son 75ème anniversaire

1. Introduction. Dans un long mémoire paru en 1915, S. Ramanujan a défini et étudié les nombres hautement composés, c'est-à-dire les nombres qui ont strictement plus de diviseurs que les nombres qui les précèdent (cf. [14], n° 15). Appelons $Q(X)$ la quantité de nombres hautement composés inférieurs ou égaux à X. S. Ramanujan a en particulier démontré que

$$\lim_{X \to +\infty} Q(X)/\log X = +\infty$$

mais n'a pas donné de majoration pour $Q(X)$. C'est pourtant un sujet qui l'intéressait puisque dans l'article écrit avec G. H. Hardy: *Asymptotic formulae for the distribution of integers of various types* (cf. [14], n° 34), il étudie les nombres de la forme:

$$2^{a_1} 3^{a_2} \dots p_k^{a_k}$$

où p_k désigne le $k^{\text{ième}}$ nombre premier, et $a_1 \geqslant a_2 \geqslant \dots \geqslant a_k$; parmi les nombres de cette forme figurent les nombres hautement composés.

En 1944, P. Erdős démontre (cf. [3]) qu'il existe $c' > 0$ tel que

$$Q(X) \geqslant (\log X)^{1+c'}$$

pour X assez grand.

Un des outils essentiels dans sa démonstration, est le résultat de Hoheisel:

LEMME 1. *Soit* $\pi(x) = \sum_{p \leqslant x} 1$. *Il existe* $\tau < 1$ *tel que*

$$\pi(x+x^\tau) - \pi(x) \sim x^\tau/\log x.$$

Nous désignerons par τ un nombre réel < 1 pour lequel on a, pour tout x assez grand:

$$\pi(x+x^\tau) - \pi(x) \geqslant x^\tau/\log x. \tag{1}$$