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A local Turán–Kubilius inequality

by

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In celebration of the seventy-fifth birthday of P. Erdős

A complex-valued function $f(n)$, defined on the positive integers, is *additive* if it satisfies $f(ab) = f(a)+f(b)$ for every pair of mutually prime integers a, b. A standard form of the *Turán–Kubilius inequality* asserts that for $x \geqslant 2$

$$x^{-1} \sum_{n \leqslant x} |f(n) - x^{-1} \sum_{m \leqslant x} f(m)|^2 \ll \sum_{q \leqslant x} |f(q)|^2 q^{-1}, \tag{1}$$

the final sum being taken over prime-powers q and the implied constant absolute.

For integral x the sum $V(f)$ estimated here may be viewed as a variance. To a certain extent it may be modelled using sums of independent random variables (e.g. Kubilius [7], Elliott [1]), so that for a wide class of additive functions the upper bound in (1) is asymptotically best possible. However, when $f(n) = \log n$ the upper bound sum is $\gg (\log x)^2$, whereas $V(\log) \ll 1$, which is much smaller. Indeed, after an application of the Cauchy–Schwarz inequality we obtain

$$V(f) \ll \lambda^2 + \sum_{q \leqslant x} |f(q) - \lambda \log q|^2 q^{-1}, \tag{2}$$

valid for all λ. That this is of an appropriate form was shown by Ruzsa [8], who proved that with a suitable choice of λ there is an inequality of this type going in the other direction. His argument combined an elaboration of the method of Halász [5], with ideas from the Theory of Probability. A possible value λ_0 for λ may be readily computed:

$$\lambda_0 \Big(1 + \sum_{q \leqslant x} (\log q)^2 q^{-1}\Big) = \sum_{q \leqslant x} q^{-1} f(q) \log q.$$

We may reappraise the inequality (2) by considering the complex space $\boldsymbol{C}^s$ of tuples $(f(2), f(3), \ldots)$, with one coordinate for each of the s prime-powers not exceeding x, and introducing the norm

$$\|y\|_0 = |\lambda_0| + \Big(\sum_{q \leqslant x} q^{-1} |f(q) - \lambda_0 \log q|^2\Big)^{1/2}.$$

Furthermore, let $C^{[x]}$ denote the standard $[x]$-dimensional space derived from the Cartesian product of copies of C, with the norm

$$\|z\| = \Big([x]^{-1} \sum_{n \leqslant x} |z_n|^2\Big)^{1/2}.$$

Then the inequality (2) with the companion obtained by Ruzsa, assert that for certain positive constants c_1, c_2, the operator $A\colon C^s \to C^{[x]}$ given by

$$(Af)(n) = \sum_{q \| n} f(q) - \sum_{q \leqslant x} q^{-1} f(q)$$

satisfies

$$c_1 \|f\|_0 \leqslant \|Af\| \leqslant c_2 \|f\|_0.$$

A is an approximate isometry.

A functional-analytic source for the logarithm can be given. Let q_0 denote the prime of which q is a power, and introduce on C^s the inner-product

$$(f, g) = \sum_{q \leqslant x} \frac{f(q)\overline{g(q)}}{q}\Big(1 - \frac{1}{q_0}\Big)$$

with its corresponding norm

$$\|f\| = \Big(\sum_{q \leqslant x} \frac{|f(q)|^2}{q}\Big(1 - \frac{1}{q_0}\Big)\Big)^{1/2}.$$

Then $\psi(q) = \|L^{-1}\log\|^{-1} L^{-1}\log q$, with $L = \log x$, is an approximate eigenfunction of the operator $T\colon C^s \to C^s$ given by

$$f(q) \mapsto \sum_{x/q < l \leqslant x} \frac{f(l)}{l}\Big(1 - \frac{1}{l_0}\Big),$$

where l runs through prime-powers. The operator T is self-adjoint with respect to this inner-product, and ψ has a corresponding approximate eigenvalue 1, which is close to the largest eigenvalue of T. A functional-analytic derivation of Ruzsa's result is sketched in Elliott [2], and a detailed justification of the above assertion is given in Elliott [4]. Whilst the scalar factor L may be removed from ψ, its presence reminds that underlying this functional-analytic point of view is the isomorphism $u \mapsto (\log u)/\log x$ between the multiplicative group of positive reals, and the additive group of all reals. In some sense this study takes place on the former of these two groups. This notation is also consistent with that of Elliott [4], save that there ψ is denoted by $\hat{\psi}_1$.

What form should an analogue of the Turán–Kubilius inequality take if we restrict the integers n in $V(f)$ to lie in an interval $x-y < n \leqslant x$? Ideally it should be an approximate isometry.

Define

$$\Delta(f) = y^{-1} \sum_{x-y<n\leqslant x} \Big| f(n) - \frac{1}{y} \sum_{x-y<t\leqslant x} f(t)\Big|^2.$$

For integral y this is once again a variance, and with the weights y^{-1} replaced by $([x]-[x-y])^{-1}$ this will be true whether y is integral or not. The following results remain true with this modified definition of $\Delta(f)$, but for notational simplicity I shall retain the use of the y^{-1}, and assume y to be an integer.

THEOREM 1. *The inequality*

$$\Delta(f) \ll |\lambda|^2 \Delta(\log) + \sum_{q \leqslant y} \frac{|g(q)|^2}{q} + \frac{1}{y} \sum_{y<q\leqslant x}{}' \Big| g(q) - \frac{1}{y} \sum_{y<q\leqslant x}{}' g(q)\Big|^2 + \frac{1}{y^2 \log y} \Big| \sum_{y<q\leqslant x}{}' g(q)\Big|^2$$

with $g(q) = f(q) - \lambda \log q$, holds for all additive functions f, for all λ, and for all $\sqrt{2} \leqslant x^{1/2} < y \leqslant x$. Here $'$ indicates that summation is confined to those prime-powers which exactly-divide at least one integer in the interval $(x-y, x]$.

If $q \leqslant y$, then some integer in the interval $(x-y, x]$ will be at least divisible by q, but for $q > y$ that need not happen. In contrast to the situation for the standard form of the Turán–Kubilius inequality ($y = x$), the values $f(n)$, $x-y < n \leqslant x$, need not completely determine f on the whole interval $[1, x]$.

It is convenient to define $f(q)$ to be zero when q does not exactly divide an integer in the interval $(x-y, x]$.

THEOREM 2. *Let c be a positive constant. In the notation of Theorem 1*

$$\Delta(f) \ll |\lambda|^2 \Delta(\log) + \sum_{q \leqslant y} \frac{|g(q)|^2}{q} + \frac{1}{y} \sum_{y<q\leqslant x}{}' |g(q)|^2,$$

provided $y > x^c$, $x \geqslant 2$.

That this gives an appropriate generalisation of the Turán–Kubilius inequality is shown by

THEOREM 3. *With $F = (f, \psi)$ the inequality*

$$|F|^2 \Delta(\psi) + \sum_{q \leqslant y} \frac{1}{q}\Big(1 - \frac{1}{q_0}\Big)|f(q) - F\psi(q)|^2 + \frac{1}{y} \sum_{y<q\leqslant x}{}' |f(q) - F\psi(q)|^2 \ll \Delta(f)$$

holds uniformly for all additive functions f, $x(\log x)^{-1/8} \log\log x \leqslant y < x$, and x absolutely large.

For these ranges of x, y, Theorem 3 is a companion to Theorem 2 with $\lambda = F(\|\psi\|L)^{-1}$. I have expressed this theorem in the language of the space C^s since that is how it naturally arises. No doubt a version of Theorem 3 appropriate to Theorem 1 exists with much weaker constraints upon the size of y, perhaps even those required in Theorem 1. However, with the present formulation I can appeal to results, of a functional analytic nature, already available.

The formulation and proof of these theorems was an exercise in the philosophy that: if an operator (or argument) shows that certain conditions force a statement to be valid, then its dual may be used in the investigation of their necessity. Such a methodology I applied to arithmetic functions in 1972 [3], and it forms part of my recent book [2] on arithmetic functions.

We begin with a version of the Turán–Kubilius inequality. For $z \geqslant 0$ let

$$M(z) = \sum_{q \leqslant z} \frac{f(q)}{q}\left(1-\frac{1}{q_0}\right).$$

The inequality

$$\frac{1}{y}\sum_{x-y<n\leqslant x}\Big|\sum_{\substack{q\| n\\ q\leqslant y^{1/2}}} f(q)-M(y^{1/2})\Big|^2 \ll \sum_{q\leqslant y^{1/2}}\frac{|f(q)|^2}{q}$$

may be obtained in the classical way of Turán, as developed by Kubilius [7]; by carrying out the squaring and inverting the order of summation. If $y > x^{1/2}$, then each integer n, not exceeding x, can have at most 3 exact divisors $q > y^{1/2}$. Applications of the Cauchy–Schwarz inequality enable us to extend to

$$(3)\qquad \frac{1}{y}\sum_{x-y<n\leqslant x}\Big|\sum_{\substack{q\| n\\ q\leqslant y}} f(q)-M(y)\Big|^2 \ll \sum_{q\leqslant y}\frac{|f(q)|^2}{q}.$$

Proof of Theorem 1. By the Cauchy–Schwarz inequality

$$\Delta(f) \leqslant 2|\lambda|^2\Delta(\log)+2\Delta(f-\lambda\log),$$

so that it will be enough to establish the theorem for the case $\lambda = 0$.

We note that

$$\alpha = y^{-1}\sum_{x-y<n\leqslant x} f(n) = \frac{1}{y}\sum_{q\leqslant x} f(q)\left(\left[\frac{x}{q}\right]-\left[\frac{x-y}{q}\right]-\left[\frac{x}{qq_0}\right]+\left[\frac{x-y}{qq_0}\right]\right)$$

$$= \sum_{q\leqslant y}\frac{f(q)}{q}\left(1-\frac{1}{q_0}\right)+O\left(\frac{1}{y}\sum_{q\leqslant y}|f(q)|\right)+\frac{1}{y}\sum_{y<q\leqslant x}{}' f(q)$$

since at most one integer in the interval $(x-y, x]$ can be exactly divisible by a $q > y$. Let the last of these sums be denoted by B.

Writing $f(n)$ in the form

$$\sum_{\substack{q\| n\\ q\leqslant y}} f(q)+\sum_{\substack{q\| n\\ q> y}}{}' f(q)$$

applying the Cauchy–Schwarz inequality and then (3), we see that

$$(4)\qquad \Delta(f) \ll \sum_{q\leqslant y}\frac{|f(q)|^2}{q}+\frac{1}{y}\sum_{x-y<n\leqslant x}\Big|\sum_{\substack{q\| n\\ q>y}}{}' f(q)-\frac{B}{y}\Big|^2+\frac{1}{y^2}\Big(\sum_{q\leqslant y}|f(q)|\Big)^2.$$

Since

$$\Big(\sum_{q\leqslant y}|f(q)|\Big)^2 \leqslant \sum_{q\leqslant y}\frac{|f(q)|^2}{q}\sum_{q\leqslant y} q$$

and this last sum is $\ll y^2/\log y$, the third of the terms in the upper bound at (4) may be omitted in favour of the first. Towards the second term in that upper bound, those integers n which are exactly-divisible by some $q > y$ contribute at most

$$\sum_{y<q\leqslant x}{}' \Big|f(q)-\frac{B}{y}\Big|^2.$$

The remaining integers give $\leqslant y^{-1}w|B|^2$, where w denotes the number of integers in the interval $(x-y, x]$ which are not exactly-divisible by any $q > y$. We can estimate this number by

$$w \leqslant \sum_{y<p^2\leqslant x}\ \sum_{\substack{x-y<n\leqslant x\\ n\equiv 0 (\mathrm{mod}\ p^2)}} 1+\sum_{x-y<p\leqslant x} 1$$

where p denotes a prime; since any integer not exceeding x which is not divisible by any prime $> y$ $(> x^{1/2})$ must be a prime. That the number of primes in an interval $(x-y, x]$, with $y \geqslant 2$, is $O(y/\log y)$ goes back to Hardy and Littlewood, and may be obtained by a sieve method (e.g. Halberstam and Richert [6]). Moreover,

$$\sum_{y^{1/2}<p\leqslant x^{1/2}}\left(\left[\frac{x}{p^2}\right]-\left[\frac{x-y}{p^2}\right]\right) \ll y\sum_{p>y^{1/2}}\frac{1}{p^2}+\frac{x^{1/2}}{\log x}$$

so that altogether $w \ll y/\log y$.

The proof of Theorem 1 is complete.

Theorem 2 may be proved in a similar manner provided we note that

$$cL\sum_{y<q\leqslant x}{}' 1 \leqslant \sum_{y<q\leqslant x}{}' \log q \leqslant \sum_{x-y<n\leqslant x}\log n \leqslant yL.$$

For $c \geqslant 1/2$ it may be deduced from Theorem 1 with applications of the Cauchy–Schwarz inequality.

Proof of Theorem 3. This is the more interesting. We dualise the norm inequality (3):

$$\sum_{q\leqslant y} q\left|\frac{1}{y}\sum_{\substack{x-y<n\leqslant x\\ q\| n}} a_n-\frac{1}{q}\left(1-\frac{1}{q_0}\right)\frac{1}{y}\sum_{x-y<n\leqslant x} a_n\right|^2 \ll \frac{1}{y}\sum_{x-y<n\leqslant x}|a_n|^2,$$

valid for all complex a_n. This inequality remains valid if we remove the factor $1-q_0^{-1}$, and introduce the condition $(n, q) = 1$ into the sum attached to it. We then set $a_n = f(n)-\alpha$, and note that the terms involving α contribute to the multiple sum an amount $\ll |\alpha|^2/\log y$. Altogether

$$\sum_{q\leqslant y} q \left|\frac{1}{y}\sum_{\substack{x-y<n\leqslant x\\ q\| n}} f(n) - \frac{1}{qy}\sum_{\substack{x-y<n\leqslant x\\ (n,q)=1}} f(n)\right|^2 \ll \Delta(f)+\frac{|\alpha|^2}{\log y}. \tag{5}$$

It is convenient to define

$$\sigma_q = \frac{1}{y}\left(\left[\frac{x}{q}\right]-\left[\frac{x-y}{q}\right]-\left[\frac{x}{qq_0}\right]+\left[\frac{x-y}{qq_0}\right]\right),$$

a function which satisfies

$$\frac{1}{2}\leqslant \sigma_q\left(\frac{1}{q}\left(1-\frac{1}{q_0}\right)\right)^{-1}\leqslant\frac{3}{2}$$

uniformly for $2\leqslant q\leqslant y/8$. For this range of q the first inner sum of (5) has the alternative representation

$$\sigma_q f(q)+\frac{1}{y}\sum_{\substack{(x-y)/q<m\leqslant x/q\\ (m,q)=1}} f(m).$$

Let H be the space of complex vectors, one coordinate for each prime-power q which exactly divides some integer in the interval $(x-y, x]$. To investigate the implications of the inequality (5) a study of the operator $H \to H$ given by

$$f(q)\mapsto \frac{1}{y\sigma_q}\left(\sum_{\substack{(x-y)/q<m\leqslant x/q\\ (m,q)=1}} f(m)-\frac{1}{q}\sum_{\substack{x-y<n\leqslant x\\ (n,q)=1}} f(n)\right) \tag{6}$$

is appropriate. Moreover, it is better to keep the terms involving α inside the square at (5), and integrate them into the treatment. However, to reduce details I relate this operator to the operator T introduced earlier, requiring that y not be too small.

The sums at (6) may be treated in the earlier manner of α, and given the alternative representation

$$\sum_{\substack{l\leqslant x/q\\ (l,q)=1}} \frac{f(l)}{l}\left(1-\frac{1}{l_0}\right)-\sum_{\substack{l\leqslant y\\ (l,q)=1}}\frac{f(l)}{l}\left(1-\frac{1}{l_0}\right)+E_q,$$

where

$$E_q \ll \frac{1}{y}\sum_{l\leqslant y}|f(l)|+\frac{1}{y}\sum_{y<q\leqslant x}{}' |f(l)|+\frac{q}{y}\sum_{l\leqslant x/q}|f(l)|$$

uniformly for $q\leqslant y/8$. In particular

$$\sum_{q\leqslant y/8}\frac{1}{q}|E_q|^2 \ll \left(\frac{\log L}{L}+\frac{x^2\log L}{y^2\sqrt{L}}\right)\left(\sum_{l\leqslant y}\frac{|f(l)|^2}{l}+\frac{1}{y}\sum_{y<l\leqslant x}{}'|f(l)|^2\right). \tag{7}$$

For example, by the Cauchy–Schwarz inequality

$$\sum_{q\leqslant y}\frac{1}{q}\left(\frac{q}{y}\sum_{l\leqslant x/q}|f(l)|\right)^2 \ll \sum_{q\leqslant y}\frac{1}{q}\cdot\frac{x^2}{y^2\max(1,\log x/q)}\sum_{l\leqslant x/q}\frac{|f(l)|^2}{l}$$
$$\ll \frac{x^2}{y^2}\sum_{l\leqslant x}\frac{|f(l)|^2}{l}\left(\sum_{q\leqslant xe^{-\sqrt{L}}}\frac{1}{q\sqrt{L}}+\sum_{xe^{-\sqrt{L}}<q\leqslant x}\frac{1}{q}\right),$$

to which we may apply well-known elementary estimates. For small values of q this argument is wasteful.

Let

$$\delta = \Delta(f)+(\log L)^{-1}\left(\sum_{q\leqslant y/8}\frac{|f(q)|^2}{q}+\frac{1}{y}\sum_{y/8<q\leqslant x}{}'|f(q)|^2\right).$$

Noting that under the hypotheses of Theorem 3 $|\alpha|^2L^{-1}\ll\delta$,

$$\sum_{q\leqslant y}\frac{1}{q}\left(\sum_{\substack{x/q<l\leqslant y\\ (l,q)>1}}\frac{|f(l)|}{l}\right)^2 \ll \sum_{l\leqslant y}\frac{|f(l)|^2}{l}\sum_{q\leqslant y}\frac{1}{q}\frac{q}{x}\ll\delta$$

and

$$\sum_{q\leqslant y}\frac{1}{q}\left(\sum_{y<l\leqslant x}{}'\frac{|f(l)|}{l}\right)^2 \ll \sum_{y<l\leqslant x}{}'|f(l)|^2\sum_{q\leqslant y}\frac{1}{q}\frac{x}{y^2L}\ll\delta,$$

we derive from (5) the partial-norm inequality

$$\sum_{q\leqslant y/8}\frac{1}{q}|f(q)-(Tf)_q|^2\ll\delta. \tag{8}$$

Define the operator S: $C^s\to C^s$ by

$$f(q)\mapsto(Tf)_q-(f,\psi)\psi(q),$$

largely removing the effect of T upon its first eigenspace. As $x\to\infty$ the spectral radius of the operator S approaches 1/2, and the resolvent operator $(I-S)^{-1}$ is bounded. This may be obtained using the argument on pages 433, 434 of the Supplement in Elliott [2]. A detailed proof of this result is given in Elliott [4].

The inequality (8) only gives information concerning Tf on the prime-powers $q\leqslant y/8$, although the definition of T involves values of f on the prime-powers all the way up to x. It is therefore convenient to employ an analogue of T defined on a slightly smaller space.

Let t denote the number of prime-powers not exceeding $y/8$. Let T_0 be the operator $C^t\to C^t$ which is defined like T, but with $y/8$ in place of x.

Corresponding to S there will be an operator S_0, with a function ψ_0 corresponding to ψ.

Applications of the Cauchy–Schwarz inequality, and of the estimate

$$\sum_{y/q<l\leqslant x/q} \frac{1}{l} \ll \frac{\log L}{\sqrt{L}}$$

which is valid uniformly for $q \leqslant xe^{-\sqrt{L}}$, allow us to write (8) in the form

$$\|(I-T_0)f\|^2 \ll \delta,$$

in the norm on C^t. Then

$$\|f-F_0(I-S_0)^{-1}\psi_0\|^2 \ll \delta$$

with $F_0 = (f, \psi_0)$. Moreover

$$\|S_0\psi_0\| = \|T_0\psi_0-\psi_0\| \ll L^{-1}$$

may be checked directly. In view of the operator relation

$$(I-S)^{-1} = I+(I-S)^{-1}S$$

we have

$$\|f-F_0\psi_0\|^2 \ll |F_0|^2 L^{-1}+\delta \ll \delta,$$

this last by applying the Cauchy–Schwarz inequality to the inner-product F_0. Since ψ, ψ_0 are each the logarithm function, rescaled to have norm 1 in C^s, C^t respectively, it is readily checked that these final inequalities remain valid with F_0, ψ_0 replaced by F, and ψ suitably restricted. Thus we reach

$$\sum_{q\leqslant y/8} \frac{1}{q}|f(q)-F\psi(q)|^2 \ll \delta. \tag{9}$$

Replacing f in the inequality (9) by $f-F\psi$, and taking note of the form of δ, gives

$$\sum_{q\leqslant y/8} \frac{1}{q}|f(q)-F\psi(q)|^2 \ll \Delta(f-F\psi)+\frac{1}{y\log L}\sum_{y<q\leqslant x}' |f(q)-F\psi(q)|^2. \tag{10}$$

We denote $f-F\psi$ by g, consistent with the notation of Theorem 1 when $\lambda = F(\|\psi\|\,L)^{-1}$. We may apply this together with the Turán–Kubilius variant (3) to the estimation of $\Delta(g)$ itself, and so obtain

$$\frac{1}{y}\sum_{x-y<n\leqslant x}\Big|\sum_{\substack{q\|n\\ q>y/8}} g(q)\Big|^2 \ll \Delta(g)+\frac{1}{y\log L}\sum_{y<q\leqslant x}' |g(q)|^2.$$

Thus

$$\frac{1}{y}\sum_{y/8<q\leqslant x}' |f(q)-F\psi(q)|^2 \ll \Delta(f-F\psi) \tag{11}$$

also holds.

We next investigate the function $\Delta(\psi)$.

Lemma 1. *The estimate*

$$\Delta(f+\varrho\psi) = \Delta(f)+|\varrho|^2\Delta(\psi)+O\left(|\varrho|\,\|f\|\frac{x^2(\log L)^3}{y^2L^{3/2}}\right)$$

holds uniformly for $2 \leqslant y < x$.

Proof. We need only consider the cross-term

$$Y = \frac{\varrho}{y}\sum_{x-y<n\leqslant x}\Big(f(n)-\frac{1}{y}\sum_{x-y<t\leqslant x} f(t)\Big)\Big(\psi(n)-\frac{1}{y}\sum_{x-y<t\leqslant x}\psi(t)\Big).$$

Since y is an integer (and only in this lemma do we use that fact), we may replace the average of ψ over the interval $(x-y, x]$ by its average

$$\beta(x) = \frac{1}{x}\sum_{t\leqslant x}\psi(t)$$

over the whole interval $[1, x]$. This enables us to take advantage of the good distribution of the logarithm function in residue classes. Direct calculation shows that for $z \geqslant 2$

$$\sum_{q\leqslant z} q\,\Big|\sum_{\substack{n\leqslant z\\ q\|n}}\Big(\log n-\frac{1}{z}\sum_{t\leqslant z}\log t\Big)\Big|^2 \ll \frac{(z\log\log z)^2}{\log z}.$$

We represent Y in the form

$$\frac{\varrho}{y}\sum_{n\leqslant x}\Big(f(n)-\frac{1}{y}\sum_{x-y<t\leqslant x} f(t)\Big)(\psi(n)-\beta(x))$$
$$-\frac{\varrho}{y}\sum_{n\leqslant x-y}\Big(f(n)-\frac{1}{y}\sum_{x-y<t\leqslant x} f(t)\Big)(\psi(n)-\beta(x-y))$$
$$+\frac{\varrho}{y}(\beta(x)-\beta(x-y))\sum_{n\leqslant x-y}\Big(f(n)-\frac{1}{y}\sum_{x-y<t\leqslant x} f(t)\Big).$$

In the first of these three sums we may omit the inner average over t and, representing f in terms of its values on prime-powers, invert the order of summation

$$\frac{\varrho}{y}\sum_{q\leqslant x} f(q)\sum_{\substack{n\leqslant x\\ q\|n}}(\psi(n)-\beta(x)).$$

Applying the Cauchy–Schwarz inequality, and the bound (10) with $z = x$, shows this expression to be $\ll |\varrho|\,\|f\|\,x\log L(yL^{3/2})^{-1}$.

The second sum may be similarly treated, with $\log(x-y)$ in place of L, giving a satisfactory bound if $x-y > x^{1/2}$. Otherwise, we argue crudely with

$$\sum_{t\leqslant x}|f(t)| \leqslant \sum_{t\leqslant x}\sum_{q\| t}|f(q)| \ll x\|f\|\sqrt{\log L},$$

and estimate it to be

$$\ll \frac{|\varrho|}{y}(x-y)\frac{x}{y}\|f\|L^{1/2},$$

which is also satisfactory.

To estimate the third and final sum representing Y, we note that from our earlier representation of α

$$\frac{1}{y}\sum_{n\leqslant x}\left(f(n)-\frac{1}{y}\sum_{x-y<t\leqslant x}f(t)\right) \ll \frac{x^2\|f\|}{y^2\sqrt{L}},$$

whilst for $x-y\geqslant 2$

$$\beta(x)-\beta(x-y) \ll \left(1-\log\left(1-\frac{y}{x}\right)\right)L^{-1}.$$

If $x-y > x\exp(-(\log L)^2)$ then altogether

$$Y \ll |\varrho|\,\|f\|\,x^2(\log L)^3(y^2L^{3/2})^{-1}.$$

Otherwise we give the third sum the same crude treatment that we gave the second, and obtain for it the bound

$$\ll \frac{|\varrho|}{y}|\beta(x)-\beta(x-y)|(x-y)\frac{x}{y}\|f\|\sqrt{L},$$

from which a sharper upper bound for Y follows.

In order to usefully apply this lemma we need a lower bound for $\Delta(\psi)$.

LEMMA 2. *We have*

$$y^2x^{-2} \ll \Delta(\log) \ll 1$$

provided $y^2(x\log x)^{-1}$ *and* x *are sufficiently large.*

Proof. Replacing the sum by an integral shows that

$$\log A = \frac{1}{y}\sum_{x-y<t\leqslant x}\log t = -\frac{x}{y}\log\left(1-\frac{y}{x}\right)+\log(x-y)-1+O\left(\frac{L}{y}\right).$$

Since $-\log(1-\theta)\leqslant\theta(1-\theta)^{-1}$ for $0\leqslant\theta<1$,

$$0\leqslant -\left(\frac{x}{y}-1\right)\log\left(1-\frac{y}{x}\right)\leqslant 1,$$

and $\log A = \log x + O(1)$ uniformly for $y\geqslant L$. To obtain an upper bound for $\Delta(\log)$ we note that

$$\frac{1}{y}\sum_{x-y<n\leqslant x}\left(\log\frac{n}{x}\right)^2 \leqslant \frac{1}{y}\int_{x-y}^{x}\left(\log\frac{t}{x}\right)^2 dt + O\left(\frac{L^2}{y}\right),$$

since the integrand decreases over the range $1\leqslant t\leqslant x$. The change of variable $t = x/u$ shows that the integral has the alternative representation

$$\frac{x}{y}\int_{1}^{(1-y/x)^{-1}}\frac{(\log u)^2}{u^2}du,$$

which, considering the cases $y > x/2$, $y\leqslant x/2$ separately, is seen to be bounded. The given upper bound for $\Delta(\log)$ is actually uniform for $L^2\leqslant y < x$.

On the other hand

$$-\frac{x}{y}\log\left(1-\frac{y}{x}\right)\geqslant -\frac{x}{y}\left(\frac{y}{x}+\frac{y^2}{2x^2}\right),$$

so that for $y^2(xL)^{-1}$ sufficiently large

$$x\geqslant A\geqslant (x-y)\exp\left(\frac{y}{4x}\right).$$

Thus

$$\Delta(\log)\geqslant\frac{1}{y}\sum_{x-y<n\leqslant A\exp(-y/8x)}\left(\log\frac{A}{n}\right)^2$$
$$\geqslant\frac{1}{y}\left(\frac{y}{8x}\right)^2\left(\left(\exp\left(\frac{y}{8x}\right)-1\right)(x-y)-1\right)$$

which for $y\geqslant x/2$ and y sufficiently large is at least $y^2(32x)^{-2}$. However, if $y\geqslant x/2$, then

$$\Delta(\log)\geqslant\frac{1}{y}\int_{x-y}^{x}\left(\log\frac{t}{A}\right)^2 dt + O\left(\frac{L^2}{y}\right).$$

Here the integral is larger than

$$\frac{A}{y}\left(\int_{3x/4A}^{x/A}+\int_{x/2A}^{5x/8A}(\log u)^2\,du\right).$$

If $x/A > 3/2$, then $\log u > 9/8$ over the first interval of integration, otherwise $-\log u > 16/15$ over the second interval. In either case we conclude that $\Delta(\log)\gg 1$.

To complete the proof of Theorem 3 we apply Lemma 2 with f replaced by $f-F\psi$, and ϱ by F. In view of Lemma 2, our hypothesis on the size of y, (10) and (11):

$$\Delta(f) = \left(1+O\left(\frac{1}{\log L}\right)\right)\left(\Delta(f-F\psi)+|F|^2\Delta(\psi)\right).$$

Note that only here do we use the full restriction on y. Up until this point the weaker $y \gg xL^{-1/4}(\log L)^2$ would have sufficed.

Concluding remarks. Introduce the norm

$$\Big(\sum_{q\leqslant y}{}' q^{-1}|x_q|^2\Big)^{1/2}+\Big(y^{-1}\sum_{y<q\leqslant x}{}' |x_q|^2\Big)^{1/2}$$

onto the space H. Let K be the space of complex vectors z_n, one for each integer in the interval $x-y<n\leqslant x$, with norm

$$\Big(\frac{1}{y}\sum_{x-y<n\leqslant x}|z_n|^2\Big)^{1/2}.$$

According to Theorem 2, the map A: $H\to K$ which takes $(f(q))$ to

$$\Big(f(n)-\sum_{q\leqslant x}{}' \frac{f(q)}{q}\Big(1-\frac{1}{q_0}\Big)\Big)$$

has a norm which is bounded independently of x, uniformly for $x^c\leqslant y<x$, any fixed $c>0$.

Let N be the norm of a typical functional $\boldsymbol{x}\mapsto(\boldsymbol{x}, \boldsymbol{b})$ in the dual space H', where the inner-product is the usual Euclidean one on H. Then

$$N\leqslant\Big(\sum_{q\leqslant y}{}' q|b_q|^2\Big)^{1/2}+\Big(y\sum_{y<q\leqslant x}{}' |b_q|^2\Big)^{1/2}\leqslant 2N.$$

The dual space of K may be represented by functionals $\boldsymbol{z}\mapsto(\boldsymbol{z}, \boldsymbol{t})$, with associated norm

$$\Big(y\sum_{x-y<n\leqslant x}|t_n|^2\Big)^{1/2}.$$

Since H, K are Hilbert spaces, both the dual A' and the adjoint A^* of the operator A are well defined, and have the same bounded norm as A. In particular the inequality

$$\sum_{q\leqslant y}{}' q\Big|\sum_{\substack{x-y<n\leqslant x\\ q\| n}} a_n-\frac{1}{q}\Big(1-\frac{1}{q_0}\Big)\sum_{x-y<n\leqslant x}a_n\Big|^2$$
$$+y\sum_{y<q\leqslant x}{}'\Big|\sum_{\substack{x-y<n\leqslant x\\ q\| n}} a_n-\frac{1}{q}\Big(1-\frac{1}{q_0}\Big)\sum_{x-y<n\leqslant x}a_n\Big|^2\ll y\sum_{x-y<n\leqslant x}|a_n|^2$$

holds for all complex a_n, $x^c\leqslant y<x$, $x\geqslant 2$.

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