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A note on a result of Erdős, Sárközy and Sós

by

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1. Introduction. Let A be an infinite sequence of positive integers $a_1 < a_2 < a_3 < \ldots$; Let

$$R(n) = \sum_{a_i+a_j=n} 1; \quad R_1(n) = \sum_{\substack{a_i+a_j=n \\ i<j}} 1; \quad R_2(n) = \sum_{\substack{a_i+a_j=n \\ i\leqslant j}} 1; \quad A(N) = \sum_{\substack{a\in A \\ a\leqslant N}} 1.$$

Then the following three theorems are due to Erdős, Sárközy and Sós [1] (our notation is different from their notation; they use R_1, R_2 and R_3 instead of our R, R_1 and R_2).

Theorem 1. *If $R(n+1) \geqslant R(n)$ for all large n, then $A(N) = N+O(1)$.*

Theorem 2. *If $R_1(n+1) \geqslant R_1(n)$ for all large n, then $A(N) = \Omega(N/\log N)$ (and a similar result for $R_2(n)$).*

Theorem 3. *For a suitable sequence A, we have, $R_1(n+1) \geqslant R_1(n)$ for all large n and $A(N) \leqslant N-cN^{1/3}$ for infinitely many N and for a suitable $c > 0$.*

The authors in [1] state that probably a theorem analogous to Theorem 3 holds for $R_2(n)$ also.

In this note, we give (what we believe) a simple proof of Theorem 1; slightly improve Theorem 2 and disprove their conjecture (?) about a result analogous to Theorem 3 for $R_2(n)$. More precisely we prove

Theorem 4. *If $R_2(n+1) \geqslant R_2(n)$ for all large n, then*

$$A(N) = N+O(\log N).$$

Theorem 5. *If $R_1(n+1) \geqslant R_1(n)$ for all large n, then*

$$\sum_{a\in A} e^{-a/N} \gg N/\log N.$$

2. Notation. In the sequel, a, a_k denote the generic elements of A;

$$f(z) = \Big(\sum_{a\in A} z^a\Big);$$

consequently

$$(f(z))^2 = \sum_{n=1}^{\infty} R(n) z^n;$$

Let α be a real number, $0 < \alpha < 1$, and α is close to 1;

$$g(z) = f(z)(1-z) = \sum_{n=1}^{\infty} b_n z^n;$$

$$\varepsilon_n = \begin{cases} 1 & \text{if } n = 2a_k \text{ or } 2a_k - 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_n = \begin{cases} 1 & \text{if } n = 2a_k, \\ 0 & \text{otherwise;} \end{cases}$$

$$\lambda_n = R(n+1) - R(n) - \varepsilon_n.$$

We write $A(z) \overset{p}{\equiv} B(z)$ to denote that $A(z) - B(z)$ is a polynomial in z.

3. Proof of Theorem 1.

LEMMA 1. (a) *We have* $R(n) \leqslant 2A((n+1)/2)$ *for all* n.

(b) *We have* $\sum_{n \leqslant m} \varepsilon_n \geqslant 2A(m/2)$ *for all* m.

(c) $R(2a_k)$ *is odd;* $R(2a_k+1)$ *and* $R(2a_k-1)$ *are even.*

Proof. We have

$$R(n) = \sum_{a_i + a_j = n} 1 \leqslant 2 \sum_{\substack{a_i \leqslant a_j \\ a_i + a_j = n}} 1 \leqslant 2 \sum_{a_i \leqslant (n+1)/2} 1 = 2A\left(\frac{n+1}{2}\right)$$

and this proves 1 (a); Lemma 1 (b) follows from the definition of ε_n; to prove 1 (c), note that all the solutions (a_i, a_j) of $a_i + a_j = n$ can be paired as (a_i, a_j) and (a_j, a_i) except when n is of the form $2a_k$, in which case, we have one extra solution (a_k, a_k).

LEMMA 2. *We have* $\lambda_n \geqslant 0$ *for all large* n.

Proof. This follows from the monotonicity of $R(n)$ and Lemma 1 (c).

LEMMA 3. *We have* $\sum_{n \leqslant m} \lambda_n$ *is bounded above for all large* n.

Proof. We have

$$\sum_{n \leqslant m} \lambda_n = \sum_{1 \leqslant n \leqslant m} (R(n+1) - R(n) - \varepsilon_n) = R(m+1) - R(1) - \sum_{n \leqslant m} \varepsilon_n$$

and the lemma follows from Lemmas 1 (a) and (b).

LEMMA 4. *We have* $\lambda_n = 0$ *for large* n.

Proof. This follows from Lemmas 2 and 3.

LEMMA 5. *We have* $(f(z))^2(1-z) = (1+z) f(z^2) + p(z)$ *for a suitable polynomial* $p(z)$.

Proof. From Lemma 4, we have $\sum \lambda_n z^{n+1} \overset{p}{\equiv} 0$. Hence

$$\sum (R(n+1) - R(n) - \varepsilon_n) z^{n+1} \overset{p}{\equiv} 0.$$

Hence

$$\left(\sum R(n) z^n\right)(1-z) \overset{p}{\equiv} \sum \varepsilon_n z^{n+1}.$$

Hence

$$(f(z))^2 (1-z) \overset{p}{\equiv} \sum (z^{2a_k} + z^{2a_k+1}) \overset{p}{\equiv} (1+z) \sum z^{2a_k} \overset{p}{\equiv} (1+z) f(z^2).$$

LEMMA 6. *We have* $(g(z))^2 = g(z^2) + p(z)(1-z)$.

Proof. This follows from Lemma 5 and the definition of $g(z)$.

LEMMA 7. *We have* $b_n = 0$ *for all large* n.

Proof. From Lemma 6, we have

$$\int_{|z|=\alpha} |g(z)|^2 dz \leqslant \int_{|z|=\alpha} |g(z^2)| dz + O(1)$$
$$\leqslant \Big(\int_{|z|=\alpha} |g(z^2)|^2 dz\Big)^{1/2} \Big(\int_{|z|=\alpha} |dz|\Big)^{1/2} + O(1).$$

Hence, allowing $\alpha \to 1$, we have

$$\sum_{n=1}^{\infty} |b_n|^2 \ll \left(\sum |b_n|^2\right)^{1/2} + O(1).$$

Consequently $\sum |b_n|^2 = O(1)$. Hence the lemma.

Now $b_n = \chi(n) - \chi(n-1)$ where χ is the characteristic function of the sequence A. Since A is an infinite sequence, $\chi(n) = 1$ for infinitely many n and hence for large n. Hence Theorem 1 follows.

4. Proof of Theorem 4.

LEMMA 8. *We have*

$$f(\alpha^2) \leqslant \frac{1-\alpha}{2\alpha} (f(\alpha))^2 + O(1).$$

Proof. We put $z = -\alpha$ in the relation $(f(z))^2 = \sum R(n) z^n$ to get

$$0 \leqslant (f(-\alpha))^2 = \sum R(n)(-1)^n \alpha^n = \sum \left(R(2n)\alpha^{2n} - R(2n+1)\alpha^{2n+1}\right).$$

This yields, using $R(n) = 2R_2(n) - \delta(n)$,

$$0 \leqslant 2\sum (R_2(2n) - R_2(2n+1))\alpha^{2n+1} + 2\sum R_2(2n)(\alpha^{2n} - \alpha^{2n+1})$$
$$- \sum (\delta(2n) - \delta(2n+1)\alpha^{2n+1}) = \Sigma_1 + \Sigma_2 + \Sigma_3 \quad \text{say.}$$

Since $R_2(2n) - R_2(2n+1) \leqslant 0$ for all large n, Σ_1 is bounded above. From the definition of $\delta(n)$ it follows that

$$\Sigma_3 = -\sum \alpha^{2a_k} = -(f(\alpha^2)).$$

Further

$$\begin{aligned}\Sigma_2 &= 2\sum R_2(2n)(\alpha^{2n}-\alpha^{2n+1}) = 2(1-\alpha)\sum R_2(2n)\alpha^{2n}\\ &\leqslant (1-\alpha)\sum (R_2(2n)+R_2(2n+1))\alpha^{2n}+O(1)\\ &\leqslant \frac{1-\alpha}{\alpha}\sum (R_2(2n)\alpha^{2n}+R_2(2n+1)\alpha^{2n+1})+O(1)\\ &= \frac{1-\alpha}{\alpha}\sum R_2(n)\alpha^n+O(1) \leqslant \frac{1-\alpha}{2\alpha}\sum R(n)\alpha^n+O(1)\\ &= \frac{1-\alpha}{2\alpha}(f(\alpha))^2+O(1)\end{aligned}$$

and the lemma follows.

LEMMA 9. *Define* $\Psi(N) = f(e^{-1/N})$. *Then*

$$(\Psi(N))^2 \geqslant 2N\Psi(N/2)+O(N).$$

Proof. We put $\alpha = e^{-1/N}$, note that $f(\alpha) = \Psi(N)$ and $f(\alpha^2) = \Psi(N/2)$. Then

$$\frac{1-\alpha}{\alpha} = \frac{1}{N}(1+O(1/N)).$$

Hence the result follows from Lemma 8.

5. Proof of Theorem 4 (continued). In this section, we assume, as we may, that

$$\Psi(N)\to\infty \quad \text{as} \quad N\to\infty.$$

LEMMA 10. *We have*

$$\prod_{j=1}^{\lambda}\left(\frac{N}{2^{j-2}}\right)^{1/2^j} \geqslant N^{1-(1/2^\lambda)}.$$

Proof. The power of N in the left side is easily seen to be $1-(1/2^\lambda)$. The power of 2 appearing in the denominator

$$= \sum_{j=1}^{\lambda}\frac{j-2}{2^j} = \sum_{j=0}^{\infty}\frac{j-2}{2^j}+2$$

$$= \sum_{j=0}^{\infty}(j+1)(\tfrac12)^j - 3\sum(\tfrac12)^j+2 = (1-\tfrac12)^{-2}-3(1-\tfrac12)^{-1}+2 = 0.$$

LEMMA 11. *We have*

$$(1-d_1)^{d_2} \geqslant 1-2d_1d_2 \quad \text{if} \quad 0<d_1<1/2;\ d_2>0.$$

Proof. We have

$$\left(-\log((1-d_1)^{d_2})\right) = -d_2\log(1-d_1) = d_2\left(d_1+\frac{d_1^2}{2}+\frac{d_1^3}{3}+\ldots\right)$$
$$\leqslant d_2(d_1+d_1^2+d_1^3+\ldots) \leqslant 2d_1d_2.$$

Hence $(1-d_1)^{d_2} \geqslant e^{-2d_1d_2} \geqslant 1-2d_1d_2$.

LEMMA 12. *For a suitable constant* $c_1>0$,

$$(\Psi(N))^2 \geqslant N\Psi(N/2) \geqslant N \quad \text{for all } N\geqslant c_1.$$

Proof. Since $\Psi(N)\to\infty$, both inequalities follow from Lemma 9.

LEMMA 13. *We have* $\Psi(N)\geqslant N$.

From Lemma 12,

$$\Psi(N) \geqslant N^{1/2}(\Psi(N/2))^{1/2} \quad \text{if } N \text{ is large.}$$
$$\Psi(N/2) \geqslant (N/2)^{1/2}(\Psi(N/4))^{1/2} \quad \text{if } N \text{ is large.}$$

Hence

$$\Psi(N) \geqslant N^{1/2}(N/2)^{1/4}(\Psi(N/4))^{1/2} \quad \text{if } N \text{ is large.}$$

Proceeding similarly and choosing λ such that

$$c_1 \leqslant N/2^{\lambda+1} \leqslant 2c_1,$$

we have

$$\Psi(N) \geqslant N^{1/2}(N/2)^{1/4}\ldots(N/2^\lambda)^{\lambda/2}(\Psi(N/2^{\lambda+1}))^{1/2^{\lambda+1}}.$$

Since $\Psi(N/2^{\lambda+1})\geqslant 1$ (Lemma 12), we are through.

LEMMA 14. *For a suitable constant* c_2,

$$(\Psi(N))^2 \geqslant 2N\Psi(N/2)(1-c_2/N).$$

Proof. This follows from Lemmas 9 and 13. In the sequel, we assume, as we may, that $c_2\geqslant c_1$.

LEMMA 15. *We have*

$$\Psi(N) \geqslant N+O(\log N).$$

Proof. From Lemma 14, we have

$$\Psi(N) \geqslant (2N)^{1/2}(\Psi(N/2))^{1/2}(1-c_2/N)^{1/2},$$
$$\Psi(N/2) \geqslant (N)^{1/2}(\Psi(N/4))^{1/2}(1-2c_2/N)^{1/2}.$$

Hence

$$\Psi(N) \geqslant (2N)^{1/2} N^{1/4} \left(\Psi(N/4)\right)^{1/4} (1-c_2/N)^{1/2} (1-c_2/N)^{1/4}$$

and using Lemma 11,

$$\Psi(N) \geqslant (2N)^{1/2} N^{1/4} \left(\Psi(N/4)\right)^{1/4} \left(1-\frac{1}{4}\cdot\frac{c_2}{N}\right)^2.$$

Proceeding similarly and choosing λ such that

$$100 \leqslant \frac{N}{2^{\lambda-1} c_2} \leqslant 200,$$

we have

$$\Psi(N) \geqslant (2N)^{1/2} N^{1/4} \left(\frac{N}{2}\right)^{1/8} \cdots \left(\frac{N}{2^{\lambda-2}}\right)^{1/2^\lambda} \left(\Psi\left(\frac{N}{2^\lambda}\right)\right)^{1/2^\lambda} \left(1-\frac{1}{4}\cdot\frac{c_2}{N}\right)^\lambda.$$

Now using $\Psi(N/2^\lambda) \geqslant 1$ and Lemma 10, we have

$$\Psi(N) > N^{1-1/2^\lambda} \left(1-\frac{1}{4}\cdot\frac{c_2}{N}\right)^\lambda$$

and hence the lemma.

To prove Theorem 4, observe that, if $\chi(n)$ is the characteristic function of the set A, then from Lemma 15, we have

$$\sum_{n=1}^{\infty} \chi(n) e^{-n/N} \geqslant N + O(\log N).$$

Hence

$$\sum_{n=1}^{\infty} \left(1-\chi(n)\right) e^{-n/N} = O(\log N).$$

Hence

$$\sum_{n \leqslant N} \left(1-\chi(n)\right) = O(\log N)$$

and this proves Theorem 4.

6. Proof of Theorem 5. The proofs of Lemmas 16, 17, 18 and 19 are already contained in [1], and we give them for the sake of completeness.

LEMMA 16. *We have*

$$\left|\sum_{n=1}^{\infty} R_1(n) z^n\right| \leqslant \frac{\left(f(\alpha)\right)^2 (1-\alpha) + O(1)}{|1-z|} \quad \textit{on the circle } |z| = \alpha.$$

Proof.

$$\begin{aligned} \left|\left(\sum R_1(n) z^n\right)(1-z)\right| &= \left|\sum \left(R_1(n) - R_1(n-1)\right) z^n\right| \\ &\leqslant \sum \left(R_1(n) - R_1(n-1)\right) \alpha^n + O(1) \\ &= \left(\sum R_1(n) \alpha^n\right)(1-\alpha) + O(1) \\ &\leqslant \left(\sum R(n) \alpha^n\right)(1-\alpha) + O(1) \\ &= \left(f(\alpha)\right)^2 (1-\alpha) + O(1). \end{aligned}$$

LEMMA 17. *We have, on the circle* $|z| = \alpha$,

$$|f(z)|^2 \leqslant 2\frac{\left(f(\alpha)\right)^2 (1-\alpha) + O(1)}{|1-z|} + |f(z^2)|.$$

Proof. We observe that

$$\left(f(z)\right)^2 = \sum R(n) z^n = 2\sum R_1(n) z^n + \sum \delta_n z^n = 2\sum R_1(n) z^n + f(z^2)$$

and the result follows from Lemma 16.

LEMMA 18. *We have*

$$f(\alpha^2) \ll \left(\left(f(\alpha)\right)^2 (1-\alpha) + O(1)\right) \log\frac{1}{1-\alpha} + \left(f(\alpha^4)\right)^{1/2}.$$

Proof. We integrate the relation in Lemma 17 on $|z| = \alpha$. We note that

$$\int_{|z|=\alpha} |f(z)|^2\, dz = \sum_{a \in A} \alpha^{2a} = f(\alpha^2),$$

$$\int_{|z|=\alpha} \frac{dz}{|1-z|} \ll \log\frac{1}{1-\alpha},$$

$$\int_{|z|=\alpha} |f(z^2)|\, dz \leqslant \Big(\int_{|z|=\alpha} |f(z^2)|^2\, dz\Big)^{1/2} \left(\int |dz|\right) \ll \Big(\sum_{a \in A} \alpha^{4a}\Big)^{1/2}.$$

LEMMA 19. *We have*

$$f(\alpha^2) \ll \left(\left(f(\alpha)\right)^2 (1-\alpha) + O(1)\right) \log\frac{1}{1-\alpha} \quad \textit{as} \quad \alpha \to 1.$$

Proof. Since $f(\alpha^4) \leqslant f(\alpha^2)$ and $f(\alpha^2) \to \infty$ as $\alpha \to 1$, the last term on the right of Lemma 17 could be dropped. Putting $\alpha = e^{-1/N}$, we have

LEMMA 20.

$$\Psi\left(\frac{N}{2}\right) \ll \left(\frac{\left(\Psi(N)\right)^2}{N} + O(1)\right) \log N.$$

LEMMA 21. *We have*

$$\Psi\left(\frac{N}{2}\right) \ll \frac{\left(\Psi(N)\right)^2}{N} \log N.$$

Proof. We observe that

$$2\sum_{n\leqslant N} R_1(n) = \sum_{n\leqslant N} (R(n)-\delta(n)) = \sum_{n\leqslant N}\sum_{a_i+a_j=n} 1 - A(N)$$
$$= \sum_{a_i+a_j\leqslant N} 1 - A(N) \leqslant \Big(\sum_{a_i\leqslant N} 1\Big)^2 - A(N) \leqslant A^2(N) - A(N).$$

Since $A(N)\to\infty$, it follows that $\sum_{n\leqslant N} R_1(n)\to\infty$. Hence $R_1(n)\geqslant 1$ for infinitely many n. But $R_1(n)$ is monotonic. Hence $R_1(n)\geqslant 1$ for all large n. Hence $\sum_{n\leqslant N} R_1(n) \gg N$. Using this lower bound in the above inequality it follows that $A^2(N)\gg N$. Hence $A(N)\gg N^{1/2}$. Hence $\Psi(N)\gg N^{1/2}$. In particular $\Psi(N/2)\gg N^{1/2}$.

Therefore from Lemma 20,

$$\left(\frac{(\Psi(N))^2}{N}+O(1)\right)\log N \gg \Psi(N/2) \gg N^{1/2}.$$

Hence $\Psi(N)\gg N^{3/4}/(\log N)^{1/2}$. Consequently

$$\frac{(\Psi(N))^2}{N}+O(1) \ll \frac{(\Psi(N))^2}{N}.$$

Hence Lemma 21 follows from Lemma 20.

Now Theorem 5 could be completed as in Theorem 4. From Theorem 5, one can deduce that

$$A(N) = \Omega(N/\log N).$$

If not, then

$$A(2^j N) = o\left(\frac{2^j N}{\log N}\right) \quad \text{for all } j\geqslant 1.$$

Then

$$\Psi(N) = \sum_{a\in A} e^{-a/N} = \sum_j \sum_{\substack{2^j N < a \leqslant 2^{j+1}N\\ a\in A}} e^{-a/N}$$

$$\leqslant \sum_j e^{-2^j} \sum_{\substack{a\leqslant 2^{j+1}N\\ a\in A}} 1 \leqslant \sum_j e^{-2^j}\left(\varepsilon\frac{2^j N}{\log N}\right) \quad \text{for a suitably small } \varepsilon>0.$$

Hence $\Psi(N)\leqslant c\varepsilon\dfrac{N}{\log N}$ for an absolute constant c, which contradicts Theorem 5.

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Dr. S. Srinivasan kindly pointed out the following result which is equivalent to Theorem 1.

Let $1 = a_0 < a_1 < a_2 < \ldots$ be an infinite sequence. Define

$$\varepsilon_r(n) = \begin{cases} 1 & \text{if } n-a_r\in[a_{2j}, a_{2j+1}) \text{ for some } j,\\ 0 & \text{otherwise.}\end{cases}$$

Then the function $f(n) = \sum(-1)^r\varepsilon_r(n)$ changes sign infinitely often.

Added in proof. We observe that the method of proof of Theorem 4 actually yields the following stronger theorem (compare with Theorem 2 of *Problems and results on additive properties of general sequences, V* by P. Erdős, A. Sárközy and V. T. Sós):

THEOREM. *If*

$$\lim_{n\to\infty}\frac{n-A(n)}{\log n} = \infty,$$

then

$$\limsup \sum_{k\leqslant N} (R_2(2k)-R_2(2k+1)) = \infty.$$

References

[1] P. Erdős, A. Sárkőzy and V. T. Sós, *Problems and results on additive properties of general sequence*, IV, Number theory proceedings, Ootacamund, India 1984, Springer-Verlag Lecture Notes 1122, pp. 85–104.

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