which gives

$$\bigcup_{r=1}^{\infty} \lambda_r-\alpha_r \varrho^{-k}+[-\delta\varrho^{-k-1}/4, 0] \cap \bigcup_{n=1}^{\infty} 1/\mu_n Z+[0, \delta\varrho^{-k-1}/4] = \emptyset.$$

If $\alpha_r = 0$, $r = 1, 2, \ldots$ the contradiction is established by Lemma 2. A slight modification of the argument will give the general case, remembering that α_r is bounded.

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On prime factors of sums of integers III

by

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Dedicated to Professor P. Erdős on the occasion of his seventy-fifth birthday

1. In [1], pp. 36–37, Erdős posed the following questions. (We use a slightly different notation and denote the number of distinct prime divisors of an integer n by $\omega(n)$.) "Let $G(k)$ be a graph of k vertices $x_1, \ldots, x_k$. Let $a_1, \ldots, a_k$ by any set of k distinct integers. Put

$$\omega(G(k)) = \min \omega\left(\prod (a_i+a_j)\right)$$

where the factor a_i+a_j occurs if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct integers $a_1, \ldots, a_k$. It is probably hopeless to determine $\omega(G(k))$ except for very simple graphs, but perhaps one can get conditions on the class of graphs for which $\omega(G(k)) \to \infty$. One conjecture which just occurs to me states: There is a $g(r) \to \infty$ as $r \to \infty$ so that if $G(k)$ has a chromatic number $\geqslant r$ then $\omega(G(k)) \geqslant g(r)$. A stronger conjecture: If $G(k)$ has more than kr edges then $\omega(G(k)) \geqslant g(r)$. Perhaps this is too optimistic".

We shall show that the latter conjecture is incorrect indeed, but that Theorem 1 implies

$$\omega\left(\prod (a_i+a_j)\right) \gg \log r \quad \text{as} \quad r \to \infty$$

under the additional assumption that $\omega(a_1 \ldots a_k)$ is bounded. Furthermore, it follows from Theorem 2 that if $G(k)$ has more than $k^{3/2} r$ edges or if $G(k)$ contains a complete bipartite subgraph of type $(r, 2)$ then $\omega(G(k)) \gg \log r$ as $r \to \infty$. Here $\gg$ denotes the Vinogradov symbol. Similar results will also be proved for the products of differences of integers. Our results are applications

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of two deep results (Lemmas 1, 2) which provide good and explicit upper bounds for the numbers of solutions of S-unit equations in two variables.

2. The following example shows that Erdős' stronger conjecture is incorrect. Let k be an odd integer, $k \geqslant 35$. Put $a_i = i-(k+1)/2$ for $i = 1, \ldots, k$. Let $p_1, \ldots, p_s$ be the smallest s prime numbers, $s \geqslant 7$. An integer n is said to be *composed* of $p_1, \ldots, p_s$ if n has no prime divisor different from $p_1, \ldots, p_s$. The number, R, of positive integers less than $k/2$ which are composed of $p_1, \ldots, p_s$ equals the number of non-negative integer solutions $z_1, \ldots, z_s$ of the inequality

$$z_1 \log p_1 + \ldots + z_s \log p_s < \log(k/2).$$

Since $p_j \leqslant 2s\log s$ for $j = 1, \ldots, s$ (see [7], formula (3.13)), R is at least the number of non-negative integer solutions $z_1, \ldots, z_s$ of the inequality

$$z_1 + \ldots + z_s < \frac{\log(k/2)}{\log(2s\log s)},$$

hence

$$R \geqslant \frac{1}{s!}\left(\frac{\log(k/2)}{\log(2s\log s)}\right)^s \geqslant 4\left(\frac{\log(k/2)}{s\log s}\right)^s.$$

For each positive integer n with $1 \leqslant n < k/2$ there are at least $(k-2)/2$ pairs (i, j) with $1 \leqslant i < j \leqslant k$ and $a_i + a_j = \pm n$. Let G be the graph with vertex set $\{x_1, \ldots, x_k\}$ such that x_i and x_j are joined by an edge if and only if $a_i + a_j$ is composed of $p_1, \ldots, p_s$. Then the number of edges of G is at least

$$\frac{k-2}{2} R \geqslant k\left(\frac{\log(k/2)}{s\log s}\right)^s.$$

This implies that the latter conjecture of Erdős is incorrect as we see by taking $r = (\log(k/2)/(s\log s))^s$ and letting $k \to \infty$, s fixed. It even follows that it is not sufficient to assume that $G(k)$ has more than $k(\log k)^A r$ edges, where A is any constant, to conclude that $\omega(G(k)) \to \infty$ as $r \to \infty$.

3. We state a special case of a result of Evertse which shall be used in the proofs of Theorems 1 and 2.

LEMMA 1 (Evertse [2]). *Let v be a non-zero integer. There are at most $3 \times 7^{2s+3}$ pairs of integers x, y each composed of $p_1, \ldots, p_s$ such that $x + y = v$.*

The crucial point of the lemma is that the upper bound only depends on s. We used this lemma in [5] to derive a lower bound for the number of distinct prime divisors in the full product $\prod_{i=1}^{k}\prod_{j=1}^{l}(a_i+b_j)$. We refer to [8] for a survey of related results.

4. As to Erdős' second conjecture it is true that if $G(k)$ has more than kr edges then $\omega(\prod(a_i+a_j)) \to \infty$ as $r \to \infty$, provided that $\omega(a_1 \ldots a_k)$ is bounded. We prove a more general result. We denote the cardinality of a set V by $|V|$.

THEOREM 1. *Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ be sets of non-zero integers. Let $\{p_1, \ldots, p_s\}$ be a set of prime numbers. Suppose $a_1, \ldots, a_k$ are all composed of $p_1, \ldots, p_s$. Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of $p_1, \ldots, p_s$ if $(i, j) \in V$. Then*

$$s \geqslant \frac{1}{4}\log\frac{|V|}{l} - 2. \tag{1}$$

Proof. Choose $j \in \{1, \ldots, l\}$ such that there are at least $|V|/l$ integers i with $(i, j) \in V$. Taking $v = b_j$ we find that the equation $x + y = v$ has at least $|V|/l$ solutions $x = a_i + b_j$, $y = -a_i$. By applying Lemma 1 we obtain

$$|V|/l \leqslant 3 \times 7^{2s+3} < 7^{2s+4}$$

whence (1). ∎

5. We now drop the condition that $a_1, \ldots, a_k$ are composed of $p_1, \ldots, p_s$ and derive a weaker lower bound for s.

THEOREM 2. *Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ be sets of integers. Let $\{p_1, \ldots, p_s\}$ be a set of prime numbers. Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of $p_1, \ldots, p_s$ if $(i, j) \in V$. If $|V| > k$ then*

$$s \geqslant \frac{1}{4}\log\frac{|V|(|V|-k)}{kl^2} - 2.$$

We note that the assumption $|V| > k$ implies $l \geqslant 2$. It follows immediately from Theorem 2 that

$$\omega\Big(\prod_{i=1}^{k}\prod_{j=1}^{l}(a_i+b_j)\Big) \geqslant \tfrac{1}{4}\log(k/2) - 2$$

which is an explicit version of Theorem 1 of [5].

Proof. Let v_i be the number of integers j such that $(i, j) \in V$, for $i = 1, \ldots, k$. Then the number of unordered pairs $(i, j_1), (i, j_2)$ in V^2 is $\sum_{i=1}^{k}\binom{v_i}{2}$. By the Cauchy–Schwarz inequality we have

$$\sum_{i=1}^{k}\binom{v_i}{2} = \frac{1}{2}\sum_{i=1}^{k}\left(v_i - \frac{1}{2}\right)^2 - \frac{k}{8} \geqslant \frac{1}{2k}\left(\sum_{i=1}^{k}\left(v_i - \frac{1}{2}\right)\right)^2 - \frac{k}{8}$$

$$= \frac{1}{2k}\left(|V| - \frac{k}{2}\right)^2 - \frac{k}{8} = \frac{|V|(|V|-k)}{2k}.$$

Since the number of unordered pairs j_1, j_2 is $\binom{l}{2}$, there exist two integers j and j' such that $(i, j) \in V$ and $(i, j') \in V$ for at least $|V|(|V|-k)/kl^2$ integers i. Taking $v = b_j - b_{j'}$, we find that the equation $x+y=v$ has at least $|V|(|V|-k)/kl^2$ solutions $x = a_i + b_j$, $y = -(a_i + b_{j'})$, each composed of $p_1, \ldots, p_s$. By applying Lemma 1 we obtain that

$$\frac{|V|(|V|-k)}{kl^2} \leqslant 3 \times 7^{2s+3}. \blacksquare$$

6. The estimates in Theorems 1 and 2 can be considerably improved when we assume that the numbers $b_1, b_2, \ldots, b_l$ are very large. Then it is possible to apply the following result which has been proved by means of estimates for linear forms in the logarithms of algebraic numbers. For a finite set $S = \{p_1, \ldots, p_s\}$ of prime numbers and for a non-zero integer a, we shall denote by $[a]_{S^\sim}$ the largest divisor of a which is not divisible by $p_1, p_2, \ldots$ or p_s.

LEMMA 2. *Let $S = \{p_1, \ldots, p_s\}$ be a set of prime numbers not exceeding P and let v be a non-zero integer for which*

$$[v]_{S^\sim} > \exp\{((2s)^s P)^c\}$$

for some appropriate effectively computable constant c. Then the equation $x+y = v$ has at most $4s+1$ solutions in integers x, y composed of $p_1, \ldots, p_s$.

This lemma is a special case of the Theorem of Győry [3].

Following the proof of Theorem 2, but using Lemma 2 instead of Lemma 1, we arrive at the following result.

THEOREM 3. *Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ be sets of integers where $l \geqslant 2$. Let $S = \{p_1, \ldots, p_s\}$ be a set of primes not exceeding P. Suppose*

$$[b_i - b_j]_{S^\sim} > \exp\{((2s)^s P)^c\}$$

for all pairs i, j with $1 \leqslant i < j \leqslant l$, where c is the constant occurring in Lemma 2. Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of $p_1, \ldots, p_s$ if $(i, j) \in V$. Then

$$s \geqslant \frac{1}{4}\frac{|V|(|V|-k)}{kl^2} - \frac{1}{4}.$$

The essence of our method is to show that the sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ have subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|B'| = 2$, $|A'|$ is "large" with respect to $|V|$ and for each $a \in A'$, $b \in B'$ the number $a+b$ is composed of $p_1, \ldots, p_s$. Then Lemmas 1 and 2 can be applied. M. Simonovits has pointed out to us that a similar method has been used by Kővári, Sós and Turán [6] to solve a problem of Zarankiewicz.

7. We return to the graph theoretical problem posed by Erdős. Let $G(k)$ be a graph and $a_1, \ldots, a_k$ integers as described in the introduction such that $\omega(G(k))$ attains its minimal value. Let $\{p_1, \ldots, p_s\}$ be the minimal set of prime numbers such that $a_i + a_j$ is composed of $p_1, \ldots, p_s$ if x_i and x_j are joined by an edge. Then $s = \omega(G(k))$. Choosing $l = k$ and $b_i = a_i$ for $i = 1, \ldots, k$, we obtain the following consequence of Theorem 2.

COROLLARY 1. *Let N be the number of edges of $G(k)$. If $N > k$ then*

$$\omega(G(k)) \geqslant \frac{1}{4}\log\frac{N(N-k)}{k^3} - 2.$$

Put

$$\omega_-(G(k)) = \min \omega\left(\prod (a_i - a_j)\right)$$

where the factor $a_i - a_j$ occurs if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct integers $a_1, \ldots, a_k$. Choosing $a_1, \ldots, a_k$ such that $\omega_-(G(k))$ attains its minimal value and applying Theorem 2 with $l = k$ and $b_i = -a_i$ for $i = 1, \ldots, k$, we obtain the following result.

COROLLARY 2. *Let N be the number of edges of $G(k)$. If $N > k$ then*

$$\omega_-(G(k)) \geqslant \frac{1}{4}\log\frac{N(N-k)}{k^3} - 2.$$

Corollaries 1 and 2 imply that if $N > k^{(3/2)+\varepsilon}$ then

$$\omega(G(k)) \gg_\varepsilon \log k \quad \text{and} \quad \omega_-(G(k)) \gg_\varepsilon \log k.$$

We conjecture that for any positive number ε the inequality $N > k^{1+\varepsilon}$ implies

$$\omega(G(k)) \gg_\varepsilon \log k \quad \text{and} \quad \omega_-(G(k)) \gg_\varepsilon \log k.$$

In our proofs the properties $\omega(G(k)) \to \infty$ and $\omega_-(G(k)) \to \infty$ as $k \to \infty$ are derived by showing that $G(k)$ contains a complete bipartite subgraph of the type $(u, 2)$ with u large. A graph is called a *complete bipartite graph of the type* (u, w) if its vertex set is the disjoint union of two distinct vertex sets U, W with $|U| = u$, $|W| = w$ such that two vertices are adjacent if and only if one of them belongs to U and the other to W.

COROLLARY 3. *Let $G(k)$ be a graph as described in the introduction and suppose that $G(k)$ contains a complete bipartite subgraph of the type $(r, 2)$ with $r \geqslant 2$. Then*

$$\min\left(\omega(G(k)), \omega_-(G(k))\right) \geqslant \tfrac{1}{4}\log(r/2) - 2. \qquad (2)$$

The lower estimate for $\omega_-(G(k))$ is a considerable improvement of the special case $K = \mathbb{Q}$, $N = 1$ of Theorem 2 of Győry [4] which concerns certain closely related arithmetical graphs. We intend to return to these arithmetical graphs in a later paper.

Proof of Corollary 3. Suppose that $\{X_{i_1}, \ldots, X_{i_r}\}$, $\{X_{j_1}, X_{j_2}\}$ are the disjoint vertex sets of a complete bipartite subgraph of the type $(r, 2)$ of $G(k)$. Let $a_1, \ldots, a_k$ be distinct integers. By applying Theorem 2 first to the sets $\{a_{i_1}, \ldots, a_{i_r}\}$, $\{a_{j_1}, a_{j_2}\}$ and then to the sets $\{a_{i_1}, \ldots, a_{i_r}\}$, $\{-a_{j_1}, -a_{j_2}\}$, (2) immediately follows in both cases. ■

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