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On the number of integers n such that $nd(n) \leqslant x$

by

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1. Introduction. In this paper, we investigate the asymptotic formula for the number of integers n such that $nd(n) \leqslant x$. The problem was first considered by Abbott and Subbarao [1] who proved that

$$\sum_{nd(n) \leqslant x} 1 \sim c \frac{x}{\sqrt{\log x}} \quad \text{for a suitable } c > 0.$$

Our aim is to improve and generalize their result.

Let $\lambda > 0$ and let $g(n)$ be a multiplicative function such that (i) $g(p) = 1/\lambda$ for all primes p, (ii) $g(n) > 0$ and (iii) $g(n) \gg n^{-1/16}$. Then we prove

Theorem 1. *The following asymptotic formula holds*:

$$\sum_{ng(n) \leqslant x} 1 \sim cx(\log x)^{\lambda-1} \quad \textit{for a suitable constant } c > 0.$$

Remark 1. It is clear from the proof that we get an asymptotic expansion with an error term

$$O\left(x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right)$$

Remark 2. Theorem 1 gives an affirmative answer to the problem raised by Professor Erdős, in one of his letters to the second author, whether

$$\sum_{n/d(n) \leqslant 2x} 1 \sim 2 \sum_{n/d(n) \leqslant x} 1 \quad \text{as} \quad x \to \infty$$

Remark 3. The condition $|g(n)| \gg n^{-1/16}$ could be weakened to

$$|g(n)| \gg n^{-1/2+\varepsilon} \quad \text{for some } \varepsilon > 0.$$

Remark 4. The method of proof will also apply to get an asymptotic formula for $\sum_{n \leqslant x} 1/g(n)$. In fact,

$$\sum_{n \leqslant x} \frac{1}{g(n)} \sim \sum_{ng(n) \leqslant x} 1.$$

2. Notation. Let $f(s) = \sum_{n=1}^{\infty} 1/(ng(n))^s$ in $\sigma > 2$; $\tau = |t|+30$; $s = \sigma + it$; A be a constant, as appearing in Lemma 1 below. x is sufficiently large; $B = \exp\left(-A(\log x)^{3/5}(\log\log x)^{-1/5}\right)$.

3. Analytic continuation of $f(s)$. Let $\Phi(s) = f(s)(\zeta(s))^{-\lambda^s}$ in $\sigma > 1$. Here we take a suitable branch of $(\zeta(s))^{-\lambda^s} = \exp(-\lambda^s \log\zeta(s))$ such that it is positive for real values of $s > 1$.

LEMMA 1. *There exists a constant A, $0 < A < 1/10000$, such that*

$$\sigma \geqslant 1 - 100A(\log\tau)^{-2/3}(\log\log\tau)^{-1/3},$$

$|s-1| \geqslant 10^{-3}$ *is free of zeros of* $\zeta(s)$. *Further in this region,* $\log(\zeta(s)(s-1))$ *can be analytically continued such that it is real for real values of $s > 1$ and further* $|\log(\zeta(s))|$ *is* $\ll (\log\tau)^{99/100}$ *in that region.*

Proof. The proof can be found in [3] or [4].

LEMMA 2. *The function* $f(s) = \Phi(s)(\zeta(s)(s-1))^{\lambda^s}(s-1)^{-\lambda^s}$ *can be analytically continued in* $\sigma \geqslant 1 - 50A(\log\tau)^{-2/3}(\log\log\tau)^{-1/3}$, $|s-1| \geqslant 10^{-3}$, $-\pi < \arg(s-1) < \pi$ *so that $f(s)$ is real for real values of $s > 1$ and further in this region* $f(s) = O(\tau^{1/2})$.

Proof. We have

$$\Phi(s) = \prod_p \left(\left(1 + \frac{1}{(pg(p))^s} + \frac{1}{(p^2 g(p^2))^s} + \ldots\right)\left(1 - \frac{\lambda^s}{p^s} + \frac{\lambda^s(\lambda^s - 1)}{2!\,p^{2s}} - \ldots\right)\right)$$

$$= \prod_p \left(1 + \sum_{n=1}^{\infty} \frac{c_n}{p^{ns}}\right), \quad \text{say.}$$

Clearly $c_1 = 0$ and

$$c_n = \sum_{r+l=n} \binom{\lambda^s}{r} \frac{(-1)^r}{(g(p^l))^s}.$$

Since $|g(p^l)| \geqslant p^{-l/16}$ and $\left|\binom{\lambda^s}{r}\right| \leqslant \binom{\lambda^\sigma + r - 1}{r}$, we have

$$|c_n| \leqslant \sum_{r+l=n} \binom{\lambda^\sigma + r-1}{r} p^{l\sigma/16} = \sum_{r=0}^{n} \binom{\lambda^\sigma + r - 1}{r} p^{(n-r)\sigma/16}$$

$$\leqslant p^{n\sigma/16} \sum_{r=0}^{\infty} \binom{\lambda^\sigma + r - 1}{r} p^{-r\sigma/16}$$

$$\leqslant p^{n\sigma/16}(1 - p^{-\sigma/16})^{-\lambda^\sigma} \leqslant p^{n\sigma/16}(1 - 2^{-\sigma/16})^{-\lambda^\sigma}.$$

Hence (with $\ll$ constant depending on λ), in $3/4 \leqslant \sigma \leqslant 2$

$$\left|\sum_{n=1}^{\infty} \frac{c_n}{p^{ns}}\right| \ll \left|\sum_{n=2}^{\infty} \frac{p^{n\sigma/16}}{p^{n\sigma}}\right| \ll p^{-15\sigma/8}.$$

Hence the product $\prod_p (1 + \sum c_n p^{-ns})$ converges in $\sigma \geqslant 3/4$ and thus defines an analytic function there. From Lemma 1, we see that $(\zeta(s)(s-1))^{\lambda^s}$ can be analytically continued in $\sigma \geqslant 1 - 50A(\log\tau)^{-2/3}(\log\log\tau)^{-1/3}$; and further we note that, in this region, $\Phi(s)$ is bounded and

$$|(\zeta(s))^{\lambda^s}| = |\exp(\lambda^s \log\zeta(s))| \leqslant \exp(|\lambda^s \log\zeta(s)|) = O(\tau^{1/4})$$

by Lemma 1. Hence (since we may restrict to $\sigma \leqslant 2$),

$$f(s) = \Phi(s)(\zeta(s))^{\lambda^s} = O(\tau^{1/2}).$$

4. The contour integration. For any $\varepsilon > 0$, we consider the following contour $c(\varepsilon)$. Let $\alpha(t) = 1 - 10A(\log\tau)^{-2/3}(\log\log\tau)^{-1/3}$. From $t = -\infty$ to $t = -\varepsilon$, we traverse along the curve $\sigma = \alpha(t)$ (denoted by $L_1(\varepsilon)$), we go along the circle with centre $s = 1$, in the anticlockwise direction till we reach $\alpha(\varepsilon) + i\varepsilon$ (denoted by $C(\varepsilon)$) and continue along the curve $\sigma = \alpha(t)$ till $t = \infty$ (denoted by $L_2(\varepsilon)$). The curves L_1, C and L_2 are obtained by allowing $\varepsilon \to 0$.

LEMMA 3. *We have*

$$\sum_{ng(n) \leqslant x} \log\frac{x}{ng(n)} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\frac{x^s}{s^2}\,ds.$$

Proof. Since

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{y^s}{s^2}\,ds = \begin{cases} \log y, & y \geqslant 1, \\ 0, & 0 < y < 1 \end{cases} \quad \text{and} \quad f(s) = \sum_n \frac{1}{(ng(n))^s},$$

the result follows.

LEMMA 4. *On L_1 and L_2,*

$$\left|\frac{x^s}{|s|^{1/4}}\right| = O(xB^6)$$

and consequently $\int_{L_j} f(s)\frac{x^8}{s^2} = O(xB^6)$.

Proof. If $\log\tau \geqslant 24A(\log x)^{3/5}(\log\log x)^{-1/5}$, then

$$\left|\frac{x^s}{|s|^{1/4}}\right| \leqslant \frac{|x|}{\tau^{1/4}}$$

and hence the estimate. If $\log\tau \leqslant 24A(\log x)^{3/5}(\log\log x)^{-1/5}$, then

$$\left|\frac{x^s}{|s|^{1/4}}\right| \ll x^\sigma = x^{\alpha(t)} = x \cdot x^{-(1-\alpha(t))}$$

and hence the estimate. Now

$$\int_{L_1} f(s)\frac{x^s}{s^2}ds \ll \max_{s\in L_1}\frac{|x^s|}{|s|^{1/4}}\int\frac{|f(s)|}{|s|^{7/4}}|ds|$$

and since $f(s) = O(|s|^{1/2})$, the integral is convergent. Thus the required estimate follows, and similarly we argue for $\int\limits_{L_2}$.

Thus we have

THEOREM 2. *There holds the asymptotic formula*

$$\sum_{ng(n)\leqslant x}\log\frac{x}{ng(n)} = \frac{1}{2\pi i}\int_C f(s)\frac{x^s}{s^2}ds+O(xB^6).$$

Proof. From Lemma 3 we have

$$\sum_{ng(n)\leqslant x}\log\frac{x}{ng(n)} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\frac{x^s}{s^2}ds.$$

Moving the line of integration, we have

$$\sum_{ng(n)\leqslant x}\log\frac{x}{ng(n)} = \frac{1}{2\pi i}\Big(\int_C+\int_{L_1}+\int_{L_2}\Big)\left(\frac{f(s)x^s}{s^2}ds\right)$$

and the result follows from Lemma 4.

5. Expansion of $f(s)$.

LEMMA 5. *On the contour C, the following expansions hold for suitable b_m, c_m, d_m and $e_{m,n}$:*

(a) $\Phi(s) = \sum\limits_{m=0}^{\infty} b_m(s-1)^m$,

(b) $\big(\zeta(s)(s-1)\big)^{\lambda} = \sum\limits_{m=0}^{\infty} c_m(s-1)^m$,

(c) *For every integer n,*

$$\left(\frac{1-\lambda^{s-1}}{s-1}\right)^n = \sum_{m=0}^{\infty} d_m(s-1)^m,\qquad d_m = d_m(n)$$

(d) $(s-1)^{\lambda-\lambda^s} = \sum\limits_{\substack{m,n,\\ n\leqslant m}} e_{m,n}(s-1)^m\big(\log(s-1)\big)^n$.

Further

$$b_m = O(8^m);\quad c_m = O(8^m);\quad d_m = O(8^m H^n)\quad \text{and}\quad e_{m,n} = O\left(\frac{\lambda^n H^n}{n!}8^m\right).$$

Here

$$H = \max_{|s-1|\leqslant 1/2}\left|\frac{1-\lambda^{s-1}}{s-1}\right|.$$

Proof. Since $\Phi(s)$ is regular in $|s-1| \leqslant 1/8$, the power series expansion in (a) is valid. Further,

$$b_m = \frac{1}{2\pi i}\int_{|s-1|=1/8}\frac{\Phi(s)}{(s-1)^{m+1}}ds = O(8^m).$$

Similarly we can prove (b) and (c). Now

$$(s-1)^{\lambda-\lambda^s} = \exp\big((\lambda-\lambda^s)\log(s-1)\big) = \sum_{j=0}^{\infty}\frac{(\lambda-\lambda^s)^j}{j!}\big(\log(s-1)\big)^j$$

$$= \sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\left(\frac{1-\lambda^{s-1}}{s-1}\right)^j(s-1)^j\big(\log(s-1)\big)^j$$

and (d) follows using (c).

LEMMA 6. *On the contour C, for suitable coefficients $a_{m,n}$ with $|a_{m,n}| \ll e^{\lambda H}10^m$, we have*

$$f(s)(s-1)^{\lambda} = \sum_{0\leqslant n\leqslant m} a_{m,n}(s-1)^m\big(\log(s-1)\big)^n.$$

Proof. Since $f(s)(s-1)^{\lambda} = \Phi(s)\big(\zeta(s)(s-1)\big)^{\lambda}(s-1)^{\lambda-\lambda^s}$, the result follows from Lemma 5.

6. More on the main term. Now we get an asymptotic formula for the main term appearing on the right side of Theorem 2.

Lemmas 7, 8 and 9 are probably well known.

LEMMA 7. *For an integer $m \geqslant 0$ and real $a \geqslant 0$,*

$$\int_{\substack{-\infty\\ \arg(s-1)=\pm\pi}}^{1}(s-1)^a\big(\log(s-1)\big)^m x^{s-1}ds = \left(\frac{d}{da}\right)^m\left(\frac{e^{\pm\pi i a}\Gamma(a+1)}{(\log x)^{a+1}}\right).$$

Proof. We consider the case $\arg(s-1) = \pi$, the other case being similar. In the special case $m = 0$, the substitution $s = 1+re^{i\pi}$ makes the integral $e^{i\pi a}\int\limits_0^{\infty} r^a x^{-r}dr$ which is $\dfrac{e^{i\pi a}\Gamma(a+1)}{(\log x)^{a+1}}$. The general case is obtained by differentiating m times with respect to a.

LEMMA 8. *Let $\mathscr{D}(\varepsilon)$ be any curve starting from $-\infty-i\varepsilon$ and ending at $-\infty+i\varepsilon$ traversing in the anticlockwise direction such that the point $s = 1$ is inside the region enclosed by $\mathscr{D}(\varepsilon)$. Let $\mathscr{D}$ be the curve obtained by allowing $\varepsilon\to 0$ in $\mathscr{D}(\varepsilon)$. Then*

$$\frac{1}{2\pi i}\int_{\mathscr{D}} x^{s-1}(s-1)^a\big(\log(s-1)\big)^m ds = \left(\frac{d}{da}\right)^m\left(\frac{\Gamma(a+1)}{(\log x)^{a+1}}\,\frac{\sin\pi a}{\pi}\right).$$

Proof. As in Lemma 7, it suffices to consider $m = 0$. In this case, since both sides are analytic functions of the complex variable a, it suffices to prove the lemma for real $a \geqslant 1$. In this case, the curve $\mathscr{D}(\varepsilon)$ could be transformed to the curve $\mathscr{D}^*(\varepsilon)$ defined as follows. From $-\infty - i\varepsilon$ to $1-\varepsilon-i\varepsilon$ by a straight line, from $1-\varepsilon-i\varepsilon$ to $1-\varepsilon+i\varepsilon$ by a circle around $s=1$ and from $1-\varepsilon+i\varepsilon$ to $-\infty+i\varepsilon$ by a straight line. The curve $\mathscr{D}^*$ is obtained as $\lim_{\varepsilon\to 0}\mathscr{D}^*(\varepsilon)$. Then

$$\frac{1}{2\pi i}\int_{\mathscr{D}} x^{s-1}(s-1)^a\,ds = \frac{1}{2\pi i}\int_{\mathscr{D}^*} x^{s-1}(s-1)^a\,ds.$$

Now as $\varepsilon \to 0$, the integral on the circular part of $\mathscr{D}^*(\varepsilon)$ tends to zero. Hence

$$\frac{1}{2\pi i}\int_{\mathscr{D}} x^{s-1}(s-1)^a\,ds = \int_{\substack{-\infty\\ \arg(s-1)=-\pi}}^{1} + \int_{\substack{1\\ \arg(s-1)=\pi}}^{-\infty}$$

and now the result follows from Lemma 7.

Lemma 9. *For integer $m \geqslant 0$ and real $a \leqslant (\log x)^{4/5}$, we have (if $|a| + m \leqslant (\log x)^{4/5}$)*

$$\frac{1}{2\pi i}\int_C x^{s-1}(s-1)^a(\log(s-1))^m\,ds = \left(\frac{d}{da}\right)^m\left(\frac{(\sin \pi a)\,\Gamma(a+1)}{\pi(\log x)^{a+1}}\right) + O(xB^3).$$

Proof. Consider the curve $\mathscr{D}(\varepsilon)$ defined as follows. From $-\infty-i\varepsilon$ to $\alpha(\varepsilon)-i\varepsilon$ by a straight line, from $\alpha(\varepsilon)-i\varepsilon$ to $\alpha(\varepsilon)+i\varepsilon$ along a circle with centre $s=1$ (that is along the curve $C(\varepsilon)$) and from $\alpha(\varepsilon)+i\varepsilon$ to $-\infty+i\varepsilon$ by a straight line. Then the curve $\mathscr{D}(\varepsilon)$ satisfies the conditions of Lemma 8. Hence, from Lemma 8,

$$\frac{1}{2\pi i}\int_{\mathscr{D}} x^{s-1}(s-1)^a(\log(s-1))^m\,ds = \left(\frac{d}{da}\right)^m\left(\frac{\Gamma(a+1)}{(\log x)^{a+1}}\,\frac{\sin \pi a}{\pi}\right).$$

Hence, defining $\alpha = \alpha(0)$,

$$\frac{1}{2\pi i}\Big(\int_{\substack{-\infty\\ \arg(s-1)=-\pi}}^{\alpha} + \int_C + \int_{\substack{\alpha\\ \arg(s-1)=\pi}}^{-\infty}\Big)\big(x^{s-1}(s-1)^a(\log(s-1))^m\,ds\big)$$
$$= \left(\frac{d}{da}\right)^m\left(\frac{\Gamma(a+1)\sin \pi a}{\pi(\log x)^{a+1}}\right).$$

Hence

$$\frac{1}{2\pi i}\int_C x^{s-1}(s-1)^a(\log(s-1))^m\,ds$$
$$= \left(\frac{d}{da}\right)^m\left(\frac{\Gamma(a+1)\sin \pi a}{\pi(\log x)^{a+1}}\right) + O\Big(\int_{\substack{-\infty\\ \arg(s-1)=\pm\pi}}^{\alpha} x^{\sigma-1}(1-\sigma)^a|\log(s-1)|^m\,d\sigma\Big)$$

and this proves the result.

Lemma 10. *For integer $m \geqslant 0$ and real $a \geqslant m/2$ such that $m+a \geqslant (\log x)^{4/5}$, we have*

$$\frac{1}{2\pi i}\int_C x^{s-1}(s-1)^a(\log(s-1))^m\,ds = O\big(xB^3(1-\alpha)^{a/2}\big),$$

where $\alpha = \alpha(0)$.

Proof. We note that the contour $C(\varepsilon)$ could be transformed into the contour defined as follows. From $\alpha(\varepsilon)-i\varepsilon$ to $1-\varepsilon-i\varepsilon$ by a straight line, from $1-\varepsilon-i\varepsilon$ to $1-\varepsilon+i\varepsilon$ by a circle around $s=1$ and from $1-\varepsilon+i\varepsilon$ to $\alpha(\varepsilon)+i\varepsilon$ by a straight line. Since a is large enough, the contribution of the circular part tends to zero as $\varepsilon\to 0$. Hence

$$\frac{1}{2\pi i}\int_C x^{s-1}(s-1)^a(\log(s-1))^m\,ds$$
$$= \frac{1}{2\pi i}\int_{\substack{\alpha\\ \arg(s-1)=-\pi}}^{1} x^{s-1}(s-1)^a(\log(s-1))^m\,ds$$
$$+\frac{1}{2\pi i}\int_{\substack{1\\ \arg(s-1)=\pi}}^{\alpha} x^{s-1}(s-1)^a(\log(s-1))^m\,ds$$
$$= O\Big(\int_{\substack{\alpha\\ \arg(s-1)=\pm\pi}}^{1} x^{\sigma-1}|1-\sigma|^a|\log(s-1)|^m\,d\sigma\Big)$$

and now trivial estimate gives the result.

Theorem 3. *The integral $\int_C \frac{f(s)x^s}{s}\,ds$ can be expressed as*

$$x(\log x)^{\lambda-1}\sum_{n\leqslant m\leqslant(\log x)^{4/5}} A_{m,n}(\log x)^{-m}(\log\log x)^n + O(xB^2)$$

where the coefficients $A_{m,n}$ are bounded by $O(100^m m^m(\log m)^m)$. A similar result holds for $\int_C \frac{f(s)x^s}{s^2}\,ds$.

Proof. By Lemma 6, $f(s)(s-1)^\lambda$ can be written as

$$\sum_{0\leqslant n\leqslant m} a_{m,n}(s-1)^m(\log(s-1))^n.$$

Hence

$$f(s) = \sum_{0\leqslant n\leqslant m} a_{m,n}(s-1)^{m-\lambda}(\log(s-1))^n.$$

Now the result follows from Lemmas 9 and 10.

7. Final result. We define

$$S(x) = \sum_{ng(n)\leqslant x} 1, \qquad M(x) = \int_C f(s)\frac{x^s}{s}\,ds,$$

$$H(x) = \int_0^x \frac{S(u)}{u}\,du = \sum_{ng(n)\leqslant x} \log\frac{x}{ng(n)};$$

let δ be a real number with $\delta = o(x)$.

LEMMA 11. *We have*

$$H(x+\delta)-H(x) \geqslant \delta S(x)/(x+\delta).$$

Proof.

$$H(x+\delta)-H(x) = \int_x^{x+\delta}\frac{S(u)}{u}\,du \geqslant \frac{S(x)}{x+\delta}\int_x^{x+\delta} du \geqslant \frac{S(x)}{x+\delta}\,\delta.$$

LEMMA 12. *We have*

$$H(x+\delta)-H(x) \leqslant \frac{\delta}{x}M(x)+O(xB^2)+O\left(\frac{\delta^2}{x}(\log x)^{\lambda+10}\right).$$

Proof. In $x\leqslant u\leqslant x+\delta$,

$$M(u) = M(x)+O\big(\delta M'(\xi)\big) \quad \text{for some } \xi,\ x\leqslant\xi\leqslant x+\delta.$$

From the method of proof of Theorem 3, $M'(\xi) = O\big((\log x)^{\lambda+10}\big)$. Hence

$$M(u) = M(x)+O\big(\delta(\log x)^{\lambda+10}\big).$$

By Theorem 2,

$$H(x) = \int_C f(s)\frac{x^s}{s^2}\,ds+O(xB^6) = \int_0^x\frac{M(u)}{u}\,du+O(xB^6),$$

$$H(x+\delta)-H(x) = \int_x^{x+\delta}\frac{M(u)}{u}\,du+O(xB^6) \leqslant \frac{1}{x}\int_x^{x+\delta}M(u)\,du+O(xB^6)$$

$$= \frac{1}{x}\int_x^{x+\delta}\big(M(x)\big)\,du+O\left(\frac{\delta^2}{x}(\log x)^{\lambda+10}\right)+O(xB^6)$$

and this proves the result.

LEMMA 13. *We have*

$$S(x) \leqslant M(x)+O\left(\frac{x^2B^6}{\delta}\right)+O\big(\delta(\log x)^{\lambda+10}\big).$$

Proof. This follows from Lemmas 11 and 12.

Similarly, by considering the interval $(x-\delta, x)$, we can deduce

LEMMA 14. *We have*

$$S(x) \geqslant M(x)+O\left(\frac{x^2B^6}{\delta}\right)+O\big(\delta(\log x)^{\lambda+10}\big).$$

From Lemmas 13 and 14, we deduce, by the proper choice of δ

THEOREM 4. *We have*

$$S(x) = M(x)+O\Big(x\exp\big(-A(\log x)^{3/5}(\log\log x)^{-1/5}\big)\Big).$$

Now Theorem 1 follows from Theorems 3 and 4.

8. Some related questions. In this section, we discuss some related problems. Mainly we are interested in relaxing the condition $g(p) = 1/\lambda$. For example, one can assume that

$$g(p) = \frac{1}{\lambda}+O\Big(\exp\big(-c(\log p)^{1/a}\big)\Big) \quad \text{for some } c>0,\ 0\leqslant a<2/3$$

and conclude that

$$\sum_{ng(n)\leqslant x} 1 = \int_C f(s)\frac{x^s}{s}\,ds+O\Big(x\exp\big(-A(\log x)^{3/5}(\log\log x)^{-1/5}\big)\Big)$$

for some $A>0$.

If, on the other hand, one assumes that

$$g(p) = \frac{1}{\lambda}+O\Big(\exp\big(-c(\log p)^{1/a}\big)\Big) \quad \text{for some } c>0,\ 2/3\leqslant a\leqslant 1,$$

then, defining as above $\Phi(s) = (\zeta(s))^{-\lambda s}f(s)$, we may not be able to prove that $|\Phi(s)| = O\big((|t|+3)^{1/2}\big)$. But we can choose, as has been done in Bateman [2], for the particular case $g(n) = \frac{\varphi(n)}{n}$, $\alpha(t) = 1-c\frac{\log\log\tau}{\log\tau}$ and move the line of integration to $\sigma = \alpha(t)$ with a detour at the point $s = 1$. This proves that

$$\sum_{ng(n)\leqslant x} 1 = \int_C f(s)\frac{x^s}{s}\,ds+O\Big(x\exp\big(-A(\log x\log\log x)^{1/a+1}\big)\Big) \quad \text{for some } A>0.$$

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The growth rate of the Dedekind Zeta-function on the critical line

by

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For Paul Erdős
on his 75th birthday

1. Introduction. Let K be an algebraic number field of degree n, and let $\zeta_K(s)$ be its Dedekind Zeta-function. Thus

$$\zeta_K(s) = \sum_A (NA)^{-s} \quad (\mathrm{Re}(s) > 1),$$

where A runs over the non-zero integral ideals of K, and NA is the absolute norm of A. The question considered in this paper is the order of magnitude of $\zeta_K(s)$ on the critical line. The trivial bound is

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/4} \quad (t \geqslant 1),$$

where the notation $\ll_K$ indicates that the implied constant may depend on K. This follows from our Lemma 2, for example. When $K = Q$, the Dedekind Zeta-function reduces to the Riemann Zeta-function $\zeta(s)$, and one has the estimate $\zeta(\frac{1}{2}+it) \ll t^{1/6+\varepsilon}$ $(t \geqslant 1)$ for any fixed $\varepsilon > 0$. Indeed, the exponent can be slightly reduced. When the field K is Abelian, $\zeta_K(s)$ factorizes as a product of $\zeta(s)$ and $n-1$ Dirichlet L-functions $L(s, \chi)$. For these one can prove an estimate

$$L(\tfrac{1}{2}+it, \chi) \ll_\chi t^{1/6+\varepsilon} \quad (t \geqslant 1). \tag{1.1}$$

(Here also it is possible to improve the exponent 1/6 in the same way as for $\zeta(s)$.) It follows that

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/6+\varepsilon} \quad (t \geqslant 1) \tag{1.2}$$

if K is Abelian. It would be of interest to make the dependence on K explicit. However, it is difficult to get a satisfactory uniform estimate even in the case of (1.1), and so we concentrate on the t-dependence in this paper. Our goal is to prove the bound (1.2) for all K, whether Abelian or not.