A note on a result of Erdős, Sárközy and Sós

by

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1. Introduction. Let $A$ be an infinite sequence of positive integers $a_1 < a_2 < a_3 < \ldots$; Let

$$R(n) = \sum_{a_i + a_j = n} 1; \quad R_1(n) = \sum_{a_i + a_j = n, \ i < j} 1; \quad R_2(n) = \sum_{a_i + a_j = n, \ i \neq j} 1; \quad A(N) = \sum_{a \in A} 1.$$

Then the following three theorems are due to Erdős, Sárközy and Sós [1] (our notation is different from their notation; they use $R_1$, $R_2$ and $R_3$ instead of our $R$, $R_1$ and $R_2$).

**Theorem 1.** If $R(n+1) \gg R(n)$ for all large $n$, then $A(N) = N + O(1)$.

**Theorem 2.** If $R_1(n+1) \gg R_1(n)$ for all large $n$, then $A(N) = \Omega(N \log N)$ (and a similar result for $R_2(n)$).

**Theorem 3.** For a suitable sequence $A$, we have, $R_1(n+1) \gg R_1(n)$ for all large $n$ and $A(N) \ll N - cN^{1/3}$ for infinitely many $N$ and for a suitable $c > 0$.

The authors in [1] state that probably a theorem analogous to Theorem 3 holds for $R_2(n)$ also.

In this note, we give (what we believe) a simple proof of Theorem 1; slightly improve Theorem 2 and disprove their conjecture (?) about a result analogous to Theorem 3 for $R_2(n)$. More precisely we prove

**Theorem 4.** If $R_2(n+1) \gg R_2(n)$ for all large $n$, then

$$A(N) = N + O(\log N).$$

**Theorem 5.** If $R_1(n+1) \gg R_1(n)$ for all large $n$, then

$$\sum_{a \in A} e^{-aN} \gg N/\log N.$$

2. Notation. In the sequel, $a, a_i$ denote the generic elements of $A$;

$$f(z) = \left( \sum_{a \in A} z^a \right).$$
consequently

$$(f(z))^2 = \sum_{n=1}^{\infty} R(n)z^n;$$

Let $\alpha$ be a real number, $0 < \alpha < 1$, and $\alpha$ is close to 1;

$$g(z) = f(z)(1 - z) = \sum_{n=1}^{\infty} b_n z^n;$$

$$r_n = \begin{cases} 1 & \text{if } n = 2a_k \\
0 & \text{otherwise}; \end{cases}$$

$$\delta_n = \begin{cases} 1 & \text{if } n = 2a_k \\
0 & \text{otherwise}; \end{cases}$$

$$\lambda_n = R(n+1) - R(n) - r_n.$$

We write $A(z) \equiv B(z)$ to denote that $A(z) - B(z)$ is a polynomial in $z$.

3. Proof of Theorem 1.

**Lemma 1.** (a) We have $R(n) \leq 2A((n+1)/2)$ for all $n$.

(b) We have $\sum r_n \geq 2A(m/2)$ for all $m$.

(c) $R(2a_k)$ is odd; $R(2a_k+1)$ and $R(2a_k-1)$ are even.

**Proof.** We have

$$R(n) = \sum_{a_i+a_j=n} 1 \leq 2 \sum_{a_i+a_j=n} 1 \leq 2 \sum_{a_i+n(n+1)/2} 1 = 2A \left( \frac{n+1}{2} \right)$$

and this proves 1 (a); Lemma 1 (b) follows from the definition of $r_n$; to prove 1 (c), note that all the solutions $(a_i, a_j)$ of $a_i + a_j = n$ can be paired as $(a_i, a_j)$ and $(a_j, a_i)$ except when $n$ is of the form $2a_k$, in which case, we have one extra solution $(a_k, a_k)$.

**Lemma 2.** We have $\lambda_n \geq 0$ for all large $n$.

**Proof.** This follows from the monotonicity of $R(n)$ and Lemma 1 (c).

**Lemma 3.** We have $\sum \lambda_n$ is bounded above for all large $n$.

**Proof.** We have

$$\sum_{n=m}^{\infty} \lambda_n = \sum_{n=m}^{\infty} (R(n+1) - R(n) - r_n) = R(m+1) - R(1) - \sum_{n=m}^{\infty} r_n$$

and the lemma follows from Lemmas 1 (a) and (b).

**Lemma 4.** We have $\lambda_n = 0$ for large $n$.

**Proof.** This follows from Lemmas 2 and 3.

**Lemma 5.** We have $(f(z))^2(1-z) = (1+z)f(z^2) + p(z)$ for a suitable polynomial $p(z)$.

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**Proof.** From Lemma 4, we have $\sum \lambda_n z^{n+1} \equiv 0$. Hence

$$\sum (R(n+1) - R(n) - \epsilon_n) z^{n+1} \equiv 0.$$ 

Hence

$$(\sum R(n)z^n)(1-z) \equiv \sum \epsilon_n z^{n+1}.$$ 

Hence

$$(f(z))^2(1-z) \equiv \sum (z^{2a_k} + z^{2a_k+1}) \equiv (1+z) \sum z^{2a_k} \equiv (1+z) f(z^2).$$

**Lemma 6.** We have $(g(z))^2 = g(z^2) + p(z)(1-z)$.

**Proof.** This follows from Lemma 5 and the definition of $g(z)$.

**Lemma 7.** We have $b_n = 0$ for all large $n$.

**Proof.** From Lemma 6, we have

$$\int_{|z|=n} \frac{|g(z)|^2 dz}{|z|^2} \leq \int_{|z|=n} \frac{|g(z^2)| dz + O(1)}{z}$$

$$\leq \left( \int_{|z|=n} \frac{|g(z^2)|^2 dz}{z} \right)^{1/2} \left( \int_{|z|=n} |dz| \right)^{1/2} + O(1).$$

Hence, allowing $z \to 1$, we have

$$\sum_{n=1}^{\infty} |b_n|^2 \leq \left( \sum |b_n|^2 \right)^{1/2} + O(1).$$

Consequently $\sum |b_n|^2 = O(1)$. Hence the lemma.

Now $b_n = \chi(n) - \chi(n-1)$ where $\chi$ is the characteristic function of the sequence $A$. Since $A$ is an infinite sequence, $\chi(n) = 1$ for infinitely many $n$ and hence for large $n$. Hence Theorem 1 follows.


**Lemma 8.** We have

$$f(x^2) \leq \frac{1-x}{2x} (f(z))^2 + O(1).$$

**Proof.** We put $z = -x$ in the relation $(f(z))^2 = \sum R(n)z^n$ to get

$$0 \leq (f(-z))^2 = \sum R(n)(-1)^n x^n = \sum (R(2n)x^{2n} - R(2n+1)x^{2n+1}).$$

This yields, using $R(n) = 2R_2(n) - \delta(n),

$$0 \leq 2(\sum (R_2(2n) - R_2(2n+1) + R_2(2n) - R_2(2n+1) + \sum R_2(n)(\delta(2n) - \delta(2n+1) + \Sigma_1 + \Sigma_2 + \Sigma_3)^2.$$ 

Since $R_2(2n) - R_2(2n+1) \leq 0$ for all large $n$, $\Sigma_1$ is bounded above. From the
definition of \( \delta(n) \) it follows that
\[
\Sigma_3 = -\sum a^{2n} = -(f(a^2)).
\]
Further
\[
\Sigma_2 = 2\sum R_2(2n)(a^{2n} - a^{2n-1}) = 2(1 - \alpha)\sum R_1(2n)a^{2n}
\leq (1 - \alpha)\sum (R_2(2n) + R_2(2n + 1))a^{2n} + O(1)
\leq \frac{1 - \alpha}{\alpha}\sum R_2(2n)a^{2n} + O(1)
= \frac{1 - \alpha}{\alpha}\sum R_2(n)a^n + O(1)
= \frac{1 - \alpha}{\alpha}(f(a))^2 + O(1)
\]
and the lemma follows.

**Lemma 9.** Define \( \Psi(N) = f(e^{-1/N}) \). Then
\[
(\Psi(N))^2 \geq 2N\Psi(N/2) + O(N).
\]

**Proof.** We put \( \lambda = e^{-1/N} \), note that \( f(a) = \Psi(N) \) and \( f(a^2) = \Psi(N/2) \).
Then
\[
\frac{1 - \lambda}{\lambda} = \frac{1}{N}(1 + O(1/N)).
\]
Hence the result follows from Lemma 8.

5. **Proof of Theorem 4 (continued).** In this section, we assume, as we may, that
\[
\Psi(N) \to \infty \quad \text{as} \quad N \to \infty.
\]

**Lemma 10.** We have
\[
\prod_{j=1}^{\lambda} \left( \frac{N - 1/2}{N} \right) \geq N^{1-1(2^j)}.
\]

**Proof.** The power of \( N \) in the left side is easily seen to be \( 1 - (1/2^j) \).
The power of 2 appearing in the denominator
\[
= \sum_{j=0}^{\lambda} \frac{j - 2}{2^j} = \frac{\lambda - 2}{2^\lambda} + 2
\]
\[
= \sum_{j=0}^{\lambda} (j + 1)(j/2 + 2) = (1 - \lambda)^{-2} - 3(1 - \lambda)^{-1} + 2 = 0.
\]

**Lemma 11.** We have
\[
(1 - d_1)^{d_2} \geq 1 - 2d_1 d_2 \quad \text{if} \quad 0 < d_1 < 1/2; \quad d_2 > 0.
\]

**Proof.** We have
\[
(-\log((1 - d_1)^{d_2})) = -d_2 \log(1 - d_1) = d_2 \left( d_1 + \frac{d_1^2}{2} + \frac{d_1^3}{3} + \ldots \right) 
\leq d_2 \left( d_1 + \frac{d_1^2}{2} + \frac{d_1^3}{3} + \ldots \right) \leq 2d_1 d_2.
\]
Hence \( (1 - d_1)^{d_2} \geq e^{-2d_1 d_2} \geq 1 - 2d_1 d_2 \).

**Lemma 12.** For a suitable constant \( c_1 > 0 \),
\[
(\Psi(N))^2 \geq N\Psi(N/2) \geq N \quad \text{for all} \quad N \geq c_1.
\]

**Proof.** Since \( \Psi(N) \to \infty \), both inequalities follow from Lemma 9.

**Lemma 13.** We have \( \Psi(N) \geq N \).

**Proof.** From Lemma 12,
\[
(\Psi(N))^2 \geq N^{1/2}(\Psi(N/2))^{1/2} \quad \text{if} \quad N \text{ is large.}
\]
\[
(\Psi(N/2))^2 \geq (N/2)^{1/2}(\Psi(N/4))^{1/2} \quad \text{if} \quad N \text{ is large.}
\]
Hence
\[
\Psi(N) \geq N^{1/2}(N/2)^{1/4}(\Psi(N/4))^{1/4} \quad \text{if} \quad N \text{ is large.}
\]
Proceeding similarly and choosing \( \lambda \) such that
\[
c_1 < N/2^{1/4} < 2c_1,
\]
we have
\[
\Psi(N) \geq N^{1/2}(N/2)^{1/4} \ldots (N/2^{1/4})(\Psi(N/2^{1/4}))^{1/2^{1/4} + 1}.
\]
Since \( \Psi(N/2^{1/4}) \geq 1 \) (Lemma 12), we are through.

**Lemma 14.** For a suitable constant \( c_2 \),
\[
(\Psi(N))^2 \geq 2N\Psi(N/2)(1 - c_3/N).
\]

**Proof.** This follows from Lemmas 9 and 13. In the sequel, we assume, as we may, that \( c_2 \geq c_1 \).

**Lemma 15.** We have
\[
\Psi(N) \geq N + O(\log N).
\]

**Proof.** From Lemma 14, we have
\[
\Psi(N) \geq (2N)^{1/2}(\Psi(N/2))^{1/2}(1 - c_3/N)^{1/2},
\]
\[
\Psi(N/2) \geq (N)^{1/2}(\Psi(N/4))^{1/2}(1 - 2c_3/N)^{1/2}.
\]
Hence
\[
\Psi(N) \geq (2N)^{1/2} N^{1/4} (\Psi(N/4))^{1/4} \left(1 - c_2/N\right)^{1/2} (1 - c_2/N)^{1/4}
\]
and using Lemma 11,
\[
\Psi(N) \geq (2N)^{1/2} N^{1/4} (\Psi(N/4))^{1/4} \left(1 - \frac{c_2}{4N}\right)^2.
\]
Proceeding similarly and choosing \(\lambda\) such that
\[
100 \leq \frac{N}{2^{9/4} c_2} \leq 200,
\]
we have
\[
\Psi(N) \geq (2N)^{1/2} N^{1/4} \left(\frac{N}{2}\right)^{1/4} \cdots \left(\frac{N}{2^{9/4-2}}\right)^{1/4} \left(\Psi\left(\frac{N}{2}\right)^{1/4} \left(1 - \frac{c_2}{4N}\right)^2.
\]
Now using \(\Psi(N/2^3) \geq 1\) and Lemma 10, we have
\[
\Psi(N) \geq N^{1/2} \left(1 - \frac{c_2}{4N}\right)^2
\]
and hence the lemma.

To prove Theorem 4, observe that, if \(\chi(n)\) is the characteristic function of the set \(A\), then from Lemma 15, we have
\[
\sum_{n=1}^{\infty} \chi(n) e^{-\pi n} \geq N + O(\log N).
\]
Hence
\[
\sum_{n=1}^{\infty} (1 - \chi(n)) e^{-\pi n} = O(\log N).
\]
Hence
\[
\sum_{n \in \mathbb{N}} (1 - \chi(n)) = O(\log N)
\]
and this proves Theorem 4.

6. Proof of Theorem 5. The proofs of Lemmas 16, 17, 18 and 19 are already contained in [1], and we give them for the sake of completeness.

**Lemma 16.** We have
\[
\left| \sum_{n=1}^{\infty} R_1(n) z^n \right| \leq \frac{(f(z))^2 (1 - \rho + O(1))}{|1 - z|} \text{ on the circle } |z| = \rho.
\]

**Proof.**
\[
\left| \sum_{n=1}^{\infty} R_1(n) z^n \right| = \left| \sum (R_1(n) - R_1(n - 1)) z^n \right|
\]
\[
\leq \left| \sum R_1(n) z^n \right| (1 - \rho) + O(1)
\]
\[
= \left| \sum R_1(n) z^n \right| (1 - \rho) + O(1)
\]
\[
= (f(z))^2 (1 - \rho) + O(1).
\]

**Lemma 17.** We have, on the circle \(|z| = \rho,
\[
|f(z)|^2 \leq 2 \frac{f(z)^2 (1 - \rho + O(1))}{|1 - z|} + |f(z)|^2.
\]

**Proof.** We observe that
\[
(f(z))^2 = \sum R(n) z^n = 2 \sum R_1(n) z^n + \sum \delta_n z^n = 2 \sum R_1(n) z^n + f(z^2)
\]
and the result follows from Lemma 16.

**Lemma 18.** We have
\[
f(z) \leq \left( \frac{(f(z))^2 (1 - \rho + O(1))}{|1 - z|} + \frac{f(z^2)^2}{|1 - z|^2} \right)^{1/2}.
\]

**Proof.** We integrate the relation in Lemma 17 on \(|z| = \rho\). We note that
\[
\int_{|z| = \rho} |f(z)|^2 \, dz \leq \sum_{n \in A} \alpha_n z^n = f(z^2),
\]
\[
\int_{|z| = \rho} \frac{dz}{1 - z} \leq \log \frac{1}{1 - \rho},
\]
\[
\int_{|z| = \rho} |f(z^2)|^2 \, dz \leq \left( \int_{|z| = \rho} |f(z^2)|^2 \, dz \right)^{1/2} \left( \int |dz| \right) \leq \left( \sum_{n \in A} |\alpha_n| \right)^{1/2}.
\]

**Lemma 19.** We have
\[
f(z^2) \leq \left( \frac{(f(z))^2 (1 - \rho + O(1))}{|1 - z|} \right)^{1/2} \text{ as } \rho \to 1.
\]

**Proof.** Since \(f(z^2) \leq f(z^2)\) and \(f(z^2) \to z\) as \(\rho \to 1\), the last term on the right of Lemma 17 could be dropped. Putting \(\rho = e^{-1/N}\), we have

**Lemma 20.**
\[
\Psi\left(\frac{N}{2}\right) \leq \left( \frac{\Psi(N)^2}{N} + O(1) \right) \log N.
\]

**Lemma 21.** We have
\[
\Psi\left(\frac{N}{2}\right) \leq \left( \frac{(\Psi(N))^2}{N} \right) \log N.
\]
Proof. We observe that
\[
2 \sum_{n \in N} R_1(n) = \sum_{n \in N} (R(n) - \delta(n)) = \sum_{n \in N} \sum_{a_j \in N} 1 - A(N) \\
= \sum_{a_j \in N} 1 - A(N) - (\sum_{a_j \in N} 1)^2 - A(N) \leq A^2(N) - A(N),
\]
Since \( A(N) \to \infty \), it follows that \( \sum_{n \in N} R_1(n) \to \infty \). Hence \( R_1(n) \geq 1 \) for infinitely many \( n \). But \( R_1(n) \) is monotonic. Hence \( R_1(n) \geq 1 \) for all large \( n \). Hence \( \sum_{n \in N} R_1(n) \geq N \). Using this lower bound in the above inequality it follows that \( A^2(N) \geq N \). Hence \( A(N) \geq N^{1/2} \). Hence \( \Psi(N) \geq N^{1/2} \). In particular \( \Psi(N/2) \geq N^{1/2} \).

Therefore from Lemma 20,
\[
\left( \frac{\Psi(N)}{N} + O(1) \right) \log N \geq \Psi(N/2) \geq N^{1/2}.
\]
Hence \( \Psi(N) \geq N^{1/4}/(\log N)^{1/2} \). Consequently
\[
\frac{\left( \frac{\Psi(N)}{N} \right)^2}{\log N} + O(1) \leq \frac{\left( \frac{\Psi(N)}{N} \right)^2}{N}.
\]
Hence Lemma 21 follows from Lemma 20.

Now Theorem 5 could be completed as in Theorem 4. From Theorem 5, one can deduce that
\[
A(N) = \Omega(N/\log N).
\]
If not, then
\[
A(2^j N) = o\left( \frac{2^j N}{\log N} \right) \quad \text{for all } j \geq 1.
\]
Then
\[
\Psi(N) = \sum_{a \in A} e^{-a/N} \sum_{a_j \in \mathbb{Z}^+ \backslash 1} e^{-a/N} \\
\leq \sum_{a \in A} \sum_{a_j \in \mathbb{Z}^+ \backslash 1} e^{-2j} \leq \sum_{a \in A} e^{-2j} \left( \frac{2^j N}{\log N} \right) \quad \text{for a suitably small } \varepsilon > 0.
\]
Hence \( \Psi(N) \leq c e^{-\varepsilon N/\log N} \) for an absolute constant \( c \), which contradicts Theorem 5.

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