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On homogeneous multiplicative hybrid problems in number theory

by

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Dedicated to Paul Erdős on his 75-th birthday

1. In the last 10 years, several authors have studied many "hybrid" problems in number theory, i.e., problems involving both special and general sequences of integers. Most of these problems have been additive problems, i.e., equations of the form

$$(1)\qquad \sum_{i=1}^{k} \alpha_i a_i = \sum_{j=1}^{l} b_j, \quad a_1 \in \boldsymbol{A}_1, \ldots, a_k \in \boldsymbol{A}_k, \quad b_1 \in \boldsymbol{B}_1, \ldots, b_l \in \boldsymbol{B}_l$$

have been studied, where $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ are general sequences ("dense" sequences), $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_l$ are fixed special sequences (sequences of the squares, of the numbers of the form $p-1$, etc.), and $\alpha_1, \ldots, \alpha_k$ are fixed real numbers. An equation of the form (1) is said to be *homogeneous* if $\sum_{i=1}^{k} \alpha_i = 0$. A typical homogeneous problem is to study the structure of difference sets of "dense" sequences of integers, i.e., to study the solvability of equations of the form

$$a - a' = b, \quad a \in \boldsymbol{A},\ a' \in \boldsymbol{A},\ b \in \boldsymbol{B},$$

(where $\boldsymbol{A}$ is a "dense" general sequence, $\boldsymbol{B}$ is a fixed special sequence); Erdős, Fürstenberg, Kamae, Mendès France, Ruzsa, Sárközy, Stewart and Tijdeman have studied problems of this type.

Recently Iwaniec and Sárközy [6], [7] have studied two multiplicative hybrid problems of inhomogeneous nature. Here our goal is to study homogeneous multiplicative problems. In fact, by a theorem of F. Behrend [1], $N > N_0$, $\boldsymbol{A} \subset \{1, 2, \ldots, N\}$ and

$$\sum_{a \in \boldsymbol{A}} \frac{1}{a} > c_1 \frac{\log N}{(\log\log N)^{1/2}}$$

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imply the solvability of $a|a'$, $a \in A$, $a' \in A$, $a \neq a'$. Thus we may expect that f certain special sequences B and for "dense" (general) sequences A t equation

$$\frac{a'}{a} = b, \quad a \in A, \ a' \in A, \ b \in B \tag{2}$$

must be solvable. The solvability of an equation of this form can be called *homogeneous multiplicative problem*, and here our goal is to look for spec sequences B for which equation (2) must be solvable (for "dense" sequenc A). In other words, here our goal is to study the structure of the integ quotients a'/a, $a \in A$, $a' \in A$.

First we will prove the following theorem:

THEOREM 1. *There exist absolute constants c_2, N_1 such that if N is positive integer with $N \geqslant N_1$, P is a set of prime numbers not exceeding with*

$$\sum_{p \in P} \frac{1}{p} > c_2, \tag{3}$$

$A \subset \{1, 2, \ldots, N\}$ and

$$\sum_{a \in A} \frac{1}{a} > 10(\log N)\left(\sum_{p \in P} \frac{1}{p}\right)^{-1/2}, \tag{4}$$

then there exist integers a, a' with

$$a \neq a', \quad a \in A, \quad a' \in A, \quad a|a', \quad \frac{a'}{a} \Big| \prod_{p | P} p. \tag{5}$$

If P is a set of prime numbers not exceeding N, then let A denote the s of the integers a with $a \leqslant N$ and

$$\sum_{\substack{p|a \\ p \in P}} 1 = \left[\sum_{p \in P} \frac{1}{p}\right].$$

It can be shown that for this sequence A we have

$$\sum_{a \in A} \frac{1}{a} > c_3 \log N \left(\sum_{p \in P} \frac{1}{p}\right)^{-1/2},$$

and, on the other hand, clearly there exists no integers a, a' with $a \neq a'$, $a \in$ $a' \in A$, $a|a'$ and $\frac{a'}{a} \Big| \prod_{p \in P} p$. This example shows that Theorem 1 is the b possible apart from the value of the constant factor on the right of (4).

Note that in the special case when P denotes the set of all the pri numbers not exceeding N, Theorem 1 gives the following slightly shar form of Behrend's theorem:

COROLLARY 1. *There exist absolute constants c_4, N_2 such that if N is a positive integer with $N \geqslant N_2$, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > c_4 \frac{\log N}{(\log\log N)^{1/2}},$$

then there exist integers a, a' ($\neq a$) such that $a \in A$, $a' \in A$, $a|a'$ and a'/a is squarefree.

We mention three other consequences of Theorem 1. First, choosing P in Theorem 1 as the set of the primes p with $p \leqslant M$, we obtain

COROLLARY 2. *There exist absolute constants c_5, c_6, N_3 such that if M, N are positive integers with $N > N_3$, $c_5 \leqslant M \leqslant N$, then for $A \subset \{1, 2, \ldots, N\}$,*

$$\sum_{a \in A} \frac{1}{a} > c_6 \frac{\log N}{(\log\log M)^{1/2}}$$

implies there exist integers a, a' ($\neq a$) such that $a \in A$, $a' \in A$, $a|a'$, a'/a is squarefree and the greatest prime of a'/a is $\leqslant M$.

(Thus if $\sum_{a \in A} \frac{1}{a} \geqslant \log N$, then there exist a, $a'(\neq a)$ with $a \in A$, $a' \in A$, $a|a'$ and $a'/a = O(1)$.)

Next, if we choose P as the set of the primes p with $p \leqslant N$ and $p \equiv 1 \pmod 4$, then we obtain

COROLLARY 3. *There exist absolute constants c_7, N_4 such that if N is a positive integer with $N \geqslant N_4$, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > c_7 \frac{\log N}{(\log\log N)^{1/2}},$$

then there exist integers a, a' ($\neq a$) such that $a \in A$, $a' \in A$, $a|a'$ and $a'/a = x^2 + y^2$ can be solved.

Finally, if we choose P as the set of the primes p with $p \leqslant N$ and $p \equiv 1 \pmod k$, then we obtain

COROLLARY 4. *There exist absolute constants c_8, N_5 such that if k, N are positive integers with $N \geqslant N_5$, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > c_8 \frac{(\varphi(k))^{1/2} \log N}{(\log\log N)^{1/2}}$$

then there exist integers, a, a' ($\neq a$) such that $a \in A$, $a' \in A$, $a|a'$ and

$$\frac{a'}{a} \equiv 1 \pmod k. \tag{6}$$

($\varphi(k)$ denotes Euler's phi function.)

In fact, it follows from Theorem 1 that there exists an integer quotient a'/a such that $p \equiv 1 \pmod k$ for all $p \mid \frac{a'}{a}$.

Note that the residue class $\equiv 1 \pmod k$ in (6) cannot be replaced by any other residue class modulo k. In fact, if A denotes the set of the integers a with $a \equiv 1 \pmod k$, $a \leqslant N$, then we have

$$\frac{a'}{a} \equiv 1 \pmod k$$

for any pair a, a' with $a \in A$, $a' \in A$, $a \mid a'$, and, on the other hand, $\sum_{a \in A} 1/a$ is "large".

As the remark after Theorem 1 shows, both Corollaries 2 and 3 are the best possible (except for the values of the constants c_5, c_6, c_7, N_3, N_4). On the other hand, Corollary 4 seems to be far from the best possible. In fact, we guess that Corollary 4 can be sharpened in the following way:

Conjecture 1. *If k is a fixed integer, N is an integer with $N \geqslant N_6(k)$, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > c_9 \frac{\log N}{(\log\log N)^{1/2}},$$

then (6) *is solvable.*

(By Behrend's theorem, this estimate would be the best possible except for the values of the constants.) This assertion would follow from the following one:

Conjecture 2. *If k is a fixed integer, N is an integer with $N \geqslant N_7(k)$, $A \subset \{1, 2, \ldots, N\}$, $a \equiv a' \pmod k$ for all $a \in A$, $a' \in A$ and*

$$\sum_{a \in A} \frac{1}{a} > c_{10} \frac{\log N}{k(\log\log N)^{1/2}},$$

then there exist integers a, a' ($\neq a$) such that $a \in A$, $a' \in A$ and $a \mid a'$.

Unfortunately, we have not been able to improve on the estimate given in Corollary 4.

Another arithmetic property of the quotients a'/a (with $a \mid a'$) can be derived easily from a theorem of Erdős, Sárközy and Szemerédi [4]. Let $\nu(n)$ denote the number of distinct prime factors of n, and let $\Omega(n)$ denote the total number of prime factors of n counted with multiplicity.

Theorem 2. (i) *If $\varepsilon > 0$, N is a positive integer with $N > N_8(\varepsilon)$, k is a positive integer, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > (1+\varepsilon) \frac{k \log N}{(2\pi \log\log N)^{1/2}}, \tag{7}$$

then there exist integers a, a' such that $a \neq a'$, $a \in A$, $a' \in A$, $a \mid a'$ and $k \mid \Omega(a'/a)$.

(ii) *If $\varepsilon > 0$, k is a fixed positive integer and $N > N_9(\varepsilon, k)$, then there exists a sequence A such that $A \subset \{1, 2, \ldots, N\}$*

$$\sum_{a \in A} \frac{1}{a} > (1-\varepsilon) \frac{k \log N}{(2\pi \log\log N)^{1/2}}, \tag{8}$$

and there exist no integers a, a' with $a \neq a'$, $a \in A$, $a' \in A$, $a \mid a'$ and $k \mid \Omega(a'/a)$.

Note that $k \mid \Omega(a'/a)$, i.e.,

$$\Omega(a'/a) \equiv 0 \pmod k$$

in Theorem 2 cannot be replaced by

$$\Omega(a'/a) \equiv b \pmod k$$

for some fixed integer b. In fact, if A denotes the set of the integers a with $k \mid \Omega(a)$, $a \leqslant N$, then we have

$$\Omega(a'/a) \equiv 0 \pmod k$$

for all $a \in A$, $a' \in A$, $a \mid a'$, and nevertheless, $\sum_{a \in A} 1/a$ is "large".

We guess that the assertion of Theorem 2 holds also with the function $\nu(n)$ in place of $\Omega(n)$. Unfortunately, we could prove only the following slightly weaker estimate:

Theorem 3. (i) *If $\varepsilon > 0$, N is a positive integer with $N > N_{10}(\varepsilon)$, k is a positive integer, $A \subset \{1, 2, \ldots, N\}$ and*

$$\sum_{a \in A} \frac{1}{a} > (C+\varepsilon) \frac{k \log N}{(2\pi \log\log N)^{1/2}} \quad \text{where} \quad C = \prod_p \left(1 + \frac{1}{(p-1)p}\right), \tag{9}$$

then there exist integers a, a' such that $a \neq a'$, $a \in A$, $a' \in A$, $a \mid a'$ and $k \mid \nu(a'/a)$.

(ii) *If $\varepsilon > 0$, k is a fixed positive integer and $N > N_{11}(\varepsilon, k)$, then there exists a sequence A such that $A \subset \{1, 2, \ldots, N\}$,*

$$\sum_{a \in A} \frac{1}{a} > (1-\varepsilon) \frac{k \log N}{(2\pi \log\log N)^{1/2}}, \tag{10}$$

and there exist no integers a, a' with $a \neq a'$, $a \in A$, $a' \in A$, $a \mid a'$ and $k \mid \nu(a'/a)$.

We mention another related conjecture: We guess that if $\varepsilon > 0$, N is sufficiently large in terms of ε, $A \subset \{1, 2, \ldots, N\}$, A consists of squarefree integers, and

$$\sum_{a \in A} \frac{1}{a} > (1+\varepsilon) \frac{6}{\pi^2} \frac{\log N}{(2\pi \log\log N)^{1/2}},$$

then there exist integers a, a' ($a \neq a'$) such that $a \in A$, $a' \in A$ and $a \mid a'$. (Compare with the theorem at the beginning of Section 3). We note that Conjecture 2 and this conjecture could be generalized; we hope to return to these questions in a subsequent paper.

If A denotes the set of integers a with $a \leqslant N$, $2|\Omega(a)$, then we have

$$\sum_{a\in A} 1/a \sim \tfrac{1}{2}\log N,$$

and $a\in A$, $a'\in A$, $a|a'$, $a'/a = p$ (prime) is not solvable.

If we replace this last equation by $a'/a = p+1$ (or $= p+(2k+1)$), then by using the fact that $p+1$ is even for $p>2$, still we may construct a "dense" sequence A so that this equation should not be solvable. On the other hand, one may guess that for "dense" sequences A, $a\in A$, $a'\in A$, $a|a'$, $a'/a = p+2$ (or $= p+2k$) must be solvable. This is not so, as the following result shows:

THEOREM 4. *For each non-zero integer k the set $A(k)$ of natural numbers divisible by at least one number of the form $p+k$ where p is prime and $|p+k|>1$ possesses asymptotic density $d(A(k))<1$.*

Thus if $A = N - A(k)$ then A is "dense" yet there do not exist $a\in A$, $a'\in A$ with $a\neq a'$, $a|a'$, and $a'/a = p+k$ for some prime p.

2. In this section, we will prove Theorem 1.

Let us write all $a\in A$ in the form

$$a = b(a)\,t^2(a),$$

where $b(a)$, $t(a)$ are positive integers and $b(a)$ is squarefree. For $t = 1, 2, \ldots$, put $A_t = \{a\colon a\in A,\ t(a) = t\}$. We are going to show that there exists a positive integer T with

$$\text{(11)}\qquad \sum_{a\in A_T}\frac{1}{b(a)} > 6(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}.$$

In fact, if we start out from the indirect assumption that there does not exist a T satisfying (11), then we get

$$\sum_{a\in A}\frac{1}{a} = \sum_{t=1}^{+\infty}\sum_{a\in A_t}\frac{1}{a} = \sum_{t=1}^{+\infty}\sum_{a\in A_t}\frac{1}{t^2 b(a)}$$
$$= \sum_{t=1}^{+\infty}\frac{1}{t^2}\sum_{a\in A_t}\frac{1}{b(a)} \leqslant \sum_{t=1}^{+\infty}\frac{1}{t^2}\cdot 6(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}$$
$$= \pi^2(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2} < 10(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}$$

which contradicts (4). This contradiction proves the existence of an integer T satisfying (11).

Let $\boldsymbol{B}$ denote the set of the positive integers b such that for some $a\in A_T$ we have $b(a) = b$. For $a\in A_T$, $a'\in A_T$ we have

$$\frac{a'}{a} = \frac{b(a')\,T^2}{b(a)\,T^2} = \frac{b(a')}{b(a)},$$

so that in order to show the solvability of (5), it suffices to show the solvability of

$$\text{(12)}\qquad b\neq b',\quad b\in\boldsymbol{B},\quad b'\in\boldsymbol{B},\quad b|b'\quad\text{and}\quad \frac{b'}{b}\Bigm|\prod_{p\in\boldsymbol{P}}p.$$

Let us write $v^+(n) = \sum_{\substack{p|n\\ p\in\boldsymbol{P}}} 1$, and for a positive integer n, let $g(n)$ denote the number of solutions of

$$bk = n,\quad b\in\boldsymbol{B},\quad v^+(k) > \tfrac{1}{2}\sum_{p\in\boldsymbol{P}}1/p.$$

We are going to show that there exists a positive integer n with

$$\text{(13)}\qquad g(n) > \tau(n)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2},\qquad N^2\leqslant n\leqslant 2N^2.$$

($\tau(n)$ denotes the divisor function.) In fact, if such an n does not exist, then for large N we have

$$\text{(14)}\qquad \sum_{n=N^2}^{2N^2} g(n) \leqslant \sum_{n=N^2}^{2N^2}\tau(n)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}$$
$$= \Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}\sum_{n=N^2}^{2N^2}\tau(n) = \Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}\bigl(1+o(1)\bigr)N^2\log N^2$$
$$< \tfrac{9}{4}N^2(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}.$$

On the other hand, if $\sum_{p\in\boldsymbol{P}}(1/p)$ and N are large enough, and $N\leqslant x$, then by a theorem of Erdős and Kac [3] we have

$$\Bigl|\Bigl\{n\colon x\leqslant n\leqslant 2x,\ v^+(n) > \tfrac{1}{2}\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr\}\Bigr| > \tfrac{2}{5}x.$$

Thus in view of (11),

$$\sum_{n=N^2}^{2N^2} g(n) = \sum_{n=N^2}^{2N^2}\ \sum_{\substack{bk=n\\ b\in\boldsymbol{B},\,v^+(k)>\frac{1}{2}\sum_{p\in\boldsymbol{P}}(1/p)}} 1$$
$$= \sum_{b\in\boldsymbol{B}}\ \sum_{\substack{\frac{N^2}{b}\leqslant k\leqslant\frac{2N^2}{b}\\ v^+(k)>\frac{1}{2}\sum_{p\in\boldsymbol{P}}(1/p)}} 1 > \sum_{b\in\boldsymbol{B}}\frac{2}{5}\frac{N^2}{b} = \frac{2}{5}N^2\sum_{b\in\boldsymbol{B}}\frac{1}{b}$$
$$> \frac{12}{5}N^2(\log N)\Bigl(\sum_{p\in\boldsymbol{P}}\frac{1}{p}\Bigr)^{-1/2}$$

which contradicts (14), and this contradiction proves the existence of an integer n satisfying (13).

Let n be an integer satisfying (13) and write $m = \prod_{p|n} p$. The elements of $\boldsymbol{B}$ are squarefree, so that $b|n$ holds if and only if $b|m$, and thus in view of (13),

$$\sum_{\substack{b|m\\ b\in \boldsymbol{B}}} 1 = \sum_{\substack{b|n\\ b\in \boldsymbol{B}}} 1 \geqslant g(n) > \tau(n)\left(\sum_{p\in \boldsymbol{P}} \frac{1}{p}\right)^{-1/2}. \tag{15}$$

Let us write

$$X = \{p\colon\ p|m\}, \quad Y = \{p\colon\ p|m,\ p\notin \boldsymbol{P}\}, \quad Z = \{p\colon\ p|m,\ p\in \boldsymbol{P}\}$$

so that

$$X = Y\cup Z, \quad Y\cap Z = \emptyset.$$

For $b|n$, put

$$M(b) = \{p\colon\ p|b\}.$$

We need the following lemma of Sárközy and Szemerédi [8]:

Lemma. *Let X be a finite set, and let $X = Y\cup Z$, $Y\cap Z = \emptyset$. Put $|Y| = y$, $|Z| = z$. Let $M_1, M_2, \ldots, M_l$ be subsets of X with*

$$l > 2^y \binom{z}{[z/2]}. \tag{16}$$

Then there exist subsets M_u and M_v $(u\neq v)$ with

$$M_u\cap Y = M_v\cap Y \quad \text{and} \quad M_u\cap Z\subset M_v\cap Z.$$

We are going to apply this lemma with the sets $M(b)$ (where $b|m$, $b\in \boldsymbol{B}$) in place of $M_1, M_2, \ldots, M_l$, so that in view of (15), the number of these subsets is

$$l > \tau(n)\left(\sum_{p\in \boldsymbol{P}} \frac{1}{p}\right)^{-1/2} \geqslant \tau(m)\left(\sum_{p\in \boldsymbol{P}} \frac{1}{p}\right)^{-1/2} = 2^{y+z}\left(\sum_{p\in \boldsymbol{P}} \frac{1}{p}\right)^{-1/2}. \tag{17}$$

On the other hand, by $g(n) > 0$, we have

$$z = \nu^+(m) = \nu^+(n) > \tfrac{1}{2}\sum_{p\in \boldsymbol{P}} \frac{1}{p}$$

so that in view of (3), for large enough c_2 (so that thus z is large) we have

$$2^y\binom{z}{[z/2]} < 2^y\cdot\frac{1}{2}\cdot\frac{2^z}{z^{1/2}} < \frac{1}{2}2^{y+z}\left(\frac{1}{2}\sum_{p\in \boldsymbol{P}}\frac{1}{p}\right)^{-1/2} < 2^{y+z}\left(\sum_{p\in \boldsymbol{P}}\frac{1}{p}\right)^{-1/2}. \tag{18}$$

Now (16) follows from (17) and (18), so that, in fact, the lemma above can be applied. By using the lemma, we obtain that there exist b, b' with $b\neq b'$,

$$M(b)\cap Y = M(b')\cap Y \quad \text{and} \quad M(b)\cap Z\subset M(b')\cap Z.$$

Clearly, these integers b, b' satisfy (12), and this completes the proof of Theorem 1.

3. To prove the first half of Theorem 2, we need the following theorem of Erdős, Sárközy and Szemerédi [4]:

If $\varepsilon > 0$, $N > N_{12}(\varepsilon)$, $A\subset\{1, 2, \ldots, N\}$ and

$$\sum_{a\in A}\frac{1}{a} > (1+\varepsilon)\frac{\log N}{(2\pi\log\log N)^{1/2}},$$

then there exist integers a, a' $(a\neq a')$ such that $a\in A$, $a'\in A$ and $a|a'$.

Assume now that (7) holds, and for $i = 0, 1, \ldots$ put $A^{(i)} = \{a\colon\ a\in A,\ \Omega(a)\equiv i \pmod k\}$. By (7) we have

$$\max_{0\leqslant i<k}\sum_{a\in A^{(i)}}\frac{1}{a} \geqslant \frac{1}{k}\sum_{a\in A}\frac{1}{a} > (1+\varepsilon)\frac{\log N}{(2\pi\log\log N)^{1/2}}$$

so that there exists an integer i_0 with $0\leqslant i_0 < k$ and

$$\sum_{a\in A^{(i_0)}}\frac{1}{a} > (1+\varepsilon)\frac{\log N}{(2\pi\log\log N)^{1/2}}.$$

Then by the theorem of Erdős, Sárközy and Szemerédi, there exist integers a, a' $(a\neq a')$ with $a\in A^{(i_0)}$, $a'\in A^{(i_0)}$ and $a|a'$. These integers satisfy also

$$\Omega(a'/a) = \Omega(a') - \Omega(a) \equiv i_0 - i_0 = 0 \pmod k$$

which completes the proof of (i) in Theorem 2.

Now we are going to prove (i) in Theorem 3. Let $\boldsymbol{M}$ denote the set of the positive integers m such that $p|m$ (p prime) implies that also $p^2|m$ holds. (In other words, $1\in \boldsymbol{M}$, and an integer $m > 1$ satisfies $m\in \boldsymbol{M}$ if and only if there exists no prime p with $p|m$, $p^2\nmid m$.) Clearly, any $a\in A$ has a unique representation of the form

$$a = m(a)n(a) \text{ where } m(a)\in \boldsymbol{M},\ |\mu(n(a))| = 1 \text{ and } (m(a), n(a)) = 1.$$

For $m\in \boldsymbol{M}$, put

$$A_m^* = \{a\colon\ a\in A,\ m(a) = m\}.$$

Then (9) implies that

$$\max_{m\in \boldsymbol{M}}\sum_{a\in A_m^*}\frac{1}{n(a)} > \left(1+\frac{\varepsilon}{C}\right)\frac{k\log N}{(2\pi\log\log N)^{1/2}} \tag{19}$$

since otherwise we would have

$$(20)\qquad \sum_{a\in A}\frac{1}{a}=\sum_{m\in M}\frac{1}{m}\sum_{a\in A_m^*}\frac{1}{n(a)}\leqslant\sum_{m\in M}\frac{1}{m}\left(1+\frac{\varepsilon}{C}\right)\frac{k\log N}{(2\pi\log\log N)^{1/2}}$$
$$=\left(1+\frac{\varepsilon}{C}\right)\frac{k\log N}{(2\pi\log\log N)^{1/2}}\sum_{m\in M}\frac{1}{m}=(C+\varepsilon)\frac{k\log N}{(2\pi\log\log N)^{1/2}}$$

since

$$\sum_{m\in M}\frac{1}{m}=\prod_p\left(1+\sum_{k=2}^{+\infty}\frac{1}{p^k}\right)=\prod_p\left(1+\frac{1}{(p-1)p}\right)=C.$$

(20) contradicts (9) which proves (19).

By (19), there exists an integer m_0 with

$$\sum_{a\in A_{m_0}^*}\frac{1}{n(a)}>\left(1+\frac{\varepsilon}{C}\right)\frac{k\log N}{(2\pi\log\log N)^{1/2}}.$$

Then by using the first half of Theorem 2 with $\{n:\ m_0 n\in A,\ |\mu(n)|=1,\ (m_0, n)=1\}$ in place of A (and with ε/C in place of ε), we obtain that there exist $a=m_0 n(a)$, $a'=m_0 n(a')$ such that $a\neq a'$, $a\in A$, $a'\in A$, $n(a)|n(a')$ (so that also $a|a'$ holds) and

$$k\Big|\Omega\left(\frac{n(a')}{n(a)}\right)=\nu\left(\frac{n(a')}{n(a)}\right)=\nu\left(\frac{m_0 n(a')}{m_0 n(a)}\right)=\nu\left(\frac{a'}{a}\right)$$

which completes the proof of (i) in Theorem 3.

We are going to complete the proofs of Theorems 2 and 3 by proving (ii) in Theorems 2 and 3 simultaneously. Let $A=\{a:\ a\leqslant N,\ \log\log n\leqslant\Omega(a)<\log\log n+k\}$, Then (8) and (10) hold by a theorem of Erdős [2]. Furthermore, $a\neq a'$, $a\in A$, $a'\in A$ and $a|a'$ imply that $1\leqslant\Omega(a'/a)\leqslant k-1$. But then neither $k|\Omega(a'/a)$ nor $k|\nu(a'/a)$ can hold which completes the proof.

4. In this section, we will prove Theorem 4.

If A is a set of positive integers, let $B(A)$ denote the set of positive multiples of members of A. Let $A_{(T)}=A\cap[1, T]$, $A^{(T)}=A\cap(T,\infty)$, and for positive integers u, M let $Z(u, M)$ denote the set of the positive integers n with $n\equiv u \pmod M$.

Lemma 1. *Let u, M be positive integers, and suppose that the set A of positive integers is such that* $\lim_{T\to+\infty}\bar{d}(B(A^{(T)}))=0$. *Then* $d(B(A)\cap Z(u, M))$ *exists and is equal to* $\lim_{T\to+\infty} d(B(A_{(T)})\cap Z(u, M))$.

($d(A)$ and $\bar{d}(A)$ denote the asymptotic density and the asymptotic upper density of A, respectively.)

Proof. For any T we have

$$(21)\qquad d(B(A_{(T)}\cap Z(u, M))\leqslant \underline{d}(B(A)\cap Z(u, M))$$
$$\leqslant\bar{d}(B(A)\cap Z(u, M))$$
$$\leqslant d(B(A_{(T)})\cap Z(u, M))+\bar{d}(B(A^{(T)})).$$

Since $d(B(A_{(T)})\cap Z(u, M))$ is monotone, it has a limit d. Thus letting $T\to+\infty$ in (21) and using the hypothesis we have

$$d(B(A)\cap Z(u, M))=d.$$

Lemma 2. *If* $\lim_{T\to+\infty}\bar{d}(B(A^{(T)}))=0$ *and* $1\notin A$, *then* $d(B(A))<1$.

Proof. First note that for any $C\subset A$, $\bar{d}(B(C^{(T)}))\leqslant\bar{d}(B(A^{(T)}))$, so by Lemma 1,

$$d(B(C)\cap Z(u, M))=\lim_{T\to+\infty} d(B(C_{(T)})\cap Z(u, M))\quad\text{ for any } u, M.$$

Also by Lemma 1, $d(B(A^{(T)}))$ exists for any T.

Let T_0 be such that $\bar{d}(B(A^{(T_0)}))=d(B(A^{(T_0)}))<1$. Let $M=\prod_{a\in A_{(T_0)}} a$ and let $C=\{a\in A:\ (a, M)=1\}$. Then if $(u, M)=1$, using $1\notin A$, we have

$$B(A)\cap Z(u, M)=B(A^{(T_0)})\cap Z(u, M)=B(C)\cap Z(u, M).$$

Thus it will suffice to show $d(B(C)\cap Z(u, M))<1/M$.

For any T let $M(T)=\prod_{c\in C_{(T)}} c$. For any $j\leqslant M(T)$ with $j\in B(C_{(T)})$, we have exactly one of $j, j+M(T), \ldots, j+(M-1)M(T)$ congruent to $u \bmod M$. But the set of $j+iM(T)$ where $j\leqslant M(T)$, $j\in B(C_{(T)})$, and $0\leqslant i\leqslant M-1$ is equal to $[1, MM(T)]\cap B(C_{(T)})$. Thus for any T,

$$(22)\qquad d(B(C_{(T)})\cap Z(u, M))=\frac{1}{M}d(B(C_{(T)})).$$

Letting $T\to+\infty$ in (22) and using Lemma 1, we have

$$d(B(C)\cap Z(u, M))=\frac{1}{M}d(B(C))\leqslant\frac{1}{M}d(B(A^{(T_0)}))<\frac{1}{M},$$

which, as we have noted, is sufficient for Lemma 2.

In order to prove Theorem 4, we need also the following result of Erdős and Wagstaff [5]:

Theorem (Erdős and Wagstaff). *If* $A=\{p-1:\ p \text{ prime}\}$, *then*

$$\lim_{T\to+\infty}\bar{d}(B(A^{(T)}))=0.$$

Replacing $p-1$ in this theorem by $p+k$ (for any fixed non-zero k), the same proof goes through.

Now combining the theorem of Erdős and Wagstaff (with $p+k$ in place of $p-1$) with Lemma 2, we obtain the assertion of Theorem 4 immediately.

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On multiples of certain real sequences

by

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Several years ago Professor Erdős suggested to me the following problem:

If $\lambda_1 < \lambda_2 < \ldots$ is a sequence of real numbers such that $\liminf N^{-1} \sum_{\lambda_n \leqslant N} 1 > 0$ is it true that, for any $\varepsilon > 0$ the inequalities $|\lambda_i - j\lambda_k| < \varepsilon$ have an infinite number of solutions in i, j, k?

If the λ_n are integers the condition reduces to $\lambda_k | \lambda_i$ and the question has a positive answer, by a well-known theorem of Davenport and Erdős ([3], Thm. 5, Ch. V). I was not able to solve this problem without a further condition on the sequence. However, it then became possible to weaken the "liminf" condition to "limsup":

Theorem 1. *If $\lambda_1 < \lambda_2 < \ldots$ is any sequence of real numbers such that*

(a) *λ_i/λ_j is irrational, $i \neq j$,*

(b) $\limsup N^{-1} \sum_{\lambda_n \leqslant N} 1 > 0$

then, for any $\varepsilon > 0$ the inequalities $|\lambda_i - j\lambda_k| < \varepsilon$ have an infinite number of solutions in i, j, k.

Here we have a situation which is quite different from the integer case. Besicovitch constructed a sequence with positive upper asymptotic density no terms of which divides any other ([3], Thm. 4, Ch. V).

Condition (a) arises from the fact that integral multiples of an irrational number are uniformly distributed modulo 1. This implies (Lemma 1) that the sets

$$\{x\colon 0 \leqslant x - n\lambda_i/y \leqslant \varepsilon/y,\ 0 < x \leqslant 1,\ n = 1, 2, \ldots\}$$

are almost independent.

I noticed that this simple lemma makes it possible to prove the following result:

Theorem 2. *If $\lambda_1 < \lambda_2 < \ldots$ is any sequence of real numbers such that, for some $\varepsilon > 0$,*

$$\limsup y^{-1} \Big|\bigcup_{n=1}^{\infty} [\lambda_n, \lambda_n + \varepsilon] \cap [0, y]\Big| > 0$$