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On Waldspurger's theorem

by

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To Paul Erdős on the occasion of his 75th birthday

1. Introduction. J.-L. Waldspurger [5], [6] showed that, under the Shimura correspondence $g = \text{Shimura}(f)$ between Hecke eigenforms $f(z)$ of weight $k = \frac{1}{2}+l$ and $g(z)$ of weight $2k-1$, the square of the nth Fourier coefficient of f, where n is square-free, is proportional to $n^{k-1} L_g(k-\frac{1}{2}, \chi_n)$. Here $L_g(s, \chi_n)$ is the L-series attached to g twisted by the real character $\chi_n(m) = \left(\frac{n}{m}\right)$ and $s = k-\frac{1}{2}$ is the center of the critical strip. The original arguments of Waldspurger use the language of representation theory. W. Kohnen [3] gave a rather explicit derivation by constructing reproducing kernels for the Shimura and the Shintani lifts.

In this note we establish a similar result in a completely elementary fashion. Our relation is essentially the Waldspurger formula averaged over a basis of the space of cusp forms. Due to such averaging we avoid speaking about the Hecke operators and the Shimura correspondence. The method of proof is conceptually direct. We first express the Fourier coefficients as a sum of generalized Kloosterman sums by an appeal to Petersson's formulas for the Poincaré series. In the case of forms of half-integral weight the Kloosterman sums in question are twisted by a real character and this makes it possible to evaluate them explicitly by means of Gauss sums. Having done this we then use Poisson's summation to get another sum involving ordinary Kloosterman sums which, in turn, are related to the Fourier coefficients of cusp forms of an integral weight.

This work is primarily of theoretical value; therefore we do not attempt to reach full generality for the sake of simplicity. In the main result (Theorem 1) we assume that $2k \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and that the level of the group $\Gamma = \Gamma_0(N)$ is $N = 4^\nu$ with $\nu \geqslant 4$.

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2. Statement of results. For z in $H = \{x+iy,\ y>0\}$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\Gamma_0(4)$ let $j(\gamma, z)$ stand for the theta multiplier;

$$j(\gamma, z) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}.$$

Let $2k$ be an integer $\geqslant 3$ and $\Gamma = \Gamma_0(N)$ with $4|N$ if $2k$ is odd. A holomorphic function $f\colon H \to C$ is a cusp form of weight k for Γ if

$$f(\gamma z) = j(\gamma, z)^{2k} f(z)$$

for all $z \in H$, $\gamma \in \Gamma$ and f vanishes at each cusp of Γ. The linear space $S_k(\Gamma)$ of cusp forms equipped with the inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash H} f(z)\overline{g(z)}\, y^{k-2}\, dxdy$$

is a finite dimensional Hilbert space spanned by the Poincaré series

$$P_m(z, k, \Gamma) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-2k} e(m\gamma z), \qquad m \geqslant 1.$$

Let

$$f(z) = \sum_1^\infty \hat{f}(n)\, e(nz)$$

be the Fourier expansion of $f \in S_k(\Gamma)$ at the cusp $i\infty$.

Petersson's formula I:

$$\hat{f}(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \langle f, P_n(\cdot, z, k)\rangle. \tag{2.1}$$

By (2.1) it follows that for $m, n \geqslant 1$

$$\sum_{f\in S_k(\Gamma)}^{*} \bar{\hat{f}}(m)\hat{f}(n) = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \hat{P}_m(n, k, \Gamma) \tag{2.2}$$

where on the left-hand side $\sum^*$ means that the summation is taken over an orthonormal basis of $S_k(\Gamma)$.

On the right-hand side $\hat{P}_m(n, k, \Gamma)$ is the nth Fourier coefficient of the mth Poincaré series for which we have another formula of Petersson.

Petersson's formula II:

$$\hat{P}_m(n, k, \Gamma) = \left(\frac{n}{m}\right)^{(k-1)/2} \left\{\delta_{mn} + 2\pi i^{-k} \sum_{c\equiv 0 (\mathrm{mod}\, N)} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) K_k(m, n; c)\right\}$$

where $J_{k-1}(x)$ is the Bessel function of order $k-1$ and $K_k(m, n; c)$ is the generalized Kloosterman sum defined by

$$K_k(m, n; c) = \sum_{d\, (\mathrm{mod}\, c)} \varepsilon_d^{-2k} \left(\frac{c}{d}\right)^{2k} e\left(\frac{m\bar{d}+nd}{c}\right)$$

and $\bar{d}$ is a solution to $d\bar{d} \equiv 1 \pmod{c}$. Notice that if k is an even integer then K_k becomes the ordinary Kloosterman sum

$$K(m, n; c) = \sum_{d (\mathrm{mod}\, c)} e\left(\frac{m\bar{d}+nd}{c}\right).$$

All the above results can be found in the book [4] of Rankin.

For m, n, N, $Q \geqslant 1$ with $(Q, N) = 1$ and $4|N$ if $2k$ is odd define

$$G_k(m, n, Q, N) = i^{-k} \sum_{\substack{c\equiv 0 (\mathrm{mod}\, N)\\ (c,Q)=1}} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) K_k(m, n; c).$$

Now we are ready to state our main results.

THEOREM 1. *Suppose $k > 1$, $2k \equiv 1 \pmod 4$, $n > 1$, $n \equiv 1 \pmod 4$, n squarefree and $N = 4^\nu$ with $\nu \geqslant 4$. We then have*

$$G_k(n, n, n, N) = 2\sum_{m=1}^\infty \left(\frac{n}{m}\right) m^{-1/2} G_{2k-1}(m, 1, n, \tfrac{1}{4}N).$$

Given a cusp form

$$g(z) = \sum_{m=1}^\infty \hat{g}(m)\, e(mz)$$

form the twisted L-series

$$L_g(s, \chi_n) = \sum_{m=1}^\infty \left(\frac{n}{m}\right) \hat{g}(m)\, m^{-s}.$$

By (2.2), (2.3) and Theorem 1 we infer

THEOREM 2. *Under the same assumptions as in Theorem* 1 *we have*

$$\sum_{d|n} \mu(d) \sum_{f\in S_k(\Gamma_0(dN))}^{*} |\hat{f}(n)|^2 = \pi^{1/2-k}\Gamma(k-\tfrac{1}{2}) n^{k-1} \sum_{d|n}\mu(d) \sum_{g\in S_{2k-1}(\Gamma_0(dN/4))}^{*} \bar{\hat{g}}(1) L_g(k-\tfrac{1}{2}, \chi_n).$$

Proof. By the Möbius formula we have

$$\sum_{(c,n)=1} = \sum_c \Big\{\sum_{\substack{d|n\\ d|c}} \mu(d)\Big\} = \sum_{d|n}\mu(d) \sum_{c\equiv 0 (\mathrm{mod}\, d)}. \tag{2.4}$$

Hence, in particular, by (2.2), (2.3) and Theorem 1 we get

$$\sum_{d|n}\mu(d)\sum_{f\in S_k(\Gamma_0(dN))}^{*}|\hat{f}(n)|^2 = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\sum_{d|n}\mu(d)\Big\{1+2\pi i^{-k}\sum_{c\equiv 0 \,(\mathrm{mod}\, dN)} c^{-1}J_{k-1}\Big(\frac{4\pi n}{c}\Big)K_k(n, n; c)\Big\}$$

$$= \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}2\pi G_k(n, n, n, N)$$

$$= 4\pi\frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\sum_{m=1}^{\infty}\Big(\frac{n}{m}\Big)m^{-1/2}G_{2k-1}(m, 1, n, \tfrac{1}{4}N).$$

Now applying again (2.2) and (2.3) in the reversed order we end up with

$$2\frac{\Gamma(2k-2)}{\Gamma(k-1)}\Big(\frac{n}{4\pi}\Big)^{k-1}\sum_{d|n}\mu(d)\sum_{m=1}^{\infty}\Big(\frac{n}{m}\Big)m^{1/2-k}\sum_{g\in S_{2k-1}(\Gamma_0(dN/4))}^{*}\bar{\hat{g}}(1)\hat{g}(m).$$

This completes the proof of Theorem 2 by the Legendre duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$.

3. Evaluation of Kloosterman sums. The main ingredient in the proof of Theorem 1 is the following

LEMMA 1. *Suppose* $2k\equiv 1 \pmod 4$, $n\equiv 1 \pmod 4$, $(c, n) = 1$, $2^8|c$. *We then have* $K_k(n, n; c) = 0$ *unless* $4^{\sigma}\|c$ *in which case*

$$K_k(n, n; c) = (1+i)c^{1/2}\Big(\frac{n}{c}\Big)\sum_{\substack{ab=c\\(a,b)=1}} e\Big(2n\Big(\frac{\bar{a}}{b}-\frac{\bar{b}}{a}\Big)\Big).$$

Proof. The essential part of the arguments is already recorded in [2]. Letting $c = qr$ with $2\nmid q$, $r = 2^{\alpha}$ we have

$$K_k(n, n; c) = K_{2k+1-q}(n\bar{q}, n\bar{q}; r)\varepsilon_q q^{1/2}\Big(\frac{n}{q}\Big)\sum_{\substack{ab=q\\(a,b)=1}} e\Big(2n\bar{r}\Big(\frac{\bar{a}}{b}-\frac{\bar{b}}{a}\Big)\Big), \tag{3.1}$$

by (3.9) of [2]. For the Kloosterman sum to modulus r we prove a general statement.

LEMMA 2. *Suppose* $n\equiv 1 \pmod 2$ *and* $r = 2^{\alpha}$ *with* $\alpha\geqslant 8$. *Then* $K_k(n, n; r) = 0$ *unless* $\alpha\equiv 0 \pmod 2$ *in which case we have*

$$K_k(n, n; r) = r^{1/2}(1+i^n)\Big[e\Big(\frac{2n}{r}\Big)-i^{k-n}e\Big(\frac{-2n}{r}\Big)\Big]. \tag{3.2}$$

Proof. Consider two cases:

Case I: $\alpha = 2\beta+1$, $\beta\geqslant 4$. Set $d = u+2^{\beta+1}v$ with $u \,(\mathrm{mod}\, 2^{\beta+1})$, $2\nmid u$ and $v \,(\mathrm{mod}\, 2^{\beta})$. Then $\bar{d}\equiv\bar{u}-\bar{u}^2v2^{\beta+1} \pmod{2^{\alpha}}$ where $\bar{u}$ is a solution to $u\bar{u}\equiv 1 \pmod{2^{\alpha}}$. This gives

$$K_k(n, n; r) = \sum_u \varepsilon_u^{-k}\Big(\frac{2}{u}\Big)e\Big(\frac{n(u+\bar{u})}{2^{\alpha}}\Big)\sum_v e\Big(\frac{nv(1-\bar{u}^2)}{2^{\beta}}\Big) = 2^{\beta}\sum_{\substack{u (\mathrm{mod}\, 2^{\beta+1})\\ u^2\equiv 1 (\mathrm{mod}\, 2^{\beta})}}\varepsilon_u^{-k}\Big(\frac{2}{u}\Big)e\Big(\frac{n(u+\bar{u})}{2^{\alpha}}\Big).$$

There are four solutions in $u \,(\mathrm{mod}\, 2^{\beta+1})$, namely

$$u\equiv 1,\quad 2^{\beta}-1,\quad 2^{\beta}+1,\quad 2^{\beta+1}-1 \pmod{2^{\beta+1}}$$

with

$$\bar{u}\equiv 1,\quad 2^{2\beta}-2^{\beta}-1,\quad 2^{2\beta}-2^{\beta}+1,\quad 2^{2\beta+1}-2^{\beta+1}-1 \pmod{2^{\alpha}}$$

respectively. Hence $u+\bar{u}\equiv 2,\ -2+2^{2\beta},\ 2+2^{2\beta},\ -2 \pmod{2^{\alpha}}$ and $\varepsilon_u^{-k}\Big(\frac{2}{u}\Big) = 1,\ -i^k,\ 1,\ -i^k$ respectively, giving

$$2^{-\beta}K_k(n, n; r) = e\Big(\frac{2n}{r}\Big)+i^k e\Big(\frac{-2n}{r}\Big)-e\Big(\frac{2n}{r}\Big)-i^k e\Big(\frac{-2n}{r}\Big) = 0.$$

Case II: $\alpha = 2\beta$, $\beta\geqslant 4$. Set $d = u+2^{\beta}v$ with $u \,(\mathrm{mod}\, 2^{\beta})$, $2\nmid u$ and $v \,(\mathrm{mod}\, 2^{\beta})$. Then $\bar{d}\equiv\bar{u}-\bar{u}^2v2^{\beta} \pmod{2^{\alpha}}$ where $\bar{u}$ is a solution to $u\bar{u}\equiv 1 \pmod{2^{\alpha}}$. This gives

$$K_k(n, n; r) = 2^{\beta}\sum_{\substack{u (\mathrm{mod}\, 2^{\beta})\\ u^2\equiv 1 (\mathrm{mod}\, 2^{\beta})}}\varepsilon_u^{-k}e\Big(n\frac{(u+\bar{u})}{2^{\alpha}}\Big).$$

There are four solutions in $u \,(\mathrm{mod}\, 2^{\beta})$, namely

$$u = 1,\quad 2^{\beta-1}-1,\quad 2^{\beta-1}+1,\quad 2^{\beta}-1 \pmod{2^{\beta}}$$

with

$$\bar{u}\equiv 1,\quad -1,\quad 2^{\beta-1}-4^{\beta-1},\quad 1-2^{\beta-1}+4^{\beta-1},\quad -1-2^{\beta} \pmod{2^{\alpha}}$$

respectively. Hence $u+\bar{u}\equiv 2,\ -2-4^{\beta-1},\ 2+4^{\beta-1},\ -2 \pmod{2^{\alpha}}$ and $\varepsilon_u^{-k} = 1,\ -i^k,\ 1,\ -i^k$ respectively, giving

$$2^{-\beta}K_k(n, n; r) = e\Big(\frac{2n}{r}\Big)-i^{k-n}e\Big(\frac{-2n}{r}\Big)+i^n e\Big(\frac{2n}{r}\Big)-i^k e\Big(\frac{-2n}{r}\Big) = (1+i^n)\Big[e\Big(\frac{2n}{r}\Big)-i^{k-n}e\Big(\frac{-2n}{r}\Big)\Big].$$

This completes the proof of Lemma 2.

For $2k\equiv n\equiv 1 \pmod 4$ and $r = 4^{\sigma}$, (3.2) gives

$$K_{2k+1-q}(n\bar{q}, n\bar{q}; r) = r^{1/2}(1+i^q)\Big[e\Big(2n\frac{\bar{q}}{r}\Big)+e\Big(-2n\frac{\bar{q}}{r}\Big)\Big]. \tag{3.3}$$

But $(1+i^q)\varepsilon_q = 1+i$, so combining (3.1) with (3.2) one completes the proof of Lemma 1.

LEMMA 3. *Let $r = 2^{2\beta+1}$ with $\beta \geqslant 2$. We then have*

$$K(1, m; r) = 0. \tag{3.4}$$

Proof. The arguments are standard and similar to those used in the proof of the previous lemma.

4. A duplication formula for Bessel's functions. We shall need the following

LEMMA 4. *Let $a, b > 0$ and $\nu \geqslant 1$. We then have*

$$\int_0^\infty e^{-i(ax+bx^{-1})} J_\nu(bx^{-1}) x^{-1/2}\,dx = 2i^{-\nu-1/2}\sqrt{\frac{\pi}{a}}\,J_{2\nu}(\sqrt{8ab})$$

and

$$\int_0^\infty e^{i(ax-bx^{-1})} J_\nu(bx^{-1}) x^{-1/2}\,dx = \pi i^{-\nu-1/2}\sqrt{\frac{\pi}{a}}\,K_{2\nu}(\sqrt{8ab}).$$

Proof. For $\alpha, \beta > 0$ and $\nu \geqslant 1$ we have (cf. [1], p. 725)

$$\int_0^\infty e^{-\alpha x-\beta x^{-1}} K_\nu(\beta x^{-1}) x^{-1/2}\,dx = 2\sqrt{\frac{\pi}{\alpha}}\,K_{2\nu}(\sqrt{8\alpha\beta}).$$

Move α and β to ia and ib respectively within the first quadrant getting

$$\int_0^\infty e^{-ia-ibx^{-1}} K_\nu(ibx^{-1}) x^{-1/2}\,dx = 2\sqrt{\frac{\pi}{ia}}\,K_{2\nu}(i\sqrt{8ab})$$

by the continuity argument. Analogously we prove

$$\int_0^\infty e^{ia-ibx^{-1}} K_\nu(ibx^{-1}) x^{-1/2}\,dx = 2\sqrt{\frac{\pi}{-ia}}\,K_{2\nu}(\sqrt{8ab}).$$

But for ν and z real we have

$$K_\nu(iz) = -\frac{\pi i}{2} e^{-\pi i\nu/2} J_\nu(z).$$

This completes the proof.

5. Proof of Theorem 1. Set $B = \{b;\ 4^\beta \| b \text{ with } \beta \geqslant \nu\}$. By Lemma 1

$$G_k(n, n, n, N) = 2i^{-k}(1+i)\sum_{b\in B}\sum_{(a,b)=1}(ab)^{-1/2}\left(\frac{n}{ab}\right)J_{k-1}\left(\frac{4\pi n}{ab}\right)\cos\left(4\pi n\left(\frac{\bar a}{b}-\frac{\bar b}{a}\right)\right)$$
$$= 2i^{-k}(1+i)\sum_{b\in B} b^{-1/2}\left(\frac{n}{b}\right)V(b),$$

say, where

$$V(b) = \operatorname{Re}\sum_{(a,b)=1} a^{-1/2}\left(\frac{n}{a}\right)J_{k-1}\left(\frac{4\pi n}{ab}\right)e\left(2n\left(\frac{\bar a}{b}-\frac{\bar b}{a}\right)\right).$$

From the 'reciprocity' formula

$$\frac{\bar a}{b}+\frac{\bar b}{a} \equiv \frac{1}{ab} \pmod 1$$

by splitting into arithmetic progressions $x \pmod{bn}$ we get

$$V(b) = \operatorname{Re}\sum_{x\,(\mathrm{mod}\,bn)}\left(\frac{n}{x}\right)e\left(\frac{4n\bar x}{b}\right)\sum_{a\equiv x\,(\mathrm{mod}\,bn)} a^{-1/2}J_{k-1}\left(\frac{4\pi n}{ab}\right)e\left(-\frac{2n}{ab}\right).$$

By Poisson's summation the innermost sum is equal to

$$(bn)^{-1/2}\sum_{m=-\infty}^{\infty} e\left(\frac{mx}{bn}\right)I(b, m)$$

where

$$I(b, m) = \int_0^\infty J_{k-1}\left(\frac{4\pi}{yb^2}\right)e\left(-\frac{2}{yb^2}-ym\right)y^{-1/2}\,dy.$$

The sum over $x \pmod{bn}$ factors into two sums; the Kloosterman sum $K(4, m; b)$ and the Gauss sum

$$\sum_{x\,(\mathrm{mod}\,n)}\left(\frac{n}{x}\right)e\left(\frac{\bar b mx}{n}\right) = \left(\frac{n}{bm}\right)n^{1/2}.$$

Collecting the above results we get

$$G_k(n, n, n, N) = 2i^{-k}(1+i)\sum_{m\neq 0}\left(\frac{n}{m}\right)\sum_{\substack{b\in B\\(b,n)=1}} b^{-1}K(4, m; b)I(b, m)$$
$$= 4\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)m^{-1/2}\sum_{\substack{b\in B\\(b,n)=1}} b^{-1}K(4, m; b)J_{2k-2}\left(\frac{8\pi\sqrt{m}}{b}\right)$$
$$+2\pi\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)m^{-1/2}\sum_{\substack{b\in B\\(b,n)=1}} b^{-1}K(4, -m; b)K_{2k-2}\left(\frac{8\pi\sqrt{m}}{b}\right),$$

by Lemma 4. We have $K(4, -m; b) = 0$ unless $m = 4m_1$. Letting $b = 4b_1$ we find $K(4, \pm m; b) = 2K(1, \pm m_1; b_1)$ and $K(1, \pm m_1; b_1) = 0$ if $2^{2\beta+1}\| b_1$ by Lemma 3. From this follows

$$G_k(n, n, n, N) = 2\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)m^{-1/2}\sum_{\substack{b\equiv 0 (\mathrm{mod}\, N/4)\\ (b,n)=1}} b^{-1}K(1, m; b)J_{2k-2}\left(\frac{4\pi\sqrt{m}}{b}\right)$$

$$+\pi\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)m^{-1/2}\sum_{\substack{b\equiv 0 (\mathrm{mod}\, N/4)\\ (b,n)=1}} b^{-1}K(1, -m; b)K_{2k-2}\left(\frac{4\pi\sqrt{m}}{b}\right).$$

Here the last sum vanishes because by (2.3) and (2.4) it is the $-m$th Fourier coefficient of a linear combination of the Poincaré series $P_1(z, 2k-1, \Gamma_0(dN/4))$ which are cusp forms. This completes the proof of Theorem 1.

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