Вычисление матрицы A требует, очевидно, времени, ограниченного полиномом от n и $\ln q$, а вычисления по формулам (9)–(12) требуют времени, ограниченного полиномом только от n (см. [4]). Лемма доказана.

Доказательство теоремы. Применяя при $q < n$ лемму 1, а при $q > n$ лемму 2, получаем утверждение теоремы.

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An awful problem about integers in base four

by

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To Paul Erdős on his 75th birthday

The set $\boldsymbol{Z}$ of all integers coincides with the language of all words on the symbols 0, 1, $\bar{1}$ and 2 interpreted as integers presented in base four; here $\bar{1}$ is a convenient contraction for the digit -1. We consider the subset $\boldsymbol{L}$ of $\boldsymbol{Z}$ omitting the digit 2; thus the language of all words on just the symbols 0, 1 and $\bar{1}$ interpreted as integers in base four. Our problem is this: *can every odd integer be written as a quotient of elements of* $\boldsymbol{L}$?

We will answer this question in the affirmative, but first we pause to remark that the matter is troublesome. For example, given an odd integer k it is not at all easy to find a nonzero multiplier m in $\boldsymbol{L}$ so that also km is in $\boldsymbol{L}$. The only method that seems efficient is exemplified by the following computation in which we discover the smallest (positive) multiplier for $k = 2\bar{1}2\bar{1}1$ $(= 477)$. We find

$$\begin{array}{cccccccccc}
2 & \bar{1} & 2 & \bar{1} & 1 & & & & & + \\
 & \underline{2} & \underline{\bar{1}} & \underline{2} & \underline{\bar{1}} & \underline{1} & & & & - \\
1 & 1 & 2 & 1 & 2 & \bar{1} & & & & \\
 & & \underline{2} & \underline{\bar{1}} & \underline{2} & \underline{\bar{1}} & \underline{1} & & & - \\
1 & 1 & 0 & 2 & 0 & 0 & \bar{1} & & & \\
 & & & \underline{2} & \underline{\bar{1}} & \underline{2} & \underline{\bar{1}} & \underline{1} & & + \\
1 & 1 & 1 & 0 & \bar{1} & 1 & 2 & 1 & & \\
 & & & & \underline{2} & \underline{\bar{1}} & \underline{2} & \underline{\bar{1}} & \underline{1} & + \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 &
\end{array}$$

so that multiplying k by $1\bar{1}\bar{1}11$ $(= 181)$ yields a product in $\boldsymbol{L}$. Roughly, the strategy at each step is to multiply by 4 and to add or subtract k or to do nothing, all the while ensuring that no digit 2 remains trapped on the left. Another example, with $k = 2\bar{1}11$ $(= 117)$:

$$\begin{array}{rrrrrrrrl}
2 & \bar{1} & 1 & 1 & & & & & + \\
 & 2 & \bar{1} & 1 & 1 & & & & - \\
\hline
1 & 1 & 2 & 0 & \bar{1} & & & & \\
 & & 2 & \bar{1} & 1 & 1 & & & - \\
\hline
1 & 1 & 0 & 0 & 2 & \bar{1} & & & \\
 & & & 2 & \bar{1} & 1 & 1 & & 0,\ - \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & \bar{1} & \bar{1} &
\end{array}$$

so $2\bar{1}11$ has the multiplier $1\bar{1}\bar{1}0\bar{1}$ ($=175$) in L. Eventually, the digits on the right seem to take care of themselves. These experiments suggest that each odd k has a smallest multiplier not a great deal longer than k; some early extreme examples are $k = 2011$ ($=133$) whose smallest multiplier is $111\bar{1}1$ ($=333$) and $k = 20\bar{1}11$ ($=501$) with smallest multiplier $1\bar{1}\bar{1}\bar{1}\bar{1}1\bar{1}$ ($=2739$).

Our question arose in the course of work of Brown and Moran which yielded [1]. The problem became known as 'that awful problem about integers in base four' to those unfortunate enough to have become obsessed, or intrigued, by it. Our solution of course developed by a more circuitous route than that we describe, but the reader will notice sufficiently many novel elements even in the 'tidied' proof to understand the original intractable appearance of the problem.

1. A theorem that solves the problem. Our principal results may appear extraneous to our objective; in fact each solves the problem:

THEOREM. *Let S be the set of integers which can be written in base four using just the digits 0 or 1, and for $n = 0, 1, 2, \ldots$ denote by S_n the subset of numbers in S with at most n digits. Let k be an odd integer, or more generally suppose that $k \equiv \pm 4^a \pmod{4^b}$ with $a < b$. Then for all sufficiently large n the set*

$$S_n + kS_n = \{s + ks' \mid s,\ s' \text{ in } S_n\}$$

has fewer than 4^n distinct elements. Moreover, these elements lie in only $O(r^n)$ distinct residue classes mod 4^n, *for some r satisfying $3 < r < 4$. Thus almost no nonnegative integer lies in the set $S + kS$.*

COROLLARY. *With k odd or, more generally, with k as described above, k can be written as a quotient of elements of the set L of integers representable in base four using just the digits 0, 1 or $\bar{1}$.*

To relieve the suspense we give an immediate proof that the Theorem does imply the Corollary. Indeed, suppose that some element of the set $S_n + kS_n$ has at least two distinct representations, say:

$$s_1 + ks_1' = s_2 + ks_2' \quad \text{with} \quad s_1,\ s_1',\ s_2,\ s_2' \text{ in } S_n.$$

Noticing that $L = S - S$, we see that

$$k(s_1' - s_2') = (s_2 - s_1)$$

gives the required multiplier for k.

The core of the relationship between the Theorem and the Corollary is thus just the observation

$$L = S - S = \{s - s' \mid s,\ s' \text{ in } S\}.$$

The second result of the Theorem yields an independent proof of the Corollary. To see this consider the generating function

$$F_k(x) = \prod_{n=0}^{\infty} (1 + x^{4^n})(1 + x^{k4^n}) = \sum_{n=0}^{\infty} r_n(k) x^n.$$

Here $r_n(k)$ is exactly the number of representations of n of the shape $s + ks'$ with s and s' in S. Thus if $r_n(k)$ is greater than 1, we have two distinct representations of n, and then, as remarked above, we readily obtain a nonzero multiplier m in L for k so that also km is in L.

But, on average, $r_n(k)$ is about $1/\sqrt{k}$; more precisely:

$$N/8\sqrt{k} < \sum_{n<N} r_n(k) < 4N/\sqrt{k}.$$

This is not difficult to see on observing that

$$\sum_{n<N} r_n(k) = \sum_{s+ks'<N} 1.$$

Then recall that there are exactly 2^n elements of S_n and suppose that $4^n \leqslant N < 4^{n+1}$, after which the allegation follows by a simple estimation. Thus our showing that, with k odd, almost all $r_n(k)$ are zero implies that some $r_n(k)$ exceed 1, again solving the problem. So also the second claim of the Theorem yields the Corollary.

2. Congruence classes and types. Recall that S_n contains 2^n numbers, representing the residue classes of S mod 4^n. We suppose that $S_n + kS_n$ contains 4^n distinct elements for each $n = 0, 1, 2, \ldots$ and obtain an eventual contradiction. A novel feature of our approach is that we study and count residue classes mod 4^n rather than studying and counting the elements *per se*.

For $n = 1$, the set $S_1 + kS_1$ consists of the 4 integers: 0, 1, k, and $k+1$, which we view as grouped into congruence classes mod 4. For example, if $k \equiv 1 \pmod 4$ we get three groups $\{0\}$, $\{1, k\}$ and $\{k+1\}$ congruent respectively to 0, 1 and 2 mod 4. To move to the next level, $n = 2$, we add the set $4(S_1 + kS_1)$ to obtain 4^2 numbers grouped into residue classes mod 4^2, and so on.

As an aside we remark that: It will be seen that we rely on the relationship:

$$S_{n+1}+kS_{n+1} = S_n+kS_n+4^n(S_1+kS_1), \quad n = 0, 1, 2, \ldots$$

Later in our proof we also use:

$$S_{n+1}+kS_{n+1} = 4(S_n+kS_n)+S_1+kS_1, \quad n = 0, 1, 2, \ldots$$

These conditions impose a severe constraint which prevents our argument being applied in contexts which otherwise may appear to differ only little from the present one.

Note that elements belonging to distinct classes mod 4^n cannot give rise to elements that lie in the same class at a higher level: For if $a+4^n(s_1+ks_1') \equiv b+4^n(s_2+ks_2') \pmod{4^{n+1}}$ then $a \equiv b \pmod{4^n}$. Thus it is enough to follow the career of a typical class mod 4^n as we raise the level.

Let $\{t_1, t_2, \ldots, t_m\}$, where $t_1 < t_2 < \ldots < t_m$ are congruent mod 4^n, be a typical such class and set

$$t_i - t_1 = 4^n r_i \quad (1 \leqslant i \leqslant m).$$

Then following the career of the class $\{t_1, t_2, \ldots, t_m\}$ from level n is equivalent to following the class $\{r_1, r_2, \ldots, r_m\}$ from level 0; this effects a useful normalisation which we shall express by saying that the class $\{t_1, t_2, \ldots, t_m\}$ is of type $(r_1, r_2, \ldots, r_m)$.

However, in moving from one level to the next, we add at most $k+1$ times the appropriate power of 4 to each element, so a t_i in a class at level n satisfies

$$t_i \leqslant (k+1)(1+4+4^2+\ldots+4^{n-1}) < (k+1)4^n/3,$$

and the corresponding normalised r_i satisfies

$$r_i < (k+1)/3.$$

Since the r_i are distinct, we have obtained our first important intermediate conclusion:

LEMMA. *For each k only finitely many different types occur in the construction described above.*

An example may be helpful. Take $k = 9$. At the trivial level, $n = 0$, we have just one class, namely

$$\{0\}.$$

At the next level, $n = 1$, we have three classes

$$\{0\}, \quad \{1, 9\}, \quad \{10\},$$

and at $n = 2$ we have ten classes

$$\{0\}, \quad \{1, 49\}, \quad \{50\}, \quad \{4, 36\}, \quad \{5, 37\},$$
$$\{40\}, \quad \{9, 41\}, \quad \{10\}, \quad \{13, 45\}, \quad \{14, 46\}.$$

But we do not need this amount of detail. We see that the singleton at $n = 0$ gives rise at $n = 1$ to two singletons and a doubleton $\{1, 9\}$ of type $(0, 2)$. The two singletons at $n = 1$ therefore produce four singletons and two doubletons of type $(0, 2)$ at $n = 2$, whilst the doubleton $\{1, 9\}$ of type $(0, 2)$ gives rise to three doubletons of type $(0, 2)$, namely $\{5, 37\}$, $\{9, 41\}$ and $\{13, 45\}$ and a further doubleton of type $(0, 3)$, namely $\{1, 49\}$. Since we have already seen what happens to the types (0) and $(0, 2)$ on raising the level we need now only study the type $(0, 3)$. But the doubleton $\{0, 3\}$ yields

$$\{0, 4, 12\}, \quad \{1, 9, 13\}, \quad \{10\}, \quad \{3\},$$

that is, two singletons and two triples: of types $(0, 1, 3)$ and $(0, 2, 3)$. The full set of classes at $n = 3$ happens then to comprise 10 singletons, 19 doubletons of type $(0, 2)$ and 5 of type $(0, 3)$, and triples of types $(0, 1, 3)$ and $(0, 2, 3)$ respectively. Happily, we do not need most of this information. To proceed to $n = 4$ we study the triple $\{0, 1, 3\}$, observing that it yields just

$$\{0, 4, 12\}, \quad \{1, 9, 13\}, \quad \{2, 10\}, \quad \{3, 11\},$$

which is fewer than the expected 12 distinct elements! So just by following the types we have demonstrated that S_4+9S_4 has fewer than 4^4 distinct elements. It follows that 9 has a nonzero multiplier m in L so that also $9m$ is in L; moreover $9m$ has at most 4 digits. In fact

$$21 \times 11 = 1\bar{1}\bar{1}1.$$

For later use we remark that the triple $\{0, 2, 3\}$ yields the classes

$$\{0, 4, 12\}, \quad \{1, 9, 13\}, \quad \{2, 10\}, \quad \{3, 11\}$$

of types $(0, 1, 3)$, $(0, 2, 3)$ and $(0, 2)$. In all, $k = 9$ yields just 5 distinct types.

3. Equations. By assumption the sets S_n+kS_n contain 4^n distinct elements and, by the Lemma, the congruence classes mod 4^n in the sets S_n+kS_n have bounded size because only finitely many different types can occur. We shall prove that these constraints are incompatible.

Take a congruence class of maximal size, say of M elements, normalised as in the discussion of type so that we can consider it to be at level 0. Denote by $N_i = N_i^{(n)}$ the number of elements of this class which are congruent to i mod 4^n. Now move to level n by adding the 4^n elements of the set S_n+kS_n to each element of this class. By our assumption, we obtain $4^n M$

distinct integers falling into various classes mod 4^n. Since M is the maximal size of a class, each of the 4^n possible congruence classes mod 4^n must contain exactly M integers.

But at level n the number of elements congruent to i mod 4^n is the sum of the quantities N_{i-t} (with the subscripts interpreted modulo 4^n) and t running through the elements of the set S_n+kS_n. Thus we have the circulating system of equations

$$\sum_{t \text{ in } S_n+kS_n} N_{i-t}^{(n)} = M \qquad (0 \leqslant i < 4^n)$$

as well as the obvious equation

$$\sum_{i \bmod 4^n} N_i^{(n)} = M.$$

For example, take $k \equiv 1$ (mod) 4) and $n = 1$ and temporarily drop the superscript on the $N_i^{(n)}$. At level 1, the number of integers congruent to 0 (mod 4) is

$$N_0+N_{-1}+N_{-k}+N_{-k-1} = N_0+N_2+2N_3$$

and similarly for the other congruence classes. The equations above become

$$N_0+N_2+2N_3 = N_1+N_3+2N_0 = N_2+N_0+2N_1 = N_3+N_1+2N_2 = M$$

whence

$$N_0 = N_2, \qquad N_1 = N_3.$$

This is already something, because it tells us that the maximal size $M = 2(N_0+N_1)$ must be even.

4. A digression on circulants. The matrix of the rather intimidating system of equations for the $N_i^{(n)}$ is a circulant of the general shape

$$C = [c_{i-j(\bmod m)}]_{0 \leqslant i,j \leqslant m-1}$$

with, in the present case, $m = 4^n$. It is well known that the corresponding determinant can be factorised with the aid of an appropriate Vandermonde determinant. Thus let $\theta_0, \theta_1, \ldots, \theta_{m-1}$ be the m distinct mth roots of unity and set

$$\Delta = [\theta_j^i]_{0 \leqslant i,j \leqslant m-1}.$$

The determinant of Δ is the difference product of the mth roots of unity, so Δ is non-singular; in fact if Δ^* denotes the transpose of the complex conjugate of Δ then

$$\Delta^{-1} = m^{-1}\Delta^*.$$

Furthermore, Δ diagonalises C in the usual sense that

$$\Delta^{-1} C \Delta = \Lambda,$$

where $\Lambda = \operatorname{diag}(\varphi_0, \varphi_1, \ldots, \varphi_{m-1})$ is the diagonal matrix whose successive entries are given by the resolvent polynomials

$$\varphi_i = \varphi(\theta_i) = c_0 + c_1\theta_i + \ldots + c_{m-1}\theta_i^{m-1} \qquad (0 \leqslant i \leqslant m-1).$$

In particular, the general solution of the system of equations

$$Cx = 0$$

is given by

$$x = \Delta y,$$

where the components of the vector $y = [y_0, y_1, \ldots, y_{m-1}]$ satisfy

$$\varphi_i y_i = 0 \qquad (0 \leqslant i \leqslant m-1);$$

so, unless $\varphi_i = 0$, $y_i = 0$.

5. Resolution of the problem. We return to the equations for the quantities $N_i^{(n)}$. Let $c_i^{(n)}$ be the number of elements of the set S_n+kS_n which are congruent to $-i$ mod 4^n, so that

$$\sum_{i \bmod 4^n} c_i^{(n)} = 4^n.$$

Then we have

$$\sum_{t \text{ in } S_n+kS_n} N_{i-t}^{(n)} = \sum_{j \bmod 4^n} c_j^{(n)} N_{i+j}^{(n)} = \sum_{j \bmod 4^n} c_{j-i}^{(n)} N_j^{(n)} = M \qquad (0 \leqslant i < 4^n)$$

and the preceding theory applies with $m = 4^n$. We need to evaluate the resolvents

$$\varphi^{(n)}(\theta) = \sum_{i \bmod 4^n} c_i^{(n)} \theta^i,$$

where θ is a 4^n-th root of unity; less precisely, in view of the discussion in Section 4, we need to know for which such θ the resolvent vanishes or fails to vanish.

The easy way to do this is to recall an earlier remark to the effect that we can obtain $S_{n+1}+kS_{n+1}$ by multiplying the elements of S_n+kS_n by 4 and adding 0, 1, k and $k+1$ to each one. In this way, each element of S_n+kS_n congruent to $-i$ mod 4^n yields four elements of $S_{n+1}+kS_{n+1}$ congruent respectively to $-4i$, $-4i+1$, $-4i+k$ and $-4i+k+1$. If η is a 4^{n+1}-th root of unity, we see that

$$\varphi^{(n+1)}(\eta) = \sum_{i \bmod 4^n} c_i^{(n)} \eta^{4i}(1+\eta^{-1}+\eta^{-k}+\eta^{-k-1}) = \varphi^{(n)}(\eta^4)(1+\eta^{-1})(1+\eta^{-k}).$$

So by induction,

$$\varphi^{(n)}(\theta) = \prod_{i=0}^{n-1} (1+\theta^{-4^i})(1+\theta^{-k4^i}).$$

At last, it has become relevant to suppose that k is odd. Then $\varphi^{(n)}(\theta) \neq 0$ when θ has exact order 4^h for some h, and $\varphi^{(n)}(\theta) = 0$ when θ has exact order $2\cdot 4^h$ for some $h < n$. By the remarks of Section 4, we can write

$$N_i^{(n)} = 4^{-n} M + \sum_j y_j \theta_j^i$$

where the θ_j in the sum run through the 4^n-th roots of unity with exact order $2\cdot 4^h$ for some $h < n$. The term $4^{-n}M$ comes from an obvious particular solution to our inhomogeneous system of equations. The point of all this is that certainly

$$N_i^{(n)} = N_{i+2\cdot 4^{n-1}}^{(n)} \quad (0 \leqslant i < 2\cdot 4^{n-1})$$

for every n, and this almost magical observation is enough to yield the contradiction towards which we have been struggling. Indeed it is easy to see by induction that at least 2^n of the $N_i^{(n)}$ are non-zero: for each i mod 4^n for which $N_i^{(n)}$ is non-zero, at least two of

$$N_i^{(n+1)}, \quad N_{i+4^n}^{(n+1)}, \quad N_{i+2\cdot 4^n}^{(n+1)} \quad \text{and} \quad N_{i+3\cdot 4^n}^{(n+1)}$$

must be non-zero.

Thus

$$M = \sum_{i \bmod 4^n} N_i^{(n)} \geqslant 2^n$$

for every n, which is absurd.

To summarise: We have shown that if k is odd and n is sufficiently large then $S_n + kS_n$ contains fewer than 4^n elements. Hence there is a non-zero multiplier m in L for k so that also km is in L. Exactly the same argument goes through if $k \equiv \pm 4^a \pmod{4^b}$ with $a < b$. If $k \equiv 2 \pmod 4$, then $\varphi^{(n)}(\theta) = 0$ for all 4^n-th roots of unity, except for $\theta = 1$. In this case, and more generally for $k \equiv 2\cdot 4^a \pmod{4^b}$ with $a < b < n$, primitive 4^n-th roots of unity survive in the above solution for the $N_i^{(n)}$ and we cannot make any dramatic assertion about the number of the $N_i^{(n)}$ which are non-zero.

6. Transition matrices. 'To gild refined gold, to paint the lilly... is', as Salisbury warns King John, 'wasteful and ridiculous excess'. Nevertheless, we continue the argument so as to say more about the number of congruence classes of $S+kS$ mod 4^n, thereby completing the proof of the Theorem.

To illustrate the situation, recall the example with $k = 9$ studied in Section 2. We obtained, as always, only finitely many different types: in this case (0), (0, 2), (0, 3), (0, 1, 3) and (0, 2, 3). Moreover, we have the data to

draw up a transition table showing the manner in which the types are obtained as we move from one level to the next.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \end{bmatrix}$$

From the first row we see that the singleton (0) gives two singletons and a doubleton of type (0, 2), from the second row that the doubleton (0, 2) gives three doubletons of type (0, 2) and one doubleton of type (0, 3), and so on. The transition matrix has non-negative integer entries and is irreducible: in the sense that each type yields every other type if we pursue its byproducts to a sufficiently high level. By the Perron–Frobenius theory for irreducible non-negative matrices, the transition matrix has a positive eigenvalue, r say, such that all other eigenvalues have absolute value at most r. Each eigenvalue of maximal absolute value is a simple root of the characteristic equation. Finally, r lies between the minimal and maximal row sums of the matrix; in this example $3 < r < 4$ because the row sums are not all equal. (For these useful facts, and much more, see [3], Volume 2, Chapter 13.)

All this is true for any odd integer k. Since only a finite number of types appear, we can obtain a transition matrix which has non-negative integer entries. The first row, corresponding to the singleton type, sums to 3, and the other row sums are at most 4 because they specify which type appears in each congruence class mod 4. It is essential that the transition matrix be irreducible and this is just what the argument of Sections 2–5 shows. In fact, we have proved that if one follows a type containing K elements to a sufficiently high level, n say, we either find fewer than $4^n K$ elements or we find a congruence class mod 4^n with more than K elements. In either case at least one of the 4^n congruence classes must contain fewer than K elements. So each type leads eventually to a smaller one amongst its byproducts and ultimately yields a singleton; this 'connectivity' of the types is the irreducibility property. As in the example, in general the transition matrix has a dominant eigenvalue, r say, satisfying $3 < r < 4$. We can find the number of congruence classes of each type in $S+kS$ mod 4^n by looking at the entries in the first row of the nth power of the transition matrix. We have therefore shown that the number of congruence classes mod 4^n represented by $S+kS$ is $O(r^n)$ as $n \to \infty$. That is, the number of distinct elements of $S+kS$ not exceeding N is $O(N^{\log r/\log 4})$ as $N \to \infty$. From the estimates of Section 1 there must be integers in $S+kS$ with an arbitrarily large number of representations, so the Corollary is obtained with quite a bit to spare. It may be worth noting that the argument fails when $k \equiv 2\cdot 4^a \pmod{4^b}$, with $a < b$, because the transition matrix has an irreducible component in which all row sums are 4; therefore the dominant eigenvalue is 4.

7. Coda. It is easy to see that base 4 is the critical base for the problem. If the base is larger than 4, there are many integers which cannot be written as a quotient of numbers with digits 0, 1 or $\bar{1}$, despite passing the obvious congruence tests. For example, in base 5, integers k satisfying $5^{m+1}/3 < k < 3\cdot 5^m$ for some m have no such representation. To see this, replace the function introduced in Section 1 by

$$\prod_{n=0}^{\infty}(1+X^{5^n})(1+X^{k5^n})$$

and show that its coefficients are either 0 or 1.

We can also generalise the problem by changing the admissible digits. If we allow the digits 0, 1, $\bar{1}$, 2 and $\bar{2}$, the critical case is base 9. Our methods will show that every k relatively prime to 9 can be written as a quotient of integers whose base 9 representations contain only the digits 0, 1, $\bar{1}$, 2 and $\bar{2}$. And *mutatis mutandis* for any base b^2.

On the other hand, on first meeting the problem in base 4, John Selfridge and Carole LaCampagne asked: can every $k \equiv \pm 1 \pmod 3$ be written as a quotient of integers which can be represented in base 3 using just the digits 1 or $\bar{1}$, and no 0's? Our experiments suggest this can always be done, but our methods do not seem to apply. If one allows the digits 0 or 1, but no 2's, there is some difficulty in describing which integers can be represented.

Our analysis of the base 4 problem proves the existence of the required representation for k, but is apparently not effective. We do not know how to find a good estimate for the smallest positive multiplier m such that m and km are both in L. We have some computational evidence suggesting that the dominant eigenvalue appearing in Section 6 is bounded away from 4. If this is so, then there is an absolute constant C so that every odd k has a multiplier less than $|k|^C$.

It should be plain to the reader that our argument was not born in the form presented here. We were impressed with the fact that the language L is generated by a finite automaton (for an introduction to that cycle of ideas see [2]), and from the start we had known (because of [4]) that $L = S - S$; but until the construction of the function

$$F_k(X) = \prod_{n=0}^{\infty}(1+X^{4^n})(1+X^{k4^n}) = \sum_{n=0}^{\infty} r_n(k)\, X^n$$

this had not seemed to help. The second argument seemed the natural attack. The principal argument appeared *en route.*

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