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On a product of sines

by

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1. Introduction. Let

$$P_N(\alpha) = \prod_{k=1}^{N} |\sin \pi k\alpha| \quad \text{and} \quad P_N = \max_{0 \leqslant \alpha \leqslant 1} P_N(\alpha).$$

The object of this note is to prove the following

THEOREM. *We have*

$$\lim_{N\to\infty} (P_N)^{1/N} = \sin \pi\alpha_0 \tag{1.1}$$

where α_0 *is the solution between* 0 *and* 1 *of the transcendental equation*

$$\int_0^{\pi x} u \cot u \, du = 0.$$

In fact, $\alpha_0 = 0.7912265710\ldots$ *and* $\sin \pi\alpha_0 = 0.6098579\ldots$

Since $P_N(\alpha) = P_N(1-\alpha)$, we see that $P_N = \max_{0 \leqslant \alpha \leqslant 1/2} P_N(\alpha)$. We note at the outset the elementary duplication formula

$$\sin \pi q\Phi = 2^{q-1} \prod_{s=0}^{q-1} \sin \pi (s/q + \Phi), \tag{1.2}$$

as well as its straightforward consequence

$$\frac{q}{2^{q-1}} = \prod_{s=1}^{q-1} \sin \frac{\pi s}{q}. \tag{1.3}$$

We shall use both these relations below, the latter on a number of occasions.

It is easy to see that our result may be stated in the alternative form

$$\lim_{N\to\infty} \Big(\max_{|z|=1} \prod_{k=1}^{N} |1-z^k|\Big)^{1/N} = 2 \sin \pi\alpha_0.$$

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Erdős and Szekeres ([1], p. 29) make the remark that "it is easy to show that [this limit] exists and is between 1 and 2". What we have done is to compute the limit. In fact, this question arose a few years later in a paper of Sudler [3], and was answered by E. M. Wright [4]; but perhaps our method is sufficiently different from Wright's as to merit description. This note may serve also to draw attention to the general problem (cf. [1]) of studying

$$M(a_1, \ldots, a_N) = \max_{|z|=1} \prod_{k=1}^{N} |1-z^{a_k}|$$

for various natural sets of exponents $(a_1, \ldots, a_N)$; for example, one might begin with $a_k = p(k)$, where $p(x)$ is a polynomial with real coefficients. In this connection we note also the result of Newman and Slater [2] that

$$\prod_{k=1}^{N} |\sin 2^{k-1}\alpha| \ll (2^{-1+\log 3/\log 4})^N.$$

We begin with a sketch of our approach. Given α, there exists a rational a/q with $0 \leqslant a \leqslant q$, $(a, q) = 1$, $1 \leqslant q \leqslant N$, such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q(N+1)}. \tag{1.4}$$

We divide our argument into three parts according to the location of α: I. $0 \leqslant \alpha \leqslant (N+1)^{-1}$; II. α in an interval (1.4) with $a \geqslant 1$ and $2 \leqslant q \leqslant N/1{,}000$; III. α in an interval (1.4) but with $N/1{,}000 < q \leqslant N$. We shall prove that the maximum in Case I occurs at $\alpha = \alpha_0/N$ and that

$$(\sin \pi\alpha_0)^N \ll P_N(\alpha_0/N) \ll N(\sin \pi\alpha_0)^N, \tag{1.5}$$

whereas for α's in Cases II and III

$$P_N(\alpha) \ll (0.6)^N. \tag{1.6}$$

Our result follows at once from these estimates. The inequalities (1.5) could be sharpened if necessary. Constants implied by use of the $\ll$-notation are absolute. N is a large positive integer; we shall assume at various stages of the argument, without always saying so explicitly, that N is sufficiently large.

2. Case I. Here

$$P_N(\alpha) = \prod_{k=1}^{N} \sin \pi k\alpha.$$

Since $P_N(0) = 0 = P_N(1/N)$ and $P_N(\alpha)$ is positive on $(0, 1/N)$, $P_N(\alpha)$ has a maximum on this interval. Write

$$P_N(\alpha) = \exp\Big(-\sum_{k=1}^{N} \log(\operatorname{cosec} \pi k\alpha)\Big) = \exp(-S_N(\alpha)) \tag{2.1}$$

say, where

$$S_N(\alpha) = \sum_{k=1}^{N} \log(\operatorname{cosec} \pi k\alpha), \qquad 0 < \alpha < 1/N. \tag{2.2}$$

The maximum of $P_N(\alpha)$ occurs at the value of α where $S_N(\alpha)$ has its minimum, i.e., where $S'_N(\alpha) = 0$. Now

$$S'_N(\alpha) = -\pi \sum_{k=1}^{N} k \cot \pi k\alpha,$$

and if we write $\alpha = x/N$, so that $0 < x < 1$, we are interested in the root x_0 of the equation

$$\sum_{k=1}^{N} \frac{k}{N} \cot \pi \frac{k}{N} = 0.$$

If $0 < x \leqslant 1/2$, the expression on the left is positive, and therefore the root x_0 of the equation lies in the interval $(\frac{1}{2}, 1)$. Moreover, for N large this root is close to the root α_0 of the equation

$$\int_0^{\pi x} u \cot u \, du = 0. \tag{2.3}$$

Since we are interested in the value of $S_N(x_0/N)$ we proceed more directly as follows. Suppose that $1/2 < x < 0.9$. For each x, $1/2 < x < 0.9$, the summand of $S_N(x/N)$ decreases monotonically as k increases to $N/(2x)$, and then increases. Hence the simplest form of Euler's summation formula is applicable, and we have

$$S_N\left(\frac{x}{N}\right) = \int_1^N \log\left(\operatorname{cosec}\frac{\pi x t}{N}\right)dt + \log\left(\operatorname{cosec}\frac{\pi x}{N}\right) - \frac{\pi x}{N}\int_1^N (t-[t]) \cot\frac{\pi x t}{N}\, dt$$

$$= N\log(\operatorname{cosec} \pi x) + \frac{\pi x}{N}\int_1^N t \cot\left(\frac{\pi x}{N} t\right)dt - \frac{\pi x}{N}\int_1^N (t-[t]) \cot\frac{\pi x t}{N}\, dt$$

after integration by parts. Hence

$$S_N\left(\frac{x}{N}\right) - N\log(\operatorname{cosec} \pi x) = \frac{N}{\pi x}\int_0^{\pi x} u \cot u\, du - \frac{\pi x}{N}\int_0^N (t-[t]) \cot\frac{\pi x t}{N}\, dt$$

and therefore

$$S_N\left(\frac{x}{N}\right) - N\log(\operatorname{cosec} \pi x) - \frac{N}{\pi x}\int_0^{\pi x} u \cot u\, du$$

$$\leqslant -\int_{\pi/2}^{\pi x} \left(\frac{Nu}{\pi x} - \left[\frac{Nu}{\pi x}\right]\right) \cot u\, du$$

$$\leqslant \int_0^{\pi(x-1/2)} \tan v\, dv \leqslant \int_0^{0.4\pi} \tan v\, dv = \log(\sec(0.4\pi)) < 1.175;$$

also

$$S_N\left(\frac{x}{N}\right)-N\log(\operatorname{cosec}\pi x)-\frac{N}{\pi x}\int_0^{\pi x}u\cot u\,du$$

$$\geqslant -\int_0^{\pi/2}\left(\frac{Nu}{\pi x}-\left[\frac{Nu}{\pi x}\right]\right)\cot u\,du \geqslant -\frac{N}{\pi x}\int_0^{\pi x/N}u\cot u\,du-\int_{\pi x/N}^{\pi/2}\cot u\,du$$

$$\geqslant -1-\log\left(\operatorname{cosec}\frac{\pi x}{N}\right)\geqslant -\log\left(e\operatorname{cosec}\frac{\pi}{2N}\right).$$

Altogether, then we have

$$-\log(e\operatorname{cosec}\pi x)\leqslant S_N\left(\frac{x}{N}\right)-N\left\{\log(\operatorname{cosec}\pi x)+\frac{1}{\pi x}\int_0^{\pi x}u\cot u\,du\right\}\leqslant 1.175.$$

Now

$$\frac{d}{dx}\left\{\log(\operatorname{cosec}\pi x)+\frac{1}{\pi x}\int_0^{\pi x}u\cot u\,du\right\} = -\frac{1}{\pi x^2}\int_0^{\pi x}u\cot u\,du = 0$$

at $x=\alpha_0$, so that the expression in parentheses has a minimum at $x=\alpha_0$ and

$$-\log\left(e\operatorname{cosec}\frac{\pi}{2N}\right)\leqslant S_N\left(\frac{\alpha_0}{N}\right)-N\log(\operatorname{cosec}\pi\alpha_0)\leqslant 1.175,$$

or

$$e^{-1.175}(\sin\pi\alpha_0)^N\leqslant P_N(\alpha_0/N)\leqslant(\sin\pi\alpha_0)^N e\operatorname{cosec}\frac{\pi}{2N}\leqslant eN(\sin\pi\alpha_0)^N. \tag{2.4}$$

This proves (1.5) in slightly more precise form.

3. Case II. We assume without loss of generality that $\beta\geqslant 0$ and consider

$$P_N(\alpha)\leqslant\prod_{0\leqslant j\leqslant N/q-1}\prod_{r=1}^{q}|\sin\pi(ar/q+\beta r+\beta qj)|$$

$$=\prod_{1\leqslant j\leqslant N/q}\sin(\pi\beta qj)\prod_{0\leqslant j\leqslant N/q-1}\prod_{r=1}^{q-1}|\sin\pi(ar/q+\beta r+\beta qj)|,$$

$$0\leqslant\beta<1/(Nq).$$

We have $\beta r<\beta q\leqslant 1/(N+1)$, and therefore do not expect to lose much by omitting βr on the right. More precisely, we show that

$$\left|\sin\pi\left(\frac{ar}{q}+\beta r+\beta qj\right)\right|\leqslant\left(1+\frac{\pi q}{2(N+1)}\right)\left|\sin\pi\left(\frac{s}{q}+\beta qj\right)\right|, \tag{3.1}$$

$$ar\equiv s\bmod q,\quad 1\leqslant s\leqslant q-1.$$

First of all,

$$\beta(r+qj)<(r+qj)/qN\leqslant 1/q.$$

Suppose that $ar\equiv s\bmod q$, $1\leqslant s\leqslant q-1$. Then

$$\left|\sin\pi\left(\frac{ar}{q}+\beta r+\beta qj\right)\right|\leqslant\sin\pi\left(\frac{s}{q}+\beta qj\right)\quad\text{if}\quad q/2\leqslant s\leqslant q-1;$$

and if $1\leqslant s<q/2$ so that $q\geqslant 3$, we have

$$\left|\sin\pi\left(\frac{ar}{q}+\beta r+\beta qj\right)-\sin\pi\left(\frac{ar}{q}+\beta qj\right)\right|$$

$$\leqslant 2\sin\frac{\pi\beta r}{2}\leqslant\pi\beta r<\pi\beta q\leqslant\frac{\pi}{N+1}$$

$$=\frac{\pi}{N+1}\,\frac{\sin\pi(s/q+\beta qj)}{\sin\pi(s/q+\beta qj)}\leqslant\frac{\pi q}{2(N+1)}\sin\pi(s/q+\beta qj).$$

Thus (3.1) is true in all cases. Hence

$$P_N(\alpha)$$

$$\leqslant\left(1+\frac{\pi q}{2(N+1)}\right)^{(q-1)(N/q-1)}\prod_{1\leqslant j\leqslant N/q}\sin(\pi\beta qj)\prod_{0\leqslant j\leqslant N/q-1}\prod_{s=1}^{q-1}\sin\pi\left(\frac{s}{q}+\beta qj\right)$$

$$\leqslant e^{\pi q/2}\prod_{s=1}^{q-1}\sin\left(\frac{\pi s}{q}\right)\prod_{1\leqslant j\leqslant N/q-1}\prod_{s=0}^{q-1}\sin\pi\left(\frac{s}{q}+\beta qj\right)$$

$$=e^{\pi q/2}\frac{q}{2^{q-1}}\prod_{1\leqslant j\leqslant N/q-1}\frac{\sin(\beta q^2 j\pi)}{2^{q-1}}$$

$$=e^{\pi q/2}q(0.5)^{(q-1)[N/q]}\prod_{1\leqslant j\leqslant N/q-1}\sin(\pi\alpha j),$$

where $0\leqslant\alpha=\beta q^2\leqslant q/(N+1)$. The last product is covered by Case I: write $M=[N/q]-1$, so that $0\leqslant\alpha<1/M$. Then this product is at most of order $M(\sin\pi\alpha_0)^M\ll\frac{N}{q}(0.6099)^{N/q}$, and altogether we arrive at

$$P_N(\alpha)\ll e^{\pi q/2}N(0.5)^{(q-1)(N/q-1)}(0.6099)^{N/q}$$

$$\ll N(2e^{\pi/2})^q(0.5)^N(1.2198)^{N/q}\ll N(9.621)^q(0.5)^N(1.2198)^{N/q}$$

and since $2 \leqslant q \leqslant N/1{,}000$ we arrive at

(3.2) $$P_N(\alpha) \ll N\{(9.621)^{.001}(0.5)(1.10445)\}^N \leqslant N(0.554)^N \ll (0.56)^N, \quad 2 \leqslant q \leqslant N/1{,}000,$$

for all large enough N. This settles Case II.

4. Case III. Here

(4.1) $$N/1{,}000 < q \leqslant N.$$

Define

(4.2) $$Q_l = \prod_{k=1}^{l}{}^{*} |\sin(\pi ak/q)| \prod_{k=l+1}^{N} |\sin \pi k\alpha|, \quad 0 \leqslant l \leqslant N,$$

where the asterisk signifies that zero factors—that is, the terms with $k \equiv 0 \bmod q$—are omitted. Note that

(4.3) $$Q_0 = P_N(\alpha).$$

We have $(l+1 \leqslant N)$

$$Q_l/Q_{l+1} = \begin{cases} |\sin \pi(l+1)\alpha / \sin \pi(a(l+1)/q)|, & l+1 \not\equiv 0 \bmod q, \\ |\sin \pi(l+1)\alpha| \leqslant 1, & l+1 \equiv 0 \bmod q. \end{cases}$$

Consider this ratio when $l+1 \not\equiv 0 \bmod q$. Here

$$\left|\sin \pi(l+1)\alpha - \sin \pi \frac{a(l+1)}{q}\right| \leqslant \pi(l+1)|\beta|$$

so that the ratio in question is at most

$$1+\frac{\pi(l+1)|\beta|}{\left|\sin \dfrac{\pi a(l+1)}{q}\right|} \leqslant 1+\frac{\pi N|\beta|}{\sin(\pi/q)} \leqslant 1+\tfrac{1}{2}\pi q N|\beta| \leqslant 1+\tfrac{1}{2}\pi < e.$$

We can do better in a number of cases. Suppose $a(l+1) \equiv s \bmod q$, $1 \leqslant s \leqslant q-1$. Then

$$\sin \pi \frac{a(l+1)}{q} = \sin \frac{\pi s}{q} \geqslant \frac{2}{\pi}\min\left(\frac{\pi s}{q}, \frac{\pi(q-s)}{q}\right) = \frac{2}{q}\min(s, q-s).$$

Let D be a large number less than q, and suppose that $\min(s, q-s) \geqslant D$. Then the ratio under consideration is at most

$$1+\frac{\pi N|\beta|}{2D/q} = 1+\frac{\pi}{2D} qN|\beta| < 1+\frac{\pi}{2D}.$$

To sum up,

$$Q_l/Q_{l+1} \leqslant \begin{cases} 1, & l+1 \equiv 0 \bmod q, \\ 1+\pi/(2D), & a(l+1) \equiv s \bmod q,\ \min(s, q-s) \geqslant D, \\ e, & \text{otherwise}. \end{cases}$$

In any complete set of residues we need to invoke the last, and worst, of these inequalities at most $2D$ times. Hence, by (4.3),

$$P_N(\alpha) = \Big(\prod_{l=0}^{N-1} Q_l/Q_{l+1}\Big) Q_N \leqslant e^{(N/q+1)2D}\left(1+\frac{\pi}{2D}\right)^N Q_N$$
$$\leqslant e^{2{,}002D+\frac{\pi}{2D}N} Q_N < e^{57N^{1/2}} Q_N$$

on choosing D so that

$$4{,}004D^2 = \pi N.$$

In view of (4.1) this choice of D is admissible if N is large enough. Thus

(4.4) $$P_N(\alpha) \leqslant e^{57N^{1/2}} Q_N.$$

By (4.2),

$$Q_N = \prod_{k=1}^{N}{}^{*}\left|\sin \pi \frac{ak}{q}\right|.$$

Let

(4.5) $$N = qm+N_1, \quad 0 \leqslant N_1 < q.$$

If $q > N/2$, $m = 1$ necessarily and $N_1 = N-q < N/2$. Hence

(4.6) $$N_1 < N/2$$

always. Consequently, by (1.3),

(4.7) $$Q_N = \prod_{k=1}^{qm}{}^{*}\left|\sin \frac{\pi ak}{q}\right| \prod_{k=1}^{N_1}\left|\sin \frac{\pi ak}{q}\right| = \left(\frac{q}{2^{q-1}}\right)^m \prod_{k=1}^{N_1}\left|\sin \frac{\pi ak}{q}\right|$$
$$= \left(\frac{q}{2^{q-1}}\right)^m P_{N_1}\left(\frac{a}{q}\right).$$

Suppose first of all that $N_1 \leqslant C_0$, some absolute constant. Then

(4.8) $$Q_N \leqslant \left(\frac{q}{2^{q-1}}\right)^m = (2^{1-q}q)^{(N-N_1)/q} = (2^{1-q}q)^{-N_1/q}\left(\tfrac{1}{2}(2q)^{1/q}\right)^N$$
$$\leqslant 2^{C_0}\left(\tfrac{1}{2}(2q)^{1/q}\right)^N < 2^{C_0}(0.55)^N,$$

say, by (4.1) (so that $(2q)^{1/q} < 1+\varepsilon$ if $N \geqslant N_0(\varepsilon)$). Hence we may suppose that N_1 is large, and now preceding arguments apply. To be precise, we suppose, as we may, that

$$\left|\frac{a}{q}-\frac{a_1}{q_1}\right| \leqslant \frac{1}{q_1(N_1+1)}, \quad (a_1, q_1) = 1, \quad 1 \leqslant q_1 \leqslant N_1,$$

and consider the two possibilities that (i) $1 \leqslant q_1 \leqslant N_1/1{,}000$ and (ii) $N_1/1{,}000 < q_1 \leqslant N_1$. The latter case is the easier to dispose of: by (4.7),

$$Q_N \leqslant \left(\frac{q}{2^{q-1}}\right)^m \left(\frac{q_1}{2^{q_1-1}}\right)^{m_1},$$

where, as above, $N_1 = q_1 m_1 + N_2$, $N_2 < \frac{1}{2}N_1 < \frac{1}{4}N$. Now

$$mq + m_1 q_1 = (N-N_1)+(N_1-N_2) = N-N_2 > N-\tfrac{1}{4}N = \tfrac{3}{4}N,$$

and each of m and m_1 is at most 1,000. Hence

$$Q_N \leqslant \frac{(2N)^{1,000}}{2^{3N/4}} \leqslant (2N)^{1,000}(0.595)^N < (0.598)^N \tag{4.9}$$

since N is large.

We come finally to (i) above. But here we are back in Case I or Case II, so that (2.4) or (3.2) applies and

$$\begin{aligned} Q_N &< (2^{1-q}q)^m (0.61)^{N_1} \\ &\leqslant \frac{(2N)^{1,000}}{2^{N-N_1}}(0.61)^{N_1} = (2N)^{1,000}\, 2^{-N}(1.22)^{N_1} \\ &< (2N)^{1,000}\left(\frac{\sqrt{1.22}}{2}\right)^N < (2N)^{1,000}(0.555)^N \\ &< (0.56)^N \end{aligned}$$

for sufficiently large N. Combining this with (4.8) and (4.9) we obtain

$$Q_N < (0.598)^N$$

in all cases and consequently, by (4.4), that

$$P_N(\alpha) < (0.6)^N, \quad N/1{,}000 < q \leqslant N.$$

This settles Case III, and our theorem is proved.

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