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The propinquity of divisors

by

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Dedicated to Professor Paul Erdős on the occasion of his 75th birthday

1. Introduction. As far as I am aware, Erdős was the first mathematician to formulate the question: how close together are the divisors of a large, random, integer? This is surprizing: the question could have been considered by Hardy and Ramanujan [8], or indeed much earlier. Erdős' famous conjecture that almost all integers have a pair of divisors d, d' such that $d < d' < 2d$ has recently been settled affirmatively by Maier and Tenenbaum [10], in fact they prove the right-hand inequality of

$$(1)\qquad (\log n)^{1-\log 3-\varepsilon} < \inf_{dd'|n, d<d'} \log d'/d < (\log n)^{1-\log 3+\varepsilon} \quad \text{p.p.,}$$

(where p.p. means for almost all n), both inequalities stated without proof by Erdős [2]. The left-hand inequality was proved in [3].

Of course there are many different ways of measuring the propinquity of divisors, and since functions involving divisors are usually highly irregular, we often have to be content with either their average, or normal orders. Erdős introduced the function

$$\tau^*(n) = \text{card}\{k\colon\ n \text{ has a divisor } d \in (2^k, 2^{k+1}]\}$$

and plainly the conjecture above is a consequence of the proposition that $\tau^*(n) < \tau(n)$ p.p. (where $\tau(n)$ denotes the number of divisors of n). It was thought at one time that $\tau^*(n)/\tau(n) \to 0$ on a sequence of asymptotic density 1, but this was disproved by Erdős and Tenenbaum [4]. They conjectured that the function

$$F(\alpha) = \lim_{x\to\infty} x^{-1}\,\text{card}\{n < x\colon\ \tau^*(n) \leqslant \alpha\tau(n)\}$$

exists and is continuous and strictly increasing on [0, 1]. See [6], Ch.4.

In 1979 Hooley introduced the function

$$\Delta(n) = \max_u \text{card}\{d\colon\ d|n,\ u < \log d \leqslant u+1\}$$

and, more generally,

$$\Delta_r(n) = \max_{u_1, u_2, \ldots, u_{r-1}} \Delta(n; u_1, u_2, \ldots, u_{r-1})$$

where

$$\Delta(n; u_1, u_2, \ldots, u_{r-1}) = \operatorname{card}\{d_1 d_2 \ldots d_{r-1} | n:\ u_i < \log d_i \leqslant u_i + 1,\ 1 \leqslant i < r\}.$$

This was in connection with the 'New Technique' [9]. At about the same time I defined

$$T(n, \alpha) = \operatorname{card}\{d, d' | n:\ |\log(d/d')| \leqslant (\log n)^\alpha\}$$

and this permits a similar generalization,

$$T_r(n, \alpha) = \operatorname{card}\{d_1 d_2 \ldots d_{r-1} | n,\ d_1' d_2' \ldots d_{r-1}' | n,\ |\log(d_i/d_i')| \leqslant (\log n)^\alpha,\ 1 \leqslant i < r\}$$

and my aim here is to give some results about this function and establish connections between it, and $\Delta_r(n)$. We assume $0 \leqslant \alpha < 1$, as $T_r(n, 1)$ is simply $\tau_r(n)^2$.

2. The average order of $T_r(n, \alpha)$.

THEOREM 1. *For every n,*

$$T_r(n, \alpha) \geqslant \max\left(\tau_r(m_\alpha(n))\tau_r(n),\ \frac{\tau_r(n)^2}{3^{r-1}((\log n)^{1-\alpha}+2)^{r-1}}\right) \tag{2}$$

where $m_\alpha(n)$ is the greatest unitary divisor of n not exceeding $\exp((\log n)^\alpha)$.

Proof. The first part is easy. Write $n = m_\alpha(n) q$, and let

$$\delta_1 \delta_2 \ldots \delta_{r-1} | m_\alpha, \quad \delta_1' \delta_2' \ldots \delta_{r-1}' | m_\alpha, \quad t_1 t_2 \ldots t_{r-1} | q.$$

We put $d_i = \delta_i t_i$, $d_i' = \delta_i' t_i$. Evidently $|\log(d_i/d_i')| \leqslant (\log n)^\alpha$, and there are $\tau_r(m_\alpha)^2 \tau_r(q)$ choices.

For the second part, we need the following lemma, which in itself presents what I believe to be an interesting open problem.

LEMMA 1. *Let $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$ be real numbers and*

$$f(x) = \operatorname{card}\{i, j:\ |x^{(i)} - x^{(j)} - x| \leqslant 1\}.$$

Then $f(x) \leqslant 3f(0)$ for all x. Similarly if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}$ are vectors in $\mathbf{R}^s$ and

$$f(\mathbf{x}) = \operatorname{card}\{i, j:\ \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)} - \mathbf{x}\| \leqslant 1\}$$

(*where* $\|(x_1, x_2, \ldots, x_s)\| = \max(|x_k|, k \leqslant s)$) *then* $f(\mathbf{x}) \leqslant 3^s f(\mathbf{0})$.

The question here is what is the best possible constant in place of 3? The example $x^{(i)} = 2i$, $x = 1$ shows that it is at least 2. If it really is 2, can we even have $f(x) = 2f(0)$? (Or more generally, is the best bound attained?).

Proof. Let

$$W(x) = \tfrac{3}{2}(1 - |x - \tfrac{2}{3}|)^+ + \tfrac{3}{2}(1 - |x + \tfrac{2}{3}|)^+$$

where $x^+ = \max(0, x)$, here and throughout the paper. We have $W(x) \geqslant 0$ for all x, and $W(x) \geqslant 1$ for $-1 \leqslant x \leqslant 1$. Hence

$$f(\mathbf{x}) \leqslant \sum_{i,j} \prod_{k=1}^{s} W(x_k^{(i)} - x_k^{(j)} - x_k) \tag{3}$$

where x_k denotes the kth coordinate of $\mathbf{x}$. Now set

$$P(\mathbf{t}) = \sum_{j=1}^{N} e^{i(t_1 x_1^{(j)} + t_2 x_2^{(j)} + \ldots + t_s x_s^{(j)})}, \qquad \mathbf{t} \in \mathbf{R}^s$$

so that the right-hand side of (3) is

$$\frac{1}{(2\pi)^s} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |P(\mathbf{t})|^2 e^{-i(t_1 x_1 + \ldots + t_s x_s)} \prod_{k=1}^{s} (\hat{W}(t_k)\, dt_k)$$

where

$$\hat{W}(t) = 3\cos\tfrac{2}{3}t \cdot \left(\frac{\sin(t/2)}{t/2}\right)^2$$

is the Fourier transform of W. The integral above does not exceed

$$\frac{1}{(2\pi)^s} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |P(\mathbf{t})|^2 \prod_{k=1}^{s} \left(3\left(\frac{\sin(t_k/2)}{t_k/2}\right)^2 dt_k\right) = 3^s \sum_{i,j=1}^{s} \prod_{k=1}^{s} (1 - |x_k^{(i)} - x_k^{(j)}|)^+ \leqslant 3^s f(\mathbf{0})$$

as required.

Now we complete the proof of (2). To each $d_1 d_2 \ldots d_{r-1} | n$ we associate a vector

$$\left(\frac{\log d_1}{(\log n)^\alpha}, \frac{\log d_2}{(\log n)^\alpha}, \ldots, \frac{\log d_{r-1}}{(\log n)^\alpha}\right) \in \mathbf{R}^{r-1}$$

so that there are $N := \tau_r(n)$ vectors altogether: we label them $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}$. In the notation of Lemma 1, we have $f(\mathbf{0}) = T_r(n, \alpha)$. Next, for all i, j the vector $\mathbf{x}^{(i)} - \mathbf{x}^{(j)}$ lies in the cube with corners $(\pm(\log n)^{1-\alpha}, \pm(\log n)^{1-\alpha}, \ldots, \pm(\log n)^{1-\alpha})$ and we cover this large cube with cubes of side 2, with centres $\mathbf{x} = (2m_1, 2m_2, \ldots, 2m_k)$, where each m_k runs from $-M$ to M and $2M+1 \geqslant (\log n)^{1-\alpha}$. The number of centres $\mathbf{x}$ is $(2M+1)^{r-1}$. For every pair i, j there is an $\mathbf{x}$ such that $f(\mathbf{x})$ counts this pair, that is

$$\tau_r(n)^2 = \sum f(\mathbf{x}) \leqslant 3^{r-1}(2M+1)^{r-1} f(\mathbf{0})$$

by the lemma. This gives the result stated.

COROLLARY. *We have*

$$\sum_{n\le x} T_r(n,\alpha)\, y^{\omega(n)} \gg \begin{cases} x(\log x)^{ry-1+r(r-1)\alpha y}, & 0<y\le 1/r,\\ x(\log x)^{r^2y-1-(r-1)(1-\alpha)}, & y\ge 1/r.\end{cases}$$

Notice that the exponent of $\log x$ may be written in the alternative form

$$ry-1+r(r-1)\alpha y+(r-1)(1-\alpha)(ry-1)^+.$$

What is perhaps rather surprizing in view of the apparently somewhat crude estimates involved in Theorem 1 is that except for one case, $y=1/r$, the right-hand side above is the correct order of magnitude of the sum on the left.

THEOREM 2. *Let $r\ge 2$, $y>0$, $\alpha\in[0,1)$. Then there exists a constant $C(r,y,\alpha)$ such that*

$$\sum_{n\le x} T_r(n,\alpha)\, y^{\omega(n)} \sim C(r,y,\alpha)$$
$$\times x(\log x)^{ry-1+r(r-1)\alpha y+(r-1)(1-\alpha)(ry-1)^+}(\log\log x)^{\delta(ry)}$$

where $\delta(u)=0$ $(u\ne 1)$, $\delta(1)=1$.

I do not give a proof of this, as the main ideas are contained in [5] but the details are complicated. Instead I prove the corresponding result with $\ll$ in place of $\sim$, as this can be done quickly and is what we need in the next section.

LEMMA 2. *Let $r\ge 2$, $n\ge 2$ and set $h=(\log n)^\alpha$. Then*

$$(4)\qquad T_r(n,\alpha) \asymp_r h^{r-1}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}|\tau(n;\theta_1,\theta_2,\ldots,\theta_{r-1})|^2\,d\theta_1\,d\theta_2\ldots d\theta_{r-1}$$

where

$$\tau(n;\theta_1,\theta_2,\ldots,\theta_{r-1}) = \sum_{d_1d_2\ldots d_{r-1}|n} d_1^{i\theta_1}d_2^{i\theta_2}\ldots d_{r-1}^{i\theta_{r-1}}.$$

Proof. Let

$$w(x)=\left(\frac{\sin(x/2)}{x/2}\right)^2 = \int_{-\infty}^{\infty}(1-|\theta|)^+e^{i\theta x}\,d\theta,$$

and put $w_1(x)=1.05w(x)\ge 1$ for $|x|\le 1$. Then

$$T_r(n,\alpha)\le \sum_{\substack{d_1d_2\ldots d_{r-1}|n\\ d_1'd_2'\ldots d_{r-1}'|n}}\prod_{k<r} w_1\left(\frac{\log(d_k/d_k')}{h}\right)$$
$$\le (1.05)^{r-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\tau(n;\theta_1/h,\theta_2/h,\ldots,\theta_{r-1}/h)|^2\prod_{k<r}(1-|\theta_k|)^+\,d\theta_k$$

and we obtain an upper bound, with implied constant $(1.05)^{r-1}$, on substituting θ_k for θ_k/h. Now put $w_2(x)=(1-|x|)^+$. We have

$$T_r(n,\alpha)\ge \sum_{\substack{d_1d_2\ldots d_{r-1}|n\\ d_1'd_2'\ldots d_{r-1}'|n}}\prod_{k<r} w_2\left(\frac{\log(d_k/d_k')}{h}\right)$$
$$\ge \frac{1}{(2\pi)^{r-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\tau(n;\theta_1/h,\theta_2/h,\ldots,\theta_{r-1}/h)|^2\prod_{k<r}\left(\frac{\sin(\theta_k/2)}{\theta_k/2}\right)^2 d\theta_k$$

and we restrict the range of integration to the cube $\max|\theta_k|\le 1$, and note that for $|\theta|\le 1$ we have $(\sin(\theta/2))^2/(\theta/2)^2\ge 2\pi/7$. We obtain the lower bound required, with implied constant $1/7^{r-1}$.

We put $h=(\log x)^\alpha$ and we have

$$\sum_{n\le x} T_r(n,\alpha)\,y^{\omega(n)}$$
$$\ll h^{r-1}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}\sum_{n\le x}|\tau(n;\theta_1,\theta_2,\ldots,\theta_{r-1})|^2y^{\omega(n)}\,d\theta_1\,d\theta_2\ldots d\theta_{r-1}$$
$$\ll \frac{xh^{r-1}}{\log x}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}\sum_{n\le x}|\tau(n;\theta_1,\theta_2,\ldots,\theta_{r-1})|^2\frac{y^{\omega(n)}}{n}\,d\theta_1\,d\theta_2\ldots d\theta_{r-1}$$
$$\ll \frac{xh^{r-1}}{\log x}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}R(1+\sigma;\theta_1,\theta_2,\ldots,\theta_{r-1};y)\,d\theta_1\,d\theta_2\ldots d\theta_{r-1}$$

where $\sigma=1/\log x$ and

$$F(s)=\sum_{n=1}^{\infty}|\tau(n;\theta_1,\theta_2,\ldots,\theta_{r-1})|^2\frac{y^{\omega(n)}}{n^s}$$
$$=\prod_p\left(1+\frac{|\tau(p;\theta_1,\theta_2,\ldots,\theta_{r-1})|^2y}{p^s}+\ldots\right)$$
$$=G(s)\zeta(s,y)^r\prod_{j<r}\zeta(s+i\theta_j,y)\zeta(s-i\theta_j,y)\prod_{k\ne l}\zeta(s+i\theta_k-i\theta_l,y),$$

where G is analytic and uniformly bounded in any half-plane $\mathrm{Re}\,s>\frac12+\delta$, and

$$\zeta(s,y):=\sum_{n=1}^{\infty}\frac{y^{\omega(n)}}{n^s}=\prod_p\left(1+\frac{1}{p^s-1}\right).$$

We use the estimate $|\zeta(s,y)|\ll|s-1|^{-y}$ in the neighbourhood of $s=1$. So the sum above is

$$\ll\frac{xh^{r-1}}{\sigma^{ry}\log x}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}\prod_{j<r}|\sigma+i\theta_j|^{-2y}\prod_{k<l<r}|\sigma+i\theta_k-i\theta_l|^{-2y}\,d\theta_1\ldots d\theta_{r-1}$$
$$\ll\frac{xh^{r}}{\sigma^{ry}\log x}\int_{-1/h}^{1/h}\cdots\int_{-1/h}^{1/h}\prod_{0\le j<k<r}|\sigma+i\theta_j-i\theta_k|^{-2y}\,d\theta_0\,d\theta_1\ldots d\theta_{r-1}$$

where we have introduced an extra variable θ_0 for the sake of extra symmetry. We can obviously do this with $\theta_0 \equiv 0$ and it is not difficult to justify the extra integration. Substituting $\theta_i \to \theta_i/h$ $(0 \leqslant i < r)$ and multiplying numerator and denominator by $h^{r(r-1)y}$ we obtain

$$\sum_{n \leqslant x} T_r(n, \alpha)\, y^{\omega(n)} \ll x(\log x)^{ry-1+r(r-1)\alpha y} J_r(\sigma h, y)$$

where

$$J_r(\varrho, x) := \int_{-1}^{1} \ldots \int_{-1}^{1} \prod_{0 \leqslant j < k < r} (\varrho + |\theta_j - \theta_k|)^{-2y}\, d\theta_0\, d\theta_1 \ldots d\theta_{r-1}.$$

LEMMA 3. *For each $r \geqslant 2$ and real $z \geqslant 0$ we have, uniformly for $\varrho \leqslant 1$, that*

$$J_r(\varrho, z) \underset{\frown}{\smile}_{r,z} \varrho^{-(r-1)(rz-1)^+} \left(\log \frac{2}{\varrho}\right)^{\delta(rz)}.$$

This is Lemma 67.2 of [6]. We apply this with $z = y$, $\varrho = \sigma h = (\log x)^{\alpha - 1}$, to obtain the result stated.

3. Inequalities between Δ_r and T_r. The inequality $T_r(n, 0) \leqslant \tau_r(n)\Delta_r(n)$ is clear. In the opposite direction, we have

THEOREM 3. *Let $n = mm'$, $(m, m') = 1$ and $r \geqslant 2$. Then*

$$\Delta_r(n)^2 \leqslant 9^{r-1} T_r(m, 0)\, T_r(m', 0). \tag{5}$$

This is very simple but I state it as a theorem for reference.

Proof. Let

$$\Delta_r(n) = \Delta(n; v_1, v_2, \ldots, v_{r-1})$$

$$\leqslant \sum_{k_1}\sum_{k_2} \ldots \sum_{k_{r-1}} \operatorname{card}\{t_1 t_2 \ldots t_{r-1} | m\colon k_i/2 \leqslant \log t_i < (k_i+1)/2,\ 1 \leqslant i < r\}$$

$$\times \operatorname{card}\{t_1' t_2' \ldots t_{r-1}' | m'\colon v_i - (k_i+1)/2 < \log t_i' \leqslant v_i + 1 - k_i/2,\ 1 \leqslant i < r\}.$$

We apply the Cauchy–Schwarz inequality. We have

$$\sum_{k_1}\sum_{k_2} \ldots \sum_{k_{r-1}} \operatorname{card}\{t_1 t_2 \ldots t_{r-1} | m\colon k_i/2 \leqslant \log t_i \leqslant (k_i+1)/2,\ 1 \leqslant i < r\}^2$$

$$\leqslant \operatorname{card}\{s_1 s_2 \ldots s_{r-1} | m,\ t_1 t_2 \ldots t_{r-1} | m\colon |\log(s_i/t_i)| \leqslant 1\} \leqslant T_r(m, 0).$$

Next, let $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{r-1})$ where each δ_i is either -1, 0 or 1. Then

$$\left(v_i - \frac{k_i+1}{2},\ v_i + 1 - \frac{k_i}{2}\right] = \bigcup_{\delta_i} \left(v_i - \frac{k_i+\delta_i}{2},\ v_i - \frac{k_i+\delta_i}{2} + \frac{1}{2}\right].$$

so we have

$$\sum_{k_1}\sum_{k_2} \ldots \sum_{k_{r-1}} \operatorname{card}\{t_1' t_2' \ldots t_{r-1}' | m'\colon v_i - (k_i+1)/2 < \log t_i' \leqslant v_i + 1 - k_i/2,\ i < r\}^2$$

$$\leqslant 3^{r-1} \sum_{k_1}\sum_{k_2} \ldots \sum_{k_{r-1}} \sum_{\delta} \operatorname{card}\{t_1' t_2' \ldots t_{r-1}' | m'\colon$$

$$v_i - (k_i+\delta_i)/2 < \log t_i' \leqslant v_i - (k_i+\delta_i)/2 + \tfrac{1}{2},\ 1 \leqslant i < r\}^2$$

$$\leqslant 3^{r-1} \sum_{\delta} \operatorname{card}\{s_1' s_2' \ldots s_{r-1}' | m',\ t_1' t_2' \ldots t_{r-1}' | m'\colon |\log(s_i'/t_i')| \leqslant 1,\ i < r\}$$

$$\leqslant 9^{r-1} T_r(m', 0),$$

and the desired result follows.

THEOREM 4. *For $r \geqslant 2$, and $y > 2/r$, we have*

$$\sum_{n \leqslant x} \Delta_r(n)^2 y^{\omega(n)} \underset{\frown}{\smile} x(\log x)^{r^2 y - 2r + 1}.$$

There is a similar result ([6], Ch. VI, Theorem 67) concerning the mean value of $\Delta_r(n)\, y^{\omega(n)}$. These theorems show that, albeit the mean-value of $\Delta_r(n)$ itself is not yet accurately determined, when suitably weighted the order of magnitude may be precisely determined, and without great difficulty.

Proof. The lower bound is trivial because we have

$$\Delta_r(n) \geqslant \max\left(1, \frac{\tau_r(n)}{(\log en)^{r-1}}\right).$$

To obtain the upper bound, put $y = 2z$ so that $z > 1/r$. Then by Theorem 3,

$$\sum_{n \leqslant x} \Delta_r(n)^2 (2z)^{\omega(n)} \leqslant \sum_{n \leqslant x} z^{\omega(n)} \sum_{\substack{d | n \\ (d, n/d) = 1}} T_r(d, 0)\, T_r(n/d, 0)$$

$$\leqslant \sum_{d \leqslant x} T_r(d, 0)\, z^{\omega(d)} \sum_{\substack{m \leqslant x/d \\ (m,d) = 1}} T_r(m, 0)\, z^{\omega(m)}.$$

We apply Theorem 2 with $\alpha = 0$, $y = z$. For convenience put $\gamma = r^2 z - r$. Then the sum above is

$$\ll \sum_{d \leqslant x} T_r(d, 0)\, z^{\omega(d)} \frac{x}{d} \left(\log \frac{x}{d}\right)^{\gamma}$$

$$\ll \sum_{d \leqslant x} T_r(d, 0)\, z^{\omega(d)} \sum_{m \leqslant x/d} (\log m)^{\gamma}$$

$$\ll \sum_{m \leqslant x} (\log m)^{\gamma} \sum_{d \leqslant x/m} T_r(d, 0)\, z^{\omega(d)}$$

$$\ll \sum_{m\leqslant x} \frac{x}{m}(\log m)^{\gamma}\left(\log\frac{x}{m}\right)^{\gamma} \ll x(\log x)^{2\gamma+1}$$

because $\gamma > -1$ (indeed $\gamma > 0$). This is the result stated.

We may now recover [6], Theorem 67.

COROLLARY. *Let* $r \geqslant 2$ *and* $y > 2$. *Then*

$$\sum_{n\leqslant x} \Delta_r(n)\, y^{\omega(n)} \underset{\cap}{\cup} x(\log x)^{ry-r}. \tag{6}$$

Again, the lower bound is easy. For the upper bound we use the Cauchy–Schwarz inequality. Here $z = y/r > 2/r$ by hypothesis. Hence

$$\Big(\sum_{n\leqslant x} \Delta_r(n)\, y^{\omega(n)}\Big)^2 \leqslant \Big(\sum_{n\leqslant x} \Delta_r(n)^2 z^{\omega(n)}\Big)\left(\sum_{n\leqslant x}\left(\frac{y^2}{z}\right)^{\omega(n)}\right)$$
$$\ll \big(x(\log x)^{r^2 z-2r+1}\big)\big(x(\log x)^{y^2/z-1}\big)$$

by Theorem 4. This yields the upper bound required.

There is a heuristic explanation why this method does not work if y is too small. It is easy to see that if a sum such as (6) be weighted with $y^{\omega(n)}$, the larger y is, the more emphasis is placed on integers with many prime factors. Now if $\omega(n)$ is large, we can argue that the number of divisors in all the various ranges in the proof of Theorem 3 will behave statistically – in this event there is little lost in the Cauchy–Schwarz inequality. When $\omega(n)$ is small, the Cauchy–Schwarz inequality is applied to a sum of the form $\sum a_j b_j$ say where a_j and b_j are usually 0 or 1, and there is no reason whatever to suppose that the 1's 'line up'. We are forced to use the much more complicated machinery of [6], Chapter VII, and of course the results are no longer sharp.

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