Elementary methods in the theory of \( L \)-functions

IX. Density theorems

by

J. Pintz (Budapest)

To Professor Paul Erdős
on the occasion of his 75th birthday

1. In the last paper of this series [11] we used elementary methods and some complex function theory to prove some well-known theorems (that of Landau, Page, Siegel, Tatuzawa) on real zeros of real \( L \)-functions. The purpose of this paper is to give simple (we may call quasi-elementary) proofs for various density theorems including the deep density theorems of Halász and Turán.

In the following let \( q = \beta + iy \) denote zeros of Dirichlet's \( L \)-function,

\[
N(\sigma, T, \chi) = \sum_{\beta > \sigma, |\beta| \leq T} 1
\]

and let \( \sum^\star \) denote summation over all primitive characters mod \( q \).

Our first result will be

**Theorem 1.** For \( \sigma > 3/4 \), \( T \geq 2 \) we have

\[
\sum_{q \in \mathbb{Q}} \sum^\star N(\sigma, T, \chi) \ll Q^{1+\epsilon} T^{1+600(1-\sigma)^{3/2}} \log^4 (QT).
\]

This was proved (in a slightly weaker form) by Halász and Turán [5] for \( \sigma = 1 \), but their method can be generalized for \( L \)-functions too [6]. Their result was proved later in a simpler way by Bombieri [1] and Montgomery [8], Corollary 12.5.

Supposing the Generalized Lindelöf Hypothesis (GLH)

\[
L(\sigma + it, q, \chi) \ll_q (q(|t|+2))^\varepsilon \quad \text{if} \quad \sigma > 1/2, \quad |\sigma + it - 1| \geq 1/4
\]

for all characters of arbitrary modulus and arbitrary \( \varepsilon > 0 \) we obtain

**Theorem 2.** Assume GLH. Then for \( \sigma > 3/4 \), \( T \geq 2 \) and any \( \varepsilon > 0 \) we have

\[
\sum_{q \in \mathbb{Q}} \sum^\star N(\sigma, T, \chi) \ll_{\varepsilon, \sigma} (QT)^{\varepsilon}.
\]
This might be expressed in the following more precise form.

**Theorem 2.** If (1.3) holds with $e^2$ in place of $e$ and for all $q \leq Q^2$, then

\[ \sum_{q \leq Q} \mathcal{S}(3/4 + \varepsilon, T, \chi) \ll_N (QT)^{\varepsilon}. \]

This was also shown (again in a slightly weaker form) by Halász and Turán [6] (for every $Q$). A simpler proof (for $Q = 1$) was given by Montgomery [8], Theorem 12.3.

Finally we give a short proof for the following

**Theorem 3.** For $\sigma > 1/2$, $T \geq 2$ we have

\[ \sum_{q \leq Q} \mathcal{S}(\sigma, T, \chi) \ll \left( Q^2 T \right)^{4\sigma(1-\sigma) + \log^2 (QT)}. \]

Our method is similar to that of Montgomery [8] and Karatsuba [7] but it reminds also for the method used in the works [9] and [10] of the author. Naturally the notion of a zero of $\zeta(s)$ in the critical strip already assumes complex function theory, so we cannot hope for completely elementary proofs. We use the inequality

\[ N(\sigma, T + 1, \chi) - N(\sigma, T, \chi) \ll \log q(T + 2) \quad (\sigma \geq 1/2) \]

which follows by Jensen's inequality. (It would be interesting to show (1.7) in an elementary way.) Accordingly we shall show Theorems 3' and 4' in place of Theorem 3 and 4 (for Theorems 3', 4 and 4' see Section 2).

But our crucial Theorem 4' really follows in a simple elementary way from Lemmas A, B and 1 (see Section 3). Lemma 1 naturally uses the notion of a complex function but can be proved in a simple way without using contour integration. For the sake of completeness in the present proofs for both Lemmas A and B, Theorem 3 follows from the large sieve Lemma C and the standard Lemmas 1 and 2. Here again Lemma 2 can be proved in a relatively simple way without contour integration; further it can be avoided if we are content with an estimate of the form $(Q^2 T)^{4\sigma(1-\sigma)}$ in place of $(Q^2 T)^{4\sigma(1-\sigma)}$.

In order to derive Theorem 1 from Theorem 4 we need only trigonometric sums estimate, e.g. an estimate of type

\[ \sum_{N \leq x < N'} (n + w)^{-it} \ll N^{1-\epsilon \log N \log |x|^2} \]

which follows by Vinogradov's method (cf. [4], Theorem 10.3).

In order to obtain explicit constants and for the sake of simplicity we used the bound (2.4) of Richert [12].

Finally to deduce Theorem 2 from Theorem 4 we use a standard contour integration technique. This might be avoided if we replace the Generalized Lindelöf Hypothesis by the following equivalent form of it (cf. (2.3)):

\[ \sum_{n=1}^{N} \frac{\chi(n)}{n^{1/2 + \varepsilon}} \ll (q(|t| + 2)^{\varepsilon} + E(\varepsilon) \sqrt{N} \frac{q}{|t| + 1} \]

for any $\varepsilon > 0$.

2. Theorems 1 and 2 will follow from

**Theorem 4.** Let us suppose that with a real $\alpha > 0$ and $M = M(\alpha, Q, T)$

\[ \geq 1 \] we have the estimate

\[ \sum_{n=1}^{N} \frac{\chi(n)}{n^{1/2 + \varepsilon}} \ll qM^{N + E(\varepsilon) \frac{N}{|t| + 1}} \]

for every $|t| \leq T$ and every character $\chi \mod q \leq Q^2$ where $E(\varepsilon) = 1$ if $\chi$ is principal and $E(\varepsilon) = 0$ if $\chi$ is non-principal. Then for $\sigma > (1 + \varepsilon)/2, T \geq 2$ we have

\[ \sum_{x \leq q \leq x + M} \mathcal{S}(\sigma, T, \chi) \ll_N M^{4(1-\varepsilon)(2\sigma - 1 - \varepsilon)} \log(MT)^{20(1-\varepsilon)(2\sigma - 1 - \varepsilon) + 7}. \]

Remark. By Abel's inequality we see that (2.1) implies the same estimate if we replace the exponent it by $\delta + it$, $\delta = \delta(\varepsilon) > 0$.

Let us denote by $N'(\sigma, T, \chi)$ the maximal number $R \leq \sigma$, $|\gamma| \leq T$ (v = 1, 2, ..., R) and with $|\gamma_\alpha - \gamma_\beta| \geq 1$ if $1 < \alpha < \mu < \beta$. Then, by Jensen's inequality, Theorem 4 is an easy consequence of

**Theorem 4'.** Theorem 4 is valid if $N(\sigma, T, \chi)$ is replaced by $N'(\sigma, T, \chi)$ and 7 by 6 in the exponent of $\log(MT)$.

Analogously, instead of Theorem 3 it is sufficient to prove

**Theorem 3'.** Theorem 3 is valid if $N(\sigma, T, \chi)$ is replaced by $N'(\sigma, T, \chi)$ and $\log^2 (QT)$ by $\log^3 (QT)$.

Now we shall show that Theorem 4 implies Theorems 1 and 2. First we remark that by Abel's inequality (2.1) will follow if we can show that

\[ \sum_{n=1}^{N} \frac{\chi(n)}{n^{1/2 + \varepsilon}} \ll_M E(\varepsilon) N^{1-\varepsilon} \frac{q}{|t| + 1}. \]

This is trivially true if $|t| \leq 2$, independently of $M$. On the other hand for $|t| \geq 2$ we have for any $w \in (0, 1)$ and $K > 0$

\[ \sum_{m=1}^{K} (m + w)^{-a - it} \ll C_1 |t|^{10(1-a)3/2} \log^{2/3} |t|. \]
from the work of Richert [12]. By (2.4) we have

\[ \left| \frac{\sum_{n=1}^{N} \chi(n)}{n^{\frac{1}{2}+\delta}} \right| \leq \frac{1}{\zeta(1+1/4)} + \frac{1}{\zeta(3/2)} \left( \sum_{m=1}^{N} \frac{1}{(m+1/q)^{1/4}} \right) \]

\[ \leq \frac{1}{\zeta(1+1/4)} + \frac{1}{\zeta(3/2)} \left( \sum_{m=1}^{N} \frac{1}{(m+1/q)^{1/4}} \right) \leq \frac{1}{\zeta(1+1/4)} + \frac{1}{\zeta(3/2)} \left( \sum_{m=1}^{N} \frac{1}{m^{1/4}} \right) \]

if \( q \leq Q^2 \), \( |t| \leq T \). Choosing \( \alpha = 4\sigma - 3 \) we obtain Theorem 1 since

\[ 1 - \alpha = 4(1 - \sigma) \quad \text{and} \quad 2\sigma - 1 - \alpha = 2(1 - \sigma). \]

In order to prove Theorem 2 let \( \sigma = 3/4 + \epsilon \), \( \alpha = 1/2 + 2\epsilon^2 \) with an \( \epsilon \leq 1/4 \). Then \( 4(1 - \sigma) \leq 1, \ 2\sigma - 1 = 2\epsilon^2 \geq 3\epsilon^2 / 2 \). Using Perron's formula with \( K = (NT)^{1/2} Q^{-2} \) we obtain for any \( \chi \) mod \( q \leq Q^2 \) and \( |t| \leq T \): \( |L(s, \chi)| = \frac{1}{2\pi i} \int_{1/2 - iT}^{1/2 + iT} \frac{L(s, \chi)}{w} \frac{N^w}{w} \, dw + O(1) \). Translating the path of integration to \( \sigma = 1/2 \) we get

\[ \sum_{n=1}^{N} \frac{\chi(n)}{n^{3/2}} = \frac{E(\chi) \varphi(q) N^{1/2} \psi(1/2 + i\epsilon)}{\psi(1-\epsilon)} + \frac{1 + \epsilon}{\psi(1-\epsilon)} \sum_{n=1}^{N} \frac{\chi(n)}{N^{3/2}} \frac{L(\sigma, \chi)}{w} \frac{N^w}{w} \, dw + O(1). \]

Here the integral is

\[ \ll \frac{1}{\psi(1-\epsilon)} \frac{N^{1/2}(\xi K)^2}{\log K} \ll N^{1/2 + 2\epsilon^2} (Q^2 T)^{1/2 - \epsilon} \approx 1. \]

Therefore we have by Theorem 4

\[ \sum_{\epsilon < \xi \chi, \xi, \chi \leq \xi} N(\sigma, T, \chi) \ll (Q^2 T)^{1/2 - \epsilon} (\log K)^{1/2 + \epsilon} \ll (Q^2 T)^{1/2 - \epsilon}. \]

3. We shall need the following lemmas.

**Lemma A** (Bombieri [2]). If \( \xi, \varphi_1, \ldots, \varphi_R \) are elements of an inner product space over the field of complex numbers, then

\[ \sum_{v=1}^{R} |\xi, \varphi_v|^2 \leq ||\xi||^2 \max_{1 \leq v < R} \sum_{w=1}^{R} |\varphi_v, \varphi_w|. \]

**Lemma B.** Let \( S(s, \chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s} \) and let \( \mathcal{A} \) be a finite set of pairs \((s, \chi) = (\sigma + it, \chi)\) where \( \chi \) is mod \( q \leq Q \) primitive, \( |t| \leq T (T \geq 2) \), \( \sigma \geq \sigma_0 \geq 3/4 \), for all pairs and \( |t - \sigma| \geq 1 \) for any distinct pairs \((\sigma + it, \chi)\) and \((\sigma' + it', \chi)\) in \( \mathcal{A} \). Supposing (2.1) we have

\[ \sum_{(s, \chi) \in \mathcal{A}} |S(s, \chi)|^2 \ll (N \log T + |\mathcal{A}| \log N^2) \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma_0}. \]

**Lemma C.** Let us suppose the conditions of Lemma B, with the exception of (2.1). Then we have

\[ \sum_{(s, \chi) \in \mathcal{A}} |S(s, \chi)|^2 \ll (N + Q^2 T) \log N \sum_{n=1}^{N} |a_n|^2 \left( 1 + \log \frac{\log 2N}{\log 2\pi} \right). \]

**Lemma 1.** For \( \sigma \geq 1/2, \ x \geq |t| \) we have

\[ \sum_{n \leq x} \frac{\chi(n)}{n^{1/2 + \epsilon}} = \frac{x^{1/2 + \epsilon}}{1 - \epsilon} + O(x^{-\epsilon}). \]

**Lemma 2.** If \( \chi \) is non-principal mod \( q \), \( \sigma \geq 1/2, \ x \geq q(|t| + 1) \) then

\[ \sum_{n \leq x} \frac{\chi(n)}{n^{\sigma - 1/2}} = L(s, \chi) + O(q^{-\epsilon}). \]

For Lemmas 1 and 2 see Theorem 4.11 of [13] and Lemma 2 in § 26 of [3], resp. Lemma B is very similar to Theorem 8.4 of [8], whilst Lemma C is Theorem 7.5 of [8]. We prove here Lemmas A and B needed in the proofs of Theorems 1 and 2.

**Proof of Lemma A.** For arbitrary complex numbers \( c_v \),

\[ \sum_{v=1}^{R} \sum_{u=1}^{R} c_v \overline{c_u} |\varphi_v, \varphi_u|^2 \leq ||\xi||^2 \sum_{v=1}^{R} |c_v|^2 \sum_{u=1}^{R} |\varphi_v, \varphi_u|^2 \]

\[ \leq ||\xi||^2 \sum_{v=1}^{R} |c_v|^2 \sum_{u=1}^{R} |\varphi_v, \varphi_u|^2 \]

\[ \leq ||\xi||^2 \sum_{v=1}^{R} |c_v|^2 \max_{1 \leq v < R} \sum_{w=1}^{R} |\varphi_v, \varphi_w|. \]

Taking \( c_v = (\xi, \varphi_v) \) we obtain

\[ \sum_{v=1}^{R} |c_v|^2 \leq ||\xi||^2 \sum_{v=1}^{R} \sum_{u=1}^{R} |\varphi_v, \varphi_u|^2 \]

\[ \leq ||\xi||^2 \sum_{v=1}^{R} |c_v|^2 \max_{1 \leq v < R} \sum_{w=1}^{R} |\varphi_v, \varphi_w|. \]

**Proof of Lemma B.** Let \( \xi = \sum_{v=1}^{R} a_v n^{-\sigma_0} \) and \( \varphi_{v, x} = \overline{\chi(n)} n^{-(\sigma - \sigma_0) + it} \) then

\[ (\xi, \varphi_{v, x}) = S(s, \chi), \quad ||\xi||^2 = N \sum_{a \neq 1} |a_n|^2 n^{-2\sigma_0}. \]
and
\[
\sum_{\nu, \nu' \in \mathcal{A}} |\langle \varphi_{\alpha_1, \nu}, \varphi_{\alpha, \nu'} \rangle| = \sum_{\nu, \nu' \in \mathcal{A}} \left| \sum_{n=1}^{N} \frac{\varphi_1(n) \bar{\varphi}_1(n)}{n^{\nu \bar{\nu} + \nu' \bar{\nu}' + (\nu - \nu') (\bar{\nu} - \bar{\nu})}} \right| \\
\leq |\mathcal{A}| N^2 M + \sum_{(\nu, \nu) \in \mathcal{A}} \left| \sum_{n=1}^{N} \frac{|\varphi_1(n)|^2}{n^{\nu \bar{\nu} + \nu' \bar{\nu}' + (\nu - \nu') (\bar{\nu} - \bar{\nu})}} \right| \\
\leq |\mathcal{A}| N^2 M + N \log T.
\]

4. For the proof of Theorem 4 let
\[(4.1) \quad \eta = 2\sigma - 1 - \alpha \quad (\sigma > 0), \quad X = C_{\alpha, \sigma} (M \log^2 (MT))^1/n, \quad \mathcal{L} = \log XT\]
with a sufficiently large constant $C_{\alpha, \sigma}$. The constants implied by the $\ll$ sign in this section may depend on $\alpha$ and $\sigma$ but not on $C_{\alpha, \sigma}$. Further we define for a zero $\vartheta$ of $L(s, \chi)$ with $\beta > \sigma$
\[(4.2) \quad a_n = \sum_{d \in \mathcal{D}} \mu(d), \quad H = H(X, \vartheta) = \sum_{n=1}^{X} \frac{\varphi(n) a_n}{n^\sigma}, \quad n \in \mathcal{X}^2 \]

So we have for any non-principal $\chi$
\[(4.3) \quad |H| = \sum_{n=1}^{X} \frac{\varphi(n) a_n}{n^\sigma} = 1 + \sum_{n=1}^{X} \frac{\varphi(n) a_n}{n^\sigma} < 1/2 \]
if $C_{\alpha, \sigma}$ is large enough.

For $q = 1$ ($\chi = \chi_0$) and $|\vartheta| > X^{3 - 3\sigma} \mathcal{L}^2$ we have by Lemma 1
\[(4.4) \quad \sum_{n=1}^{X} \frac{\varphi(n) a_n}{n^\sigma} \ll X^{2 - \sigma} \mathcal{L}^2. \quad \text{if} \quad |\vartheta| \ll Z \quad (Z = X^2). \]

If $X^{2/d} < |\vartheta|$, using (4.4) we obtain from (2.1) (where the second term may be deleted if $N \ll |\vartheta| + 1$) similarly to (4.3)
\[(4.5) \quad \sum_{n=1}^{X} \frac{\varphi(n) a_n}{n^\sigma} = \sum_{m=1}^{X^{2/d}} \frac{1}{m^\sigma} \ll X^{2/d}(X^{1 - \sigma} \mathcal{L}^2 + X^{2 - \sigma} \mathcal{L}^2). \]

So we obtain for $\chi = \chi_0$, $\vartheta \in \mathcal{A}'$
\[(4.6) \quad H = H(X, \vartheta) \ll X^{1 - \sigma} \left\{ \frac{1}{X^{1 - \sigma} \mathcal{L}^2} + \frac{M}{X^{2 - \sigma} \mathcal{L}^2} \right\} \ll \frac{1}{2} \]
where $\mathcal{A}' = \mathcal{A} \setminus \{(\vartheta, \chi_0) \in \mathcal{A}; \vartheta \ll \mathcal{L}^2 Y^{1 - \sigma}\}$. Finally, similarly to (4.7)–(4.8) we get with an $N \in [X, X^2]$ and
\[(4.7) \quad \sum_{n=1}^{N} \left| \sum_{\nu = 1}^{N} \frac{\varphi(n) a_n}{n^\sigma} \right|^2 \ll \frac{|\mathcal{A}|}{2 \mathcal{L}^2}. \]

Applying Lemma B to the above partial sums we obtain
\[(4.8) \quad \frac{|\mathcal{A}|}{2 \mathcal{L}^2} \ll (N \mathcal{L} + |\mathcal{A}|) N^2 M \mathcal{L}^3 N^{1 - 2\sigma} \ll \mathcal{L}^4 X^{4 - 4\sigma} + |\mathcal{A}| \ll \mathcal{L}^3 M X^{-\eta}. \]

But if $C_{\alpha, \sigma}$ in (4.1) is large enough then the second term on the right-hand side can be deleted and we obtain
\[(4.9) \quad |\mathcal{A}| \ll \mathcal{L}^4 X^{4 - 4\sigma} \Rightarrow |\mathcal{A}| \ll \mathcal{L}^6 X^{6 - 4\sigma}. \]

5. In order to prove Theorem 3 let
\[(5.1) \quad D = Q^2 T, \quad Y = C D^2, \quad \mathcal{L} = \log D \]

where $C$ is a sufficiently large absolute constant. The constants implied by the $\ll$ signs are absolute and independent of $C$. Let us define $a_\alpha$ as in (4.2) and let
\[(5.2) \quad H = H(X, Y, \vartheta) = \sum_{n=1}^{X} \frac{\varphi(n) a_\alpha}{n^\sigma} = 1 + \sum_{n=1}^{X} \frac{\varphi(n) a_\alpha}{n^\sigma} \ll X^{1 - \sigma} \mathcal{L}^2. \]

Since $Y/X = CD > QT$ we obtain by Lemma 2 for $\chi \neq \chi_0$
\[(5.3) \quad |H| \ll \sum_{d \in \mathcal{D}} \frac{\varphi(n) a_\alpha}{n^\sigma} \ll \sum_{d \in \mathcal{D}} \frac{\varphi(n) a_\alpha}{n^\sigma} \ll \frac{q}{Y^d} \ll \frac{q}{Y^d} \ll C^{1/2} D^{1/2} 2e - 1 - 2d \ll C^{1/2} < 1/2. \]

In case of $q = 1$, $\chi = \chi_0$ we have here by $Y/d > T$ from (4.4)
\[(5.4) \quad \frac{1}{M} \ll \frac{\varphi(n) a_\alpha}{n^\sigma} \ll \frac{1}{M} \ll \frac{Y^{1 - \sigma}}{d^{1/2} |\vartheta|} \ll \frac{1}{|\vartheta|} \ll \frac{1}{2}, \]

if $q \neq \mathcal{A}' \equiv \mathcal{A}' \setminus \{(\vartheta, \chi_0) \in \mathcal{A}; |\vartheta| \ll \mathcal{L}^2 Y^{1 - \sigma}\}$. Finally, similarly to (4.7)–(4.8) we get with an $N \in [X, X^2]$ from Lemma C
\[(5.5) \quad |\mathcal{A}'| \ll \mathcal{L}^4 (D + N) \mathcal{L}^3 N^{1 - 2\sigma} \ll \mathcal{L}^4 (D X^{1 - 2\sigma} + Y^{2 - 2\sigma}), \]
\[(5.6) \quad |\mathcal{A}'| \ll \mathcal{L}^6 D^{4\sigma(1 - \sigma)} \Rightarrow |\mathcal{A}'| \ll \mathcal{L}^6 D^{4\sigma(1 - \sigma)}. \]
Nombres hautement composés

par

JEAN-LOUIS NICOLAS (Limoges)

A Paul Erdős
pour fêter son 75ème anniversaire

1. Introduction. Dans un long mémoire paru en 1915, S. Ramanujan a défini et étudié les nombres hautement composés, c’est-à-dire les nombres qui ont strictement plus de diviseurs que les nombres qui les précèdent (cf. [14], n° 15). Appelons \( Q(X) \) la quantité de nombres hautement composés inférieurs ou égaux à \( X \). S. Ramanujan a en particulier démontré que

\[
\lim_{X \to +\infty} \frac{Q(X)}{\log X} = +\infty
\]

mais n’a pas donné de majoration pour \( Q(X) \). C’est pourtant un sujet qui l’intéressait puisque dans l’article écrit avec G. H. Hardy: Asymptotic formulae for the distribution of integers of various types (cf. [14], n° 34), il étudie les nombres de la forme:

\[2^{a_1}3^{a_2} \ldots p_k^{a_k}\]

où \( p_k \) désigne le \( k \)ème nombre premier, et \( a_1 \geq a_2 \geq \cdots \geq a_k \); parmi les nombres de cette forme figurent les nombres hautement composés.

En 1944, P. Erdős démontre (cf. [23]) qu’il existe \( c' > 0 \) tel que

\[Q(X) \geq (\log X)^{1+c'}\]

pour \( X \) assez grand.

Un des outils essentiels dans sa démonstration, est le résultat de Hohelsel:

**Lemme 1.** Soit \( \pi(x) = \sum_{p \leq x} 1 \). Il existe \( \tau < 1 \) tel que

\[\pi(x + x') - \pi(x) \sim x'/\log x.\]

Nous désignerons par \( \tau \) un nombre réel \( \tau < 1 \) pour lequel on a, pour tout \( x \) assez grand:

\[\pi(x + x') - \pi(x) \geq x'/\log x.\]