Replacing $p-1$ in this theorem by $p+k$ (for any fixed non-zero k), the same proof goes through.

Now combining the theorem of Erdős and Wagstaff (with $p+k$ in place of $p-1$) with Lemma 2, we obtain the assertion of Theorem 4 immediately.

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On multiples of certain real sequences

by

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Several years ago Professor Erdős suggested to me the following problem:

If $\lambda_1 < \lambda_2 < \ldots$ is a sequence of real numbers such that $\liminf N^{-1} \sum_{\lambda_n \leqslant N} 1 > 0$ is it true that, for any $\varepsilon > 0$ the inequalities $|\lambda_i - j\lambda_k| < \varepsilon$ have an infinite number of solutions in i, j, k?

If the λ_n are integers the condition reduces to $\lambda_k | \lambda_i$ and the question has a positive answer, by a well-known theorem of Davenport and Erdős ([3], Thm. 5, Ch. V). I was not able to solve this problem without a further condition on the sequence. However, it then became possible to weaken the "liminf" condition to "limsup":

THEOREM 1. *If $\lambda_1 < \lambda_2 < \ldots$ is any sequence of real numbers such that*

(a) *λ_i/λ_j is irrational, $i \neq j$,*

(b) $\limsup N^{-1} \sum_{\lambda_n \leqslant N} 1 > 0$

then, for any $\varepsilon > 0$ the inequalities $|\lambda_i - j\lambda_k| < \varepsilon$ have an infinite number of solutions in i, j, k.

Here we have a situation which is quite different from the integer case. Besicovitch constructed a sequence with positive upper asymptotic density no terms of which divides any other ([3], Thm. 4, Ch. V).

Condition (a) arises from the fact that integral multiples of an irrational number are uniformly distributed modulo 1. This implies (Lemma 1) that the sets

$$\{x:\ 0 \leqslant x - n\lambda_i/y \leqslant \varepsilon/y,\ 0 < x \leqslant 1,\ n = 1, 2, \ldots\}$$

are almost independent.

I noticed that this simple lemma makes it possible to prove the following result:

THEOREM 2. *If $\lambda_1 < \lambda_2 < \ldots$ is any sequence of real numbers such that, for some $\varepsilon > 0$,*

$$\limsup y^{-1} \Big| \bigcup_{n=1}^{\infty} [\lambda_n, \lambda_n + \varepsilon] \cap [0, y] \Big| > 0$$

then for any $\delta > 0$ and any sequence $\Delta_1, \Delta_2, \ldots$ of intervals of length δ the set X of x such that

$$\lambda_n x \notin \Delta_n \pmod 1, \quad n = 1, 2, \ldots$$

is at most denumerable.

Note that the condition on λ_n is satisfied if, for some $\varepsilon > 0$,

$$\lambda_{n+1} - \lambda_n > \varepsilon, \quad n = 1, 2, \ldots \quad \text{and} \quad \limsup N^{-1} \sum_{\lambda_n \leqslant N} 1 > 0.$$

J. P. Kahane and Y. Amice proved [4], [1] (independently) that if λ_n is a sequence of positive integers with positive upper asymptotic density and Δ is any interval, then the set X of x such that

$$\lambda_n x \notin \Delta, \quad n = 1, 2, \ldots$$

is finite.

On the other hand, if $\lambda_{n+1} > (1+c)\lambda_n$, for some constant $c > 0$, it is not hard to show that X is uncountable.

Notation. If $A, B \subset \mathbf{R}$ we shall write $A+B = \{a+b\colon a \in A,\ b \in B\}$ and if $\lambda \in \mathbf{R}$, $\lambda A = \{\lambda a\colon a \in A\}$.

We shall drop brackets on singletons, so that λ may mean $\{\lambda\}$. $|A|$ denotes the Lebesgue measure of A.

Lemma 1. *For any $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta > 0$, such that α/β is irrational, we have*

$$|\alpha \mathbf{Z} + [\gamma, \gamma+\varepsilon] \cap \beta \mathbf{Z} + [\delta, \delta+\zeta] \cap [0, y]| = y\varepsilon\zeta/\alpha\beta + o(y).$$

We show

$$|\alpha \mathbf{Z} + [\gamma, \gamma+\varepsilon] \cap \beta \mathbf{Z} + [\delta, \delta+\zeta] \cap [0, y]| \leqslant y\varepsilon\zeta/\alpha\beta + o(y)$$

since we may apply this inequality to complements.

Then we have

$$\begin{aligned}
&\beta^{-1} |\alpha \mathbf{Z} + [\gamma, \gamma+\varepsilon] \cap \beta \mathbf{Z} + [\delta, \delta+\zeta] \cap [0, y]| \\
&\quad = |\alpha/\beta \mathbf{Z} + [0, \varepsilon/\beta] \cap \mathbf{Z} + (\delta-\gamma)/\beta + [0, \zeta/\beta] \cap [-\gamma/\beta, (y-\gamma)/\beta]| \\
&\quad = |\alpha/\beta \mathbf{Z} + [0, \varepsilon/\beta] \cap \mathbf{Z} + (\delta-\gamma)/\beta + [0, \zeta/\beta] \cap [0, y/\beta]| + o(y) \\
&\quad \leqslant \varepsilon/\beta \operatorname{card}(\alpha/\beta \mathbf{Z} \cap [0, y/\beta] \cap \mathbf{Z} + (\delta-\gamma)/\beta + [0, \zeta/\beta]) \\
&\quad = y\varepsilon\zeta/\beta^2\alpha + o(y).
\end{aligned}$$

Lemma 2. *If $A \subset \mathbf{R}$, $\limsup y^{-1}|A \cap [0, y]| > 0$, μ_n is any sequence of positive numbers such that μ_i/μ_j is irrational, $i \neq j$ and $\sum_{n=1}^{\infty} 1/\mu_n = \infty$, then*

$$A \cap \bigcup_{n=1}^{\infty} \mu_n \mathbf{Z} + [0, \varepsilon] \neq \emptyset.$$

Proof of Lemma 2. This follows from the fact that

$$|[0, y] \setminus \bigcup_{n=1}^{\infty} \mu_n \mathbf{Z} + [0, \varepsilon]| = o(y)$$

which follows from Lemma 1 and any of the standard quasi-independence arguments, a slight modification of the Borel Cantelli lemma or the lemma of Paley and Zygmund will suffice.

Proof of Theorem 1. If, for all $\varepsilon > 0$, $\lambda_{i+1} - \lambda_i < \varepsilon$ for infinitely many i, there is nothing to prove.

Otherwise, there is a $\delta > 0$ such that $\lambda_{n+1} - \lambda_n > \delta$ for all $n = 1, 2, \ldots$ So, for all $\varepsilon \in]0, \delta]$,

$$\limsup y^{-1} \Big| \bigcup_{n=1}^{\infty} \lambda_n + [0, \varepsilon] \cap [0, y] \Big| = \varepsilon \limsup y^{-1} \sum_{\lambda_n < y} 1 > 0.$$

Then, noting that

$$\sum_{n=1}^{\infty} 1/\lambda_{2n} = \infty,$$

Lemma 2 gives, for any $N > 0$,

$$\bigcup_{n > N}^{\infty} \lambda_{2n+1} + [0, \varepsilon/2] \cap \bigcup_{n=1}^{\infty} \lambda_{2n} \mathbf{Z} + [0, \varepsilon/2] \neq \emptyset.$$

Proof of Theorem 2. In the case where the λ_n are integers $\lambda_n(x+\mathbf{Z}) + \mathbf{Z} = \lambda_n x + \mathbf{Z}$ so that $X = X + \mathbf{Z}$.

In this case X is not necessarily periodic modulo 1. However we can assume without loss of generality that X consists of positive numbers.

We assume X is uncountable and derive a contradiction.

Let $\varrho = 1 + \delta/2$. Then there is a $k \in \mathbf{Z}$ such that $[\varrho^k, \varrho^{k+1}] \cap X$ is uncountable. So there exists a sequence $\mu_n \in [\varrho^k, \varrho^{k+1}] \cap X$ such that μ_i/μ_j is irrational, $i \neq j$ and $\lambda_r \mu_n \cap \mathbf{Z} + \Delta_r = \emptyset$, $n, r = 1, 2, \ldots$

Without loss of generality Δ_r may be closed, so let $\alpha_r \in]0, 1]$ be such that

$$[\alpha_r, \alpha_r + \delta] + \mathbf{Z} = \mathbf{Z} + \Delta_r.$$

Then for any r, n

$$\lambda_r \mu_n \cap \mathbf{Z} + [\alpha_r, \alpha_r + \delta] = \emptyset,$$

$$\lambda_r \cap 1/\mu_n \mathbf{Z} + [\alpha_r/\mu_n, (\alpha_r+\delta)/\mu_n] = \emptyset,$$

$$\lambda_r \cap 1/\mu_n \mathbf{Z} + [\alpha_r \varrho^{-k}, (\alpha_r+\delta)\varrho^{-k-1}] = \emptyset$$

and so

$$\lambda_r - \alpha_r \varrho^{-k} \cap 1/\mu_n \mathbf{Z} + [0, \delta\varrho^{-k-1}/2] = \emptyset$$

which gives

$$\bigcup_{r=1}^{\infty} \lambda_r - \alpha_r \varrho^{-k} + [-\delta\varrho^{-k-1}/4, 0] \cap \bigcup_{n=1}^{\infty} 1/\mu_n Z + [0, \delta\varrho^{-k-1}/4] = \emptyset.$$

If $\alpha_r = 0$, $r = 1, 2, \ldots$ the contradiction is established by Lemma 2. A slight modification of the argument will give the general case, remembering that α_r is bounded.

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On prime factors of sums of integers III

by

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Dedicated to Professor P. Erdős
on the occasion of his
seventy-fifth birthday

1. In [1], pp. 36–37, Erdős posed the following questions. (We use a slightly different notation and denote the number of distinct prime divisors of an integer n by $\omega(n)$.) "Let $G(k)$ be a graph of k vertices $x_1, \ldots, x_k$. Let $a_1, \ldots, a_k$ by any set of k distinct integers. Put

$$\omega(G(k)) = \min \omega\left(\prod (a_i + a_j)\right)$$

where the factor $a_i + a_j$ occurs if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct integers $a_1, \ldots, a_k$. It is probably hopeless to determine $\omega(G(k))$ except for very simple graphs, but perhaps one can get conditions on the class of graphs for which $\omega(G(k)) \to \infty$. One conjecture which just occurs to me states: There is a $g(r) \to \infty$ as $r \to \infty$ so that if $G(k)$ has a chromatic number $\geqslant r$ then $\omega(G(k)) \geqslant g(r)$. A stronger conjecture: If $G(k)$ has more than kr edges then $\omega(G(k)) \geqslant g(r)$. Perhaps this is too optimistic".

We shall show that the latter conjecture is incorrect indeed, but that Theorem 1 implies

$$\omega\left(\prod (a_i + a_j)\right) \gg \log r \quad \text{as} \quad r \to \infty$$

under the additional assumption that $\omega(a_1 \ldots a_k)$ is bounded. Furthermore, it follows from Theorem 2 that if $G(k)$ has more than $k^{3/2} r$ edges or if $G(k)$ contains a complete bipartite subgraph of type $(r, 2)$ then $\omega(G(k)) \gg \log r$ as $r \to \infty$. Here $\gg$ denotes the Vinogradov symbol. Similar results will also be proved for the products of differences of integers. Our results are applications

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