On a product of sines

by

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1. Introduction. Let

\[ P_N(\alpha) = \prod_{k=1}^{N} |\sin \pi k\alpha| \quad \text{and} \quad P_N = \max_{0 < \alpha < 1} P_N(\alpha). \]

The object of this note is to prove the following

Theorem. We have

(1.1) \[ \lim_{N \to \infty} (P_N)^{1/N} = \sin \pi \alpha_0 \]

where \( \alpha_0 \) is the solution between 0 and 1 of the transcendental equation

\[ \int_{0}^{\pi} u \cot u \, du = 0. \]

In fact, \( \alpha_0 = 0.7912265710 \ldots \) and \( \sin \pi \alpha_0 = 0.6098579 \ldots \)

Since \( P_N(\alpha) = P_N(1-\alpha) \), we see that \( P_N = \max_{0 < \alpha < 1/2} P_N(\alpha) \). We note at the outset the elementary duplication formula

(1.2) \[ \sin \pi q\Phi = 2^{q-1} \prod_{s=0}^{q-1} \sin \pi (s/q + \Phi), \]

as well as its straightforward consequence

(1.3) \[ \frac{q}{2^{q-1}} = \prod_{s=1}^{q-1} \frac{\sin \pi s}{q}. \]

We shall use both these relations below, the latter on a number of occasions.

It is easy to see that our result may be stated in the alternative form

(1.4) \[ \lim_{N \to \infty} \left( \prod_{|z|=1}^{N} |1-z^k| \right)^{1/N} = 2 \sin \pi \alpha_0. \]

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Erdős and Szekeres ([1]), p. 29 make the remark that "it is easy to show that [this limit] exists and is between 1 and 2". What we have done is to compute the limit. In fact, this question arose a few years later in a paper of Sudler [3], and was answered by E. M. Wright [4]; but perhaps our method is sufficiently different from Wright’s as to merit description. This note may serve also to draw attention to the general problem (cf. [1]) of studying

\[ M(a_1, \ldots, a_N) = \max \prod_{|k| = 1}^N |1 - z^{a_k}| \]

for various natural sets of exponents \((a_1, \ldots, a_N)\); for example, one might begin with \(a_k = p(k)\), where \(p(x)\) is a polynomial with real coefficients. In this connection we note also the result of Newman and Slater [2] that

\[ \prod_{k=1}^N |\sin 2^{k-1} a| \leq (2^{-1 + \log 3/\log 4})^N. \]

We begin with a sketch of our approach. Given \(a\), there exists a rational \(a/q\) with \(0 \leq a \leq q, (a, q) = 1, 1 \leq q \leq N\), such that

\[ \left| a - \frac{a}{q} \right| \leq \frac{1}{q(N+1)}. \]

We divide our argument into three parts according to the location of \(a\):
I. \(0 \leq a \leq (N+1)^{-1}\); II. \(a\) in an interval (1.4) with \(a \geq 1\) and \(1 \leq q \leq N/1000\); III. \(a\) in an interval (1.4) but with \(N/1000 < q \leq N\). We shall prove that the maximum in Case I occurs at \(a = a_0/N\) and that

\[ \sin \pi a_0/N \ll N^{-1/2} N \ll N \sin \pi a_0/N, \]

whereas for \(a\)'s in Cases II and III

\[ P_N(a) \ll (0.6)^N. \]

Our result follows at once from these estimates. The inequalities (1.5) could be sharpened if necessary. Constants implied by use of the \(\ll\) notation are absolute. \(N\) is a large positive integer; we shall assume at various stages of the argument, without always saying so explicitly, that \(N\) is sufficiently large.

2. Case I. Here

\[ P_N(a) = \prod_{k=1}^N \sin \pi k a. \]

Since \(P_N(0) = 0 = P_N(1/N)\) and \(P_N(a)\) is positive on \((0, 1/N)\), \(P_N(a)\) has a maximum on this interval. Write

\[ P_N(a) = \exp \left( - \sum_{k=1}^N \log \csc \pi k a \right) = \exp \left( - S_N(a) \right) \]

say, where

\[ S_N(a) = \sum_{k=1}^N \log \csc \pi k a, \quad 0 < a < 1/N. \]

The maximum of \(P_N(a)\) occurs at the value of \(a\) where \(S_N(a)\) has its minimum, i.e., where \(S_N(a) = 0\). Now

\[ S_N(a) = -\pi \sum_{k=1}^N k \cot \pi k a, \]

and if we write \(a = x/N\), so that \(0 < x < 1\), we are interested in the root \(x_0\) of the equation

\[ \sum_{k=1}^N \frac{k}{N} \cot \frac{k}{N} x = 0. \]

If \(0 < x \leq 1/2\), the expression on the left is positive, and therefore the root \(x_0\) of the equation lies in the interval \((1/2, 1)\). Moreover, for \(N\) large this root is close to the root \(x_0\) of the equation

\[ \int_0^1 u \cot u \ du = 0. \]

Since we are interested in the value of \(S_N(x_0/N)\) we proceed more directly as follows. Suppose that \(1/2 < x < 0.9\). For each \(x, 1/2 < x < 0.9\), the summand of \(S_N(x/N)\) decreases monotonically as \(k\) increases to \(N/(2x)\), and then increases. Hence the simplest form of Euler’s summation formula is applicable, and we have

\[ S_N \left( \frac{x}{N} \right) = \frac{N}{\pi} \log \left( \csc \frac{\pi x}{N} \right) + \sum_{k=1}^{x} \frac{\pi x}{N} \cot \left( \frac{\pi x}{N} \right) - \frac{\pi x}{N} \left( \frac{1}{2} - [x \pi] \right) \cot \left( \frac{\pi x}{N} \right) \]

after integration by parts. Hence

\[ S_N \left( \frac{x}{N} \right) = N \log \left( \csc \frac{\pi x}{N} \right) - \frac{N}{\pi} \int_0^{x} u \cot u \ du - \frac{N}{\pi} \int_{[x \pi]}^{N/(2x)} (t-[t]) \cot \left( \frac{\pi x}{N} \right) \]

and therefore

\[ S_N \left( \frac{x}{N} \right) = N \log \left( \csc \frac{\pi x}{N} \right) - \frac{N}{\pi} \int_0^{x} u \cot u \ du \]

\[ \leq - \frac{N}{\pi} \int \left( \frac{Nu}{\pi x} - \frac{N}{\pi x} \right) \cot u \ du \]

\[ \leq \frac{\pi x}{N} \left[ \log \left( \frac{N}{\pi x} \right) \right] \leq 0.4 \pi x \]

\[ \leq \int_0^{\pi x} \tan v \ dv \leq \int_0^{0.4 \pi} \tan v \ dv = \log (\sec(0.4 \pi)) < 1.775; \]
also
\[
S_N \left( \frac{x}{N} \right) - N \log(\csc \pi x) - N \frac{\pi}{\pi x} \int_0^{\pi x} u \cot u \, du \\
\geq - \frac{\pi^2}{6} \left( \frac{\pi x}{\pi x} \right) \left( \frac{\pi x}{\pi x} \right) \int_0^{\pi x} u \cot u \, du \\
\geq -1 - \log \left( \csc \pi x \right) \geq - \log \left( e \csc \pi \frac{\pi x}{2} \right).
\]

Altogether, then we have
\[
- \log(e \csc \pi x) \leq S_N \left( \frac{x}{N} \right) - N \left\{ \log(\csc \pi x) + \frac{1}{\pi x} \int_0^{\pi x} u \cot u \, du \right\} \leq 1.175.
\]

Now
\[
\frac{d}{dx} \left\{ \log(\csc \pi x) + \frac{1}{\pi x} \int_0^{\pi x} u \cot u \, du \right\} = - \frac{1}{\pi x^2} \left( \frac{\pi x}{\pi x} \right) \int_0^{\pi x} u \cot u \, du = 0
\]
at \( x = \alpha_0 \), so that the expression in parentheses has a minimum at \( x = \alpha_0 \) and
\[
- \log \left( e \csc \pi \frac{\pi x}{2N} \right) \leq S_N \left( \frac{x}{N} \right) - N \log(\csc \pi x_0) \leq 1.175,
\]
or
\[
e^{-1.175} (\sin \pi x_0)^N \leq P_N(\pi \alpha_0 / N) \leq (\sin \pi x_0)^N e^{\pi \pi x_0} / 2N \leq eN (\sin \pi x_0)^N.
\]

This proves (1.5) in slightly more precise form.

3. Case II. We assume without loss of generality that \( \beta \geq 0 \) and consider
\[
P_N(x) \leq \prod_{0 \leq j \leq N(q-1)} \prod_{r=1}^q |\sin \pi(x/q + \beta r + \beta q)|
\]
\[
= \prod_{1 \leq j \leq N/q} \sin(\pi x/2 q) \prod_{0 \leq j \leq N(q-1)-1} \prod_{r=1}^{q-1} |\sin(\pi(x/q + \beta r + \beta q)|
\]
\[
0 \leq \beta < 1/(Nq).
\]

We have \( \beta r < \beta q \leq 1/(N+1) \), and therefore do not expect to lose much by omitting \( \beta r \) on the right. More precisely, we show that
\[
(3.1) \quad |\sin \pi(\frac{x}{q} + \beta r + \beta q)| \leq \left( 1 + \frac{\pi q}{2(N+1)} \right) |\sin \pi(\frac{x}{q} + \beta q)|,
\]

First of all,
\[
\beta(r + q) < (r + q)/N \leq 1/q.
\]

Suppose that \( \beta r \equiv s \mod q, 1 \leq s \leq q-1 \). Then
\[
|\sin \pi(\frac{x}{q} + \beta r + \beta q)| \leq |\sin \pi(\frac{s}{q} + \beta q)| \quad \text{if} \quad q/2 \leq s \leq q-1;
\]

and if \( 1 \leq s < q/2 \) so that \( q \geq 3 \), we have
\[
|\sin \pi(\frac{x}{q} + \beta r + \beta q)| = |\sin \pi(\frac{s}{q} + \beta q)| \leq \frac{\pi}{N+1} \sin \pi(\frac{s}{q} + \beta q).
\]

Thus (3.1) is true in all cases. Hence
\[
P_N(x) \leq \left( 1 + \frac{\pi q}{2(N+1)} \right) \prod_{1 \leq j \leq N/q} \sin(\pi \beta q) \prod_{0 \leq j \leq N(q-1)-1} \prod_{s=1}^{q-1} |\sin \pi(\frac{s}{q} + \beta q)|
\]
\[
\leq e^{\pi/2 \pi q} \prod_{s=1}^{q-1} \sin \left( \frac{s \pi}{q} \right) \prod_{1 \leq j \leq N(q-1)-1} \prod_{s=1}^{q-1} |\sin \pi(\frac{s}{q} + \beta q)|
\]
\[
= e^{\pi/2 \pi q} \prod_{1 \leq j \leq N(q-1)-1} \sin(\beta q^2 \pi j) \prod_{s=0}^{q-1} \sin \left( \frac{s \pi}{q} \right)
\]
\[
= e^{\pi/2 \pi q} (0.5)^{q-1} (\cos \beta q)^N \prod_{1 \leq j \leq N(q-1)-1} \sin(\beta q),
\]

where \( 0 \leq \beta = \beta q \leq q/(N+1) \). The last product is covered by Case I: write \( M \equiv \lfloor N/q \rfloor - 1 \), so that \( 0 \leq q < 1/M \). Then this product is at most of order
\[
M(\sin \pi x_0)^M \leq \left( 0.6099 \right)^N,
\]
and altogether we arrive at
\[
P_N(x) \leq e^{\pi/2 \pi} (0.5)^{1/2} (0.6099)^N = N(2e^{\pi/2} (0.5)^{(1.2198)^N} \leq N(0.6099)^N = N(0.6099)^N.
\]
and since $2 \leq q \leq N/1000$, we arrive at

$$P_N(x) \leq N \left( (9.621)^{0.001} (0.5) (1.10445)^N \right) \leq N (0.554)^N \leq (0.56)^N,$$

for all large enough $N$. This settles Case II.

4. Case III. Here

$$N/1000 < q \leq N,$$

Define

$$Q_i = \prod_{k=1}^{l} |\sin(\pi k/q)| \prod_{k=l+1}^{N} |\sin \pi k|, \quad 0 \leq l \leq N,$$

where the asterisk signifies that zero factors—that is, the terms with $k \equiv 0 \mod q$—are omitted. Note that

$$Q_0 = P_N(x).$$

We have $(l+1 \leq N)$

$$Q_i/Q_{i+1} = \begin{cases} |\sin \pi (l+1)/(l+1)/q|, & l+1 \not\equiv 0 \mod q, \\ |\sin \pi (l+1)/q|, & l+1 \equiv 0 \mod q. \end{cases}$$

Consider this ratio when $l+1 \not\equiv 0 \mod q$. Here

$$|\sin \pi (l+1)/q| \leq \pi (l+1)/|q|,$$

so that the ratio in question is at most

$$1 + \frac{\pi (l+1)/|q|}{\sin \pi (l+1)/q} \leq 1 + \frac{\pi N/|q|}{\sin \pi (l+1)/q} \leq 1 + \frac{\pi q N/|q|}{\sin \pi (l+1)/q} \leq 1 + \frac{\pi q}{2} < e.$$

We can do better in a number of cases. Suppose $a(l+1) = s \mod q$, $1 \leq s \leq q$. Then

$$\sin \frac{a(l+1)}{q} = \sin \frac{\pi s}{q} \geq \frac{2}{\pi} \min \left( \frac{\pi s, \pi (q-s)}{q} \right) = \frac{2}{q} \min(s, q-s).$$

Let $D$ be a large number less than $q$, and suppose that $\min(s, q-s) \geq D$. Then the ratio under consideration is at most

$$1 + \frac{\pi N/|q|}{2D} = 1 + \frac{\pi}{2D} q N/|q| < 1 + \frac{\pi}{2D}.$$

To sum up,

$$Q_i/Q_{i+1} \leq \begin{cases} 1, & l+1 \equiv 0 \mod q, \\ 1 + \pi/(2D), & a(l+1) = s \mod q, \quad \min(s, q-s) \geq D, \\ e, & \text{otherwise}. \end{cases}$$

In any complete set of residues we need to invoke the last, and worst, of these inequalities at most $2D$ times. Hence, by (4.3),

$$P_N(x) = \left( \prod_{l=0}^{N-1} Q_i/Q_{i+1} \right) Q_N \leq e^{\pi (q+1)/2D} \left( 1 + \frac{\pi}{2D} \right) Q_N \leq e^{2.002D + \pi/2D} Q_N < e^{5.7N^{1/2}} Q_N$$

on choosing $D$ so that

$$4.004D^2 = \pi N.$$

In view of (4.1) this choice of $D$ is admissible if $N$ is large enough. Thus

$$P_N(x) \leq e^{5.7N^{1/2}} Q_N.$$  

By (4.2),

$$Q_N = \prod_{k=1}^{N} |\sin \pi k/q|.$$

Let

$$N = qm + N_1, \quad 0 \leq N_1 < q.$$

If $q > N/2$, $m = 1$ necessarily and $N_1 = q < N/2$. Hence

$$N_1 < N/2$$

always. Consequently, by (1.3),

$$Q_N = \prod_{k=1}^{m} \left( \frac{\pi k/q}{\sin \pi k/q} \right) \prod_{k=m+1}^{N} \left( \frac{\pi k/q}{\sin \pi k/q} \right) = \left( \frac{q}{2^{N_1-1}} \right)^{N_1} \prod_{k=1}^{N} \left( \sin \pi k/q \right)$$

$$= \left( \frac{q}{2^{N_1-1}} \right)^{N_1} P_{N_1} \left( \frac{a}{q} \right).$$

Suppose first of all that $N_1 \leq C_0$, some absolute constant. Then

$$Q_N \leq \left( \frac{q}{2^{N_1-1}} \right)^{N_1} = (2^{1-s} q)^{(N-N_1)/q} = (2^{1-s} q)^{-N_1/q} (1/2^{1/q} q)^N$$

$$\leq 2 C_0 (1/2^{1/q} q)^N < 2 C_0 (0.55)^N,$$

say, by (4.1) (so that $(2q)^{1/q} < 1 + s$ if $N \geq N_0(s)$). Hence we may suppose that $N_1$ is large, and now preceding arguments apply. To be precise, we suppose,
as we may, that

\[
\frac{|a - q_1|}{q_1} \leq \frac{1}{q_1(N_1 + 1)}, \quad (a_1, q_1) = 1, \quad 1 \leq q_1 \leq N_1,
\]

and consider the two possibilities that (i) \(1 \leq q_1 \leq N_1/1,000\) and (ii) \(N_1/1,000 < q_1 \leq N_1\). The latter case is the easier to dispose of: by (4.7),

\[
Q_N \leq \left( \frac{q}{2^{N-1}} \right)^m \left( \frac{q_1}{2^{N_1-1}} \right)^{m_1},
\]

where, as above, \(N_1 = q_1 m_1 + N_2\), \(N_2 < \frac{1}{2} N_1 < \frac{1}{2} N\). Now

\[
mg + m_1 q_1 = (N-N_1) + (N_1 - N_2) = N - N_2 > N - \frac{1}{4} N = \frac{3}{4} N,
\]

and each of \(m\) and \(m_1\) is at most 1,000. Hence

\[
Q_N \leq \frac{(2N)^{1,000}}{2^{3N/4}} \leq (2N)^{1,000}(0.595)^N < (0.598)^N
\]

since \(N\) is large.

We come finally to (i) above. But here we are back in Case I or Case II, so that (2.4) or (3.2) applies and

\[
Q_N \leq (2^{1-g} q)^m (0.61)^{m_1}
\]

\[
\leq \frac{(2N)^{1,000}}{2^{N-N_1}} (0.61)^N \leq (2N)^{1,000} 2^{-N} (1.22)^N
\]

\[
< (2N)^{1,000} \left( \frac{\sqrt{1.22}}{2} \right)^N < (2N)^{1,000} (0.555)^N
\]

\[
< (0.56)^N
\]

for sufficiently large \(N\). Combining this with (4.8) and (4.9) we obtain

\[
Q_N < (0.598)^N
\]

in all cases and consequently, by (4.4), that

\[
P_N(a) < (0.6)^N, \quad N/1,000 < q \leq N.
\]

This settles Case III, and our theorem is proved.

References