Prime-like sequences

by

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Dedicated to Pál Erdős on the occasion of his 75th birthday

In this note we answer in the affirmative a question raised by Erdős.

**Theorem.** Let \( \mathcal{A} \) be a set of positive integers,

\[
A(x) = \text{card } \{ a \in \mathcal{A} : a \leq x \},
\]

and suppose that

(1) \[ A(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty. \]

Put

\[
f(n) = \sum_{a \in \mathcal{A}} \frac{1}{n-a}.
\]

Then the number 1 is a limit point of the sequence \( \{f(n)\} \).

Under the stated hypotheses it is easy to see that \( f(n) \) has mean value 1, since

(2) \[
\sum_{n=1}^{N} f(n) = \sum_{a \leq N} \sum_{m=1}^{N-a} \frac{1}{m} = \sum_{a \leq N} \log(N-a) + O(A(N)) \sim N.
\]

**Proof.** We argue by contradiction. Suppose that there is a \( \delta > 0 \) such that

(3) \[ |f(n) - 1| \leq \delta \quad \text{for all large } n. \]

First we show that there exists an arbitrarily large \( n_0 \) such that

(4) \[ f(n_0) \leq 1 - \delta. \]

and

(5) \[ f(n) \geq 1 + \delta \quad \text{for} \quad n_0 - n_0^{3/2} \leq n < n_0. \]

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To see this, let \( f(n, u) = \sum_{a \leq u, n - u} 1/(n - a) \). Then

\[
\sum_{n=1}^{N} f(n, n^{1/2}) = \sum_{a \leq n/2} \sum_{n \leq N} 1/(n - a).
\]

The root of the equation \( x - x^{1/2} = a \) lies in the interval \((a + a^{1/2}, a + a^{1/2} + 1)\). Hence by the integral test the inner sum above is

\[
\int_{a^{1/2}}^{a} (1/du) + O(1/n) = \log(N - a) + \frac{1}{2} \log a + O(1/n).
\]

Then by partial summation and (1) we deduce that

\[
\sum_{n=1}^{N} f(n, n^{1/2}) \approx (1 - \delta/2) N \quad \text{as} \quad N \to \infty.
\]

Thus there exist arbitrarily large values of \( n \) for which \( f(n, n^{1/2}) > 1 - \delta \), say \( f(n_1, n_1^{1/2}) > 1 - \delta \). If \( n_1 - n_1^{1/2} \leq n \leq n_1 \) then

\[
f(n) \geq \sum_{a \leq n_1 - n_1^{1/2}} 1/(n - a) \geq f(n_1, n_1^{1/2}) > 1 - \delta.
\]

Hence by (3) we see that \( f(n) \geq 1 + \delta \) when \( n_1 - n_1^{1/2} \leq n \leq n_1 \). Now let \( n_0 \) be the least \( n \geq n_1 \) such that \( f(n_0) < 1 \). Such an \( n_0 \) must exist, in view of (2) and (3). Then \( f(n_0) \leq 1 - \delta \), and \( f(n) \geq 1 + \delta \) for \( n_1 - n_1^{1/2} \leq n \leq n_0 \). Hence we have (4) and (5).

We now show that if \( f(n_0) \leq 1 - \delta \), if \( f(n) \geq 1 + \delta \) for \( n_0 - 2x \leq n \leq n_0 - x \), and if \( x \geq 4/\delta \), then

\[
A(n_0 - x) - A(n_0 - 4x/\delta) \geq \delta x/(4 \log x).
\]

To derive this, we first note that if \( n \leq n_0 \), \( u > 0 \), and \( a < n - u \) then

\[
\frac{1}{n - a} \leq \frac{n_0 - n + u}{u} \cdot \frac{1}{n_0 - a}.
\]

On summing this over \( a < n - u \), \( a \leq \delta \), we deduce that

\[
f(n, u) \leq \frac{n_0 - n + u}{u} f(n_0, n_0 - n + u).
\]

But \( f(n_0, v) \leq f(n_0) \) for any \( v \geq 0 \), so by (4) the above is

\[
\leq \frac{n_0 - n + u}{u} (1 - \delta).
\]

If we take \( u = (n_0 - n)/\delta \) then this is

\[
(1 + \delta)(1 - \delta) - 1 - \delta^2 < 1,
\]

so that if \( f(n) \geq 1 + \delta \) then

\[
\sum_{n_0 - (n_0 - n)/\delta \leq a < n_0 - n} 1/(n - a) \geq \delta.
\]

We sum this over all \( n \in (n_0 - 2x, n_0 - x) \) to see that

\[
(x - 1) \delta \leq \sum_{n_0 - 2x \leq a < n_0 - x} \sum_{n_0 - n \leq \delta a + n_0 - x} \frac{1}{n - a} = \sum_{a \leq n_0 - x} \sum \frac{1}{n - a}
\]

where the outer sum, over \( a \), is subject to the constraint \( n_0 - 2 (1 + 1/\delta) x < a < n_0 - x \), and the inner sum, over \( n \), is subject to the two constraints \( a < n \leq (\delta a + n_0)/(1 + \delta) \), \( n_0 - 2x \leq n < n_0 - x \). If we drop the latter of these two constraints then the inner sum is made larger (or at least not decreased). By appeal to the inequality \( \sum 1/k \leq 1 + \log x \), which holds for all \( x \geq 1 \), we conclude that the inner sum above is

\[
\leq 1 + \log n_0 - a + 1 + \log(2x/\delta).
\]

Thus the double sum is

\[
\leq (A(n_0 - x) - A(n_0 - 4x/\delta)) \log(2x/\delta),
\]

which gives (6).

We now use (5) and (6) to derive a lower bound for \( f(n) \). Let \( K = (1/2 \log n_0)/(4 \log(4/\delta)) \). The intervals \( I_k = (n_0 - (4/\delta)^k, n_0 - (4/\delta)^k) \), \( 1 \leq k \leq K \), are disjoint and lie in the range \([n_0 - n_0/2, n_0] \). Also, if \( a \in I_k \) then

\[
\frac{1}{(n_0 - a)} \geq \frac{1}{(4/\delta)^{k+1}}.
\]

Hence

\[
f(n_0) \geq \sum_{k=1}^{K} \sum_{a \in I_k} 1/(n_0 - a)
\]

\[
\geq \sum_{k=1}^{K} \frac{1}{(4/\delta)^{k+1}} (A(n_0 - (4/\delta)^k) - A(n_0 - (4/\delta)^{k+1})).
\]

From (5) and (6) we see that this is

\[
\geq \frac{\delta^2}{16 \log(4/\delta)} \sum_{k=1}^{K} \frac{1}{(4/\delta)^{k+1}} \log \left( \frac{\delta \log n_0}{(4 \log(4/\delta))} \right).
\]

That is, \( f(n_0) \geq 1/\log \log n_0 \) for large \( n_0 \). This contradicts (4), so the proof is complete.

From the hypothesis (1) alone it is not possible to derive a quantitative upper bound for the rate at which the limit point 1 is approached. For as was observed by Erdős, if \( a_n = [(1 - \epsilon_n) n \log n] \) and \( \epsilon_n \to 0 \) very slowly, then the limit point 1 is also approached very slowly. Also, from (1) it does not
follow that

\[ \sum_{a \leq x} f(n)^2 \sim x \quad \text{as} \quad x \to \infty. \]

On the other hand, it is not hard to show that if there is an \( h = h(x) \) such that \( \log h = o((\log x)^{1/2}) \) and

\[ \frac{1}{2} \left( A(u + h) - A(u) - \frac{h}{\log u} \right)^2 du = o(h^2 x (\log x)^{-2}), \]

then both (1) and (7) hold. From (1) and (7) we see that

\[ \sum_{a \leq x} (f(n) - 1)^2 = o(x), \]

from which it follows that \( f(n) \) is near 1 for almost all \( n \).

If we take \( \mathcal{P} \) to be the set of prime numbers then we have (1), since this is the prime number theorem. If the Riemann Hypothesis is assumed, then (8) holds for prime numbers with \( h = \exp((\log x)^{1/2}) \), for example. (See [1].)

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Reference


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On an additive property of squares and primes

by

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To Uncle Paul, with epsilonial love

1. Introduction. The additive property in the title is that of being an essential component. Essential components are traditionally defined via the Schnirelmann density. The Schnirelmann density \( \sigma(A) \) of a set \( A \) of integers is defined by the formula

\[ \sigma(A) = \inf \frac{A(n)}{n}, \]

where \( n \) runs over the natural numbers and we use \( A(n) \) to denote the counting function of our set \( A \), that is, the number of its elements between 1 and \( n \) (the nonpositive elements are not taken into account).

This concept of density was introduced by and named after L. G. Schnirelmann [10], who proved the inequality

\[ \sigma(A + B) \geq \sigma(A) + \sigma(B) \]

and used it to show that every set of positive density is a basis, and that the set of primes is an asymptotic basis (that is, the sumset \( P + \ldots + P \) with a sufficiently large number of summands contains all large integers), which was the first unconditional result concerning the Goldbach conjecture.

A set \( H \) is called a (Schnirelmann) essential component if \( \sigma(A + H) > \sigma(A) \) whenever \( 0 < \sigma(A) < 1 \). By (1.1), sets of positive density always have this property. The first essential component of density 0 was discovered by Khintchine [4]; it was the set \( Q \) of squares. A few years later Erdős [1] found that every basis is an essential component; he proved this in the effective form

\[ \sigma(A + H) \geq \sigma(A) + \sigma(A)(1 - \sigma(A))(2h), \]

if \( H \) is a basis of order \( h \). A much stronger version of (1.2) was found by Plünnecke [8]; he proved

\[ \sigma(A + H) \geq \sigma(A)^{1 - 1/h}, \]

which is, in this generality, the best possible order of magnitude.